Selected topics in Functional Equations

One Variable Functional Equations

In this part we introduce some applied methods for tackling non-elementary problems in functional equations. Indeed the reader can receive the explanation power after reviewing this lecture and his ability to solve similar problem is the best explanandum for it.

Problem-1: Find all surjective functions f, from $\mathbb{R}^{+}$to $\mathbb{R}^{+}$such that:

$$f(x)\left(x+f\big(f(x)\big)\right)=2xf(f(x))$$

Brazilian Math Olympiads

Solution: first we guess the identity function satisfy the problem statement then we try to establish our hypothesis by this method, assume there exist positive number a such that $f(a)=a-d$for some nonzero real number d. indeed if the existence of a be refuted we are done. Then we use the surjectivity criteria to receive that , there must exist positive real number x_0 such that $f(x_0)=a$ now we rewrite the problem statement as: $\frac{1}{x}+\frac{1}{f(f(x))}=\frac{2}{f(x)}$ and set $x=x_0$ to receive that:

$$x_0=-\frac{a(a-d)}{d}$$

By the assumption the quantities $a,a-d$are positive thus we must have $d<0$ and the without less of generality we can assume that there exist positive real number a, and positive real number for it likewise d such that: $f(a)=a+d$ then the value of $x_0$changed to be $x_0=\frac{a(a+d)}{d}$ .now by setting the value of x to be a we see that: $f(a+d)=\frac{a(a+d)}{a-d}$ and inductively $f\left(\frac{a(a+d)}{a-(n-1)d}\right)=\frac{a(a+d)}{a-nd}$ for all positive integers n. but by the Archimedean laws of real numbers there exist positive integer n such that: $nd>a$ in this sense the value of $f\left(\frac{a(a+d)}{a-(n-1)d}\right)$ must be negative. Which is absurd thus it cant happen only if d=0. And we are done.

Problem-2: find all increasing function f from $\mathbb{R}$ to $\mathbb{R}$ such that:

$$f\big(f(x)\big) + x = 2f(x)$$

Solution: by the method applied in the previous problem assume there exist number a, such that: $f(a) = a + d$ for some nonzero real d and lets assume d be positive(the other case is same as this one) thus inductively we set that: $f(a + nd) = a + (n + 1)d$ now by the injectivity of f we receive that $f(x_0) = x_0 + kd$ iff $x_0 = a + (k - 1)d$. Now set the real number a_1 in the interval $(a, a + d)$and set $f(a_1) = a_1 + d_1$ for some real numberd_1 . inductively we can see that: $f(a_1 + nd_1) = a_1 + (n + 1)d_1$ and we can prove that: for all n, $a + nd < a_1 + nd_1 < a + (n + 1)d$. Then rewrite the latter inequality as: $-\frac{a+d-a_1}{n} < d - d_1 < \frac{a_1-a}{n}$ as n tend to infinity sides of inequality vanishes an it set that $d = d_1$ thus for all numbers in the interval $(a, a + d)$we have$(a_1) = a_1 + d$ as we can cover the real numbers with the union of this intervals we can see that $f(x) = x + d$ for all real number x.

At the border of Completion Axiom

The reader need to know about elementary set theory and the definition of Supremum and infimum of the set indeed by the definition of the supremum and infimum one can find that if for a set S, we know $\mathrm{Sup(S)}=\mathrm{S}$, then for any $\varepsilon > 0$there exist an element of S, likewise x such that: $x \geq S - \varepsilon$. And if we know $\mathrm{Inf(S)}=\mathrm{I}$. then for any $\varepsilon > 0$there exist an element of S, likewise x such that: $x \leq \mathrm{I}+\varepsilon$. We need this two notion to use as a useful judgment. For finding the function. But for use of this tool we need to define a set which has Supremum and infimum for this reason we instantiate the Axiom named as “Completeness” it stated that:

If a non-empty set of real numbers to be bounded above , it has Supremum. And if to be bounded below, it necessarily has infimum.

Problem-3: find all functions $f: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$and we know $\forall x \geq 0$ we have $4f(x) \geq 3x$, and $f(4f(x) - 3x) = x$. Turkish TST

Solution: first we can find that $f(0)=0$, indeed $0=f(4f(0))\geq 3f(0)$. Now define the set $A=\left\{\frac{f(x)}{x}\middle| x\in\mathbb{R}^+\right\}$ by the problem assumption we see that A has to be bounded from below, and we can observe that:

$x=f(4f(x)-3x)\geq\frac{3}{4}(4f(x)-3x)$which is reduced to $f(x)\leq\frac{13}{12}x$. thus the set A has Supremum and infimum name S,I. thus there exist points like x_0,x_1 such that $\frac{f(x_0)}{x_0}\leq I+\varepsilon,\frac{f(x_1)}{x_1}\geq S-\varepsilon$ for any positive real ε. Thus we can establish the inequality: $x=f(4f(x)-3x)\leq S(4f(x)-3x)$or $\frac{f(x)}{x}\geq\frac{3S+1}{4S}$ $(*)$ for any positive real ε, we have: $I+\varepsilon\geq\frac{3S+1}{4S}$ indeed put $x=x_0$ in (*)and by the same argument we can set that for any positive real ε, we have:$\frac{3I+1}{4I}\geq S-\varepsilon$. As ε tends to zero we receive this system of inequalities:

$$\frac{3I+1}{4I}\geq S\ ,I\geq\frac{3S+1}{4S}$$

As I,S are positive we receive that $I=S$,or equivalently the set A must be an one element set conclude that $\frac{f(x)}{x}=c$ for all positive reals, by use of the equation of the function we see that: c=1.

Note: we can establish the system of inequality without the notion of ε indeed by the definition of Supremum and infimum I must be greater than $\frac{3S+1}{4S}$ if not the maximality of I was refuted.

Problem-4: find all surjective and strictly increasing function f from $\mathbb{R}$to $\mathbb{R}$ such that: $f\big(f(x)\big)=f(x)+12x$ Vietnam-Olympiad

Solution: at first we see that f(0)=0. And that for all x>0 we have $f(x)>0$ and otherwise $f(x)<0$. Indeed $f(x)$and x has same sign. This lead to the following inequalities. First $f\big(f(x)\big)>f(x)$ for all positive reals and by the surjectivity of f we receive that: $f(x)>x$, thus we have: $f\big(f(x)\big)<13f(x)$lead to $f(x)<13x$which help us to see that the set $A=\left\{\frac{f(x)}{x}\middle| x\in\mathbb{R}^+\right\}$is bounded an then has supremum and infimum namely S,I. Then we know there exist points like x_0,x_1 such that $\frac{f(x_0)}{x_0}\leq I+\varepsilon,\frac{f(x_1)}{x_1}\geq S-\varepsilon$ for any positive real ε. Thus we can establish the inequality, by the surjectivity of f, we know there exist positive real number t_0 such that $f(t_0)=x_0$and exist positive real number t_1 such that $f(t_1)=x_1$and then we receive that:

$$I + \varepsilon \geq \frac{f(f(t_0))}{f(t_0)} = 1 + 12\frac{t_0}{f(t_0)} \geq 1 + \frac{12}{S}$$

$$S - \varepsilon \leq \frac{f(f(t_1))}{f(t_1)} = 1 + 12\frac{t_1}{f(t_1)} \leq 1 + \frac{12}{I}$$

Then by the same argument as the preceding problem we can see that: $I \geq 1 + \frac{12}{S}$ and $S \leq 1 + \frac{12}{I}$ Which lead to I=S. And we receive $f(x) = 4x$ by the same argument for negative reals we are done.

Problem-5: find all monotone functions $f: \mathbb{R} \to \mathbb{R}$ and we have

$$f(x) + x = f(f(x))$$

Solution: here we start another way of tackling this type of problems based on recursive sequences. We divide the problem in two case but before it we can see that $f(0) = 0$. Now we introduce two cases, first the function be decreasing and the second the function be increasing, by the first case we have $f(x) > 0$ for all $x < 0$ thus for all negative x, f(x) must be positive and then $f\left(f(x)\right) < 0$. Then $f(x) < -x$ thus we can set that: $|f(x)| < |x|$ and by the same argument $\left|f^{(n)}(x)\right| < |x|$. we also can prove inductively $f^{(n)}(x) = F_{n-1}x + F_n f(x)$. fix the value of x and tend n to infinity we can see that: $\frac{f(x)}{x} + \frac{F_{n-1}}{F_n} = \frac{1}{F_n}.\frac{f^{(n)}(x)}{x}$ and we receive that: $\frac{f(x)}{x} = -\lim_{n\to\infty}\frac{F_{n-1}}{F_n} = \frac{1-\sqrt{5}}{2}$. In the second case we have f(x) and x both drom the same sign implies that $|x| < |f(x)|$. then if we use the inverse function g of f we see that: $|g(x)| < |x|$ and $g\left(g(x)\right) + g(x) = x$ also by induction we ensure that: $(-1)^{n+1}g^{(n)}(x) = F_n g(x) - F_{n-1}x$ continuing this works lead us to the function: $f(x) = \frac{1+\sqrt{5}}{2}x$.

Comment: try to solve this by the method of solving problem-4.

Problem-6: find all pairs of positive reals a,b such that there exist a function $f: \mathbb{R}^+ \to \mathbb{R}^+$. Such that. $f\left(f(x)\right) = af(x) - bx$ Kazakhstan Olympiad

Solution: first assume such function exists. once again we will use the notion of Infimum. Consider the set $A = \left\{\frac{f(x)}{x}\middle| x \in \mathbb{R}^+\right\}$ by the problem assumption we see that A has to be bounded from below(since $\frac{f(x)}{x} > 0$) define its infimum value as I. and then see that: $f\left(f(x)\right) \geq If(x) \geq I^2x$ and for any $\varepsilon > 0$ there exist $x_0$such that $f(x_0) \leq (I + \varepsilon)x_0$. Thus we receive the following inequality:

$$I^2 x \leq (aI + a\varepsilon - b)x$$

Or $I^2 - aI + b \leq 0$ this occurred only if$a^2 - 4b \geq 0$. Then the equation$I^2 - aI + b = 0$ has two positive solutions. Name one of them as $c(a,b)$. Then the function $f(x) = c(a,b)$x satisfy the problem condition.

Problem-7: Let for any positive reals a, b, c we define the set $H(a,b,c)$ as set of functions $h: \mathbb{R}^+ \to \mathbb{R}^+$such that $h(x) \geq h\big(h(ax)\big) + h(bx) + cx$ prove that $H(a,b,c)$is non-empty set iff $b \leq 1, 4ac \leq (b-1)^2$. Crux

Solution : if $b \leq 1, 4ac \leq (b-1)^2$ the equation $ax^2 + (b-1)x + c = 0$ has positive solution $d(a,b,c)$. And the function $h(x) = d(a,b,c)$x satisfy the problem statement. Now assume such function exist and define the set set $A = \left\{\frac{f(x)}{x} \middle| x \in \mathbb{R}^+\right\}$ by the problem assumption we see that A has to be bounded from below(since $\frac{f(x)}{x} > 0$) define its infimum value as I. and then see that: $h\big(h(ax)\big) \geq I^2 ax, h(bx) \geq Ibx$ thus by the method solving previous problem we see that: $I \geq aI^2 + bI + c$ and by previous problem we are done.

Problem-8. Let function f defined by the equation $f(2x) = 2f^2(x) - 1$if we know there exist a real number $x_0$such that $f(2^n x_0) \leq 0$. Find all possible value of $f(x_0)$. Singapore Selection Test

Solution: we know that the functions f and –f satisfy the problem statement thus without less of generality we assume $f(x_0) < 0$ .and define the set $A = \{f(2^n x_0) | n \in \mathbb{Z}, n \geq 0\}$ this set is bounded above then have supremum say S. thus we can find that $S \leq 0$. Then for elements of A, we have the inequality $2S^2 - 1 \leq S$ Or $\frac{-1}{2} \leq S \leq 0$. Thus for all elements of A, like t we have $f(t) \leq \frac{-1}{2}$. This leads to the inequality $2f^2\big(2^{k-1}x_0\big) - 1 \leq \frac{-1}{2}$or $f\big(2^{k-1}x_0\big) \in [-\frac{1}{2}, \frac{1}{2}]$. Which leads to the argument that A has one element which is$-\frac{1}{2}$.

Problem-9: Find all function $f:\mathbb{R}^+ \to \mathbb{R}^+$ which is strictly increasing for $\mathbb{R}^{>1}$.and satisfy in the following conditions:

$$f(x) = f\left(\frac{1}{x}\right)$$

- $f(x^2) = f^2(x) - 2$

$$f(x^3) = f^3(x) - 3f(x)$$

Adapted After Polish training Camp

Solution: if we set $A = \{f(x)|x \in \mathbb{R}^+\}$ we can set I ,as its infimum. Which both inequalities $I \le I^2 - 2$and $I \ge I^2 - 2$Satisfied. Which leads to $I = 2$. Meant that for all positive reals x we must have $f(x) \ge 2$. Then there exist positive function g, forwhich $f(x) = g(x) + \frac{1}{g(x)}$ which leads to $g(x) = \frac{f(x)+\sqrt{f(x)^2-4}}{2}$or $g(x) = \frac{f(x)-\sqrt{f(x)^2-4}}{2}$ which the first will be strictly increasing in $\mathbb{R}^{>1}$and the later will be strictly decreasing in $\mathbb{R}^{>1}$(since the product of them is constant). If we regard the second equation by g, we receive that $g(x^2) - g^2(x) = \frac{g(x^2)-g^2(x)}{g(x^2)g^2(x)}$

If $g(x^2)g^2(x) = 1$for some x>1.we receive that one of the quantities $g(x)$or $g(x^2)$must be greater than unit. But as g(1)=1(Since f(1)=2) being greater then unit of once implies the other. Thus we must have $g(x^2) = g^2(x)$. by use of the third we receive that: $g(x^3) = g^3(x)$. thus for any rational number of the form $r = 2^n 3^m$ where m,n be integers. we have $g(x^r) = g^r(x)$.

Problem-10: Find all function $f:\mathbb{R}^+ \to \mathbb{R}^+$ which is strictly increasing for $\mathbb{R}^{>1}$.and satisfy in the following condition:

$$f(x)f(y) = f(xy) + f\left(\frac{x}{y}\right)$$

Adapted after Petersburg Olympiad

Solution: one can see that the function f satisfies the first and the second conditions of the problem-9. Thus we can define such function g and assume it will be strictly increasing. Thus we receive that $g(x^2) = g^2(x)$. now we inductively will prove that: $g(x^n) = g^n(x)$. indeed set $y = x^n$ to receive that $f(x)f(x^n) = f(x^{n+1}) + f(x^{n-1})$ and by induction hypothesis we have: $f(x^k) = g^k(x) + \frac{1}{g^k(x)}$.for all $k \le n$. Then we receive that: $g(x^{n+1}) = g^{n+1}(x)$. now see that $g^n\left(x^{\frac{m}{n}}\right) = g(x^m) = g^m(x)$. thus for any rational number r, we have $g(x^r) = g^r(x)$. then fix $t > 1$ and set $g(t) = t^s$ for some real s.(indeed we set $s = \log_t g(t)$) we now g(t)>1. Then we obtain s>0. We can see that for all rational number r, we have $g(t^r) = t^{rs}$. Then set two sequence of rationales. p_n, q_n such that $p_n \leq y \leq q_n$ for some real number y. then by the ascending behavior of g. we have $g(t^{p_n}) \leq g(t^y) \leq g(t^{q_n})$or $t^{sp_n} \leq g(t^y) \leq t^{sq_n}$if we set the limit point of both sequence to be y. then $g(t^y) = t^{sy}$ now for arbitrary x, set $y = \log_t x$ we see that $g(x) = x^s$for all x>1. And now $f(x) = x^s + x^{-s}$Then by the equation $f(x) = f\left(\frac{1}{x}\right)$we can find the equation of f. for all the positive real line.

Comment: by the approach of this problem you can solve this problem which proposed for the IMO-2003.

Find all function $f: \mathbb{R}^+ \to \mathbb{R}^+$ which is strictly increasing for $\mathbb{R}^{>1}$.and satisfy in the following condition:

$$f(xyz) + f(x) + f(y) + f(z) = f\left(\sqrt{xy}\right)f\left(\sqrt{yz}\right)f\left(\sqrt{zx}\right)$$

Problem-11: find all strictly increasing functions $f: \mathbb{R}^+ \to \mathbb{R}^+$ such that for all positive reals we have:

$$f\left(xf(y) + yf(x)\right) = xy + f(x)f(y)$$

Solution: lets define $g(x) = f(x) - x$and $h(x) = f(x) + x$and we find that h is strictly increasing and $g(x) + x$ is so. Then we can find the following relations. By adding $xf(y) + yf(x)$ to the both side. We receive that $h\left(xf(y) + yf(x)\right) = h(x)h(y)$and the expression $xf(y) + yf(x)$ is equal to $\frac{h(x)h(y) - g(x)g(y)}{2}$ then we find that$h\left(\frac{h(x)h(y) - g(x)g(y)}{2}\right) = h(x)h(y)$ and by the same argument $g\left(\frac{h(x)h(y) - g(x)g(y)}{2}\right) = g(x)g(y)$. If we set $x = y$we receive the equations: $g\left(\frac{h^2(x) - g^2(x)}{2}\right) = g^2(x), h\left(\frac{h(x)^2 - g^2(x)}{2}\right) = h^2(x)$now define the function $P(t) = \frac{h^t(x) - g^t(x)}{2}$. Then we find that $g\left(P(2)\right) = g^2(x), h\left(P(2)\right) = h^2(x)$ then set $y = P(2)$obtain that$g\left(\frac{h(x)h\left(P(2)\right) - g(x)g\left(P(2)\right)}{2}\right) = g\left(\frac{h(x)h^2(x) - g(x)g^2(x)}{2}\right) = g(x)g^2(x) = g^3(x)$. implies that for any positive integer n we have :

$$g\left(\frac{h^n(x) - g^n(x)}{2}\right) = g^n(x), h\left(\frac{h(x)^n - g^n(x)}{2}\right) = h^n(x)$$

Now we extend this for rational number by induction on $\frac{p}{q}$. If $p > q$. We have:

$$g\left(\frac{h^{\frac{p-q}{q}}(x) - g^{\frac{p-q}{q}}(x)}{2}\right) = g^{\frac{p-q}{q}}(x), h\left(\frac{h(x)^{\frac{p-q}{q}} - g^{\frac{p-q}{q}}(x)}{2}\right) = h^{\frac{p-q}{q}}(x)$$

Thus $g\left(P\left(\frac{p-q}{q}\right)\right) = g^{\frac{p-q}{q}}(x), h\left(P\left(\frac{p-q}{q}\right)\right) = h^{\frac{p-q}{q}}(x)$. then set y= $P\left(\frac{p-q}{q}\right)$. And recive that$g\left(P\left(\frac{p}{q}\right)\right) = g^{\frac{p}{q}}(x), h\left(P\left(\frac{p}{q}\right)\right) = h^{\frac{p}{q}}(x)$. then r be a positive rational number such that $r > 1$we have $g(P(r))$

Problem-12:we know about the function $f: \mathbb{R} \to \mathbb{R}$ such that :$|f(x+y) - f(x) - f(y)| \leq 1$prove there exist function $g: \mathbb{R} \to \mathbb{R}$ such that $|f(x) - g(x)| \leq 1$ and g is additive. Turkish TST

Solution: first we can see that $|f(2^{m+1}x_0) - 2f(2^m x_0)| \leq 1$and we see that: $\left|\frac{f(2^{m+1}x_0)}{2^{m+1}} - \frac{f(2^m x_0)}{2^m}\right| \leq \frac{1}{2^{m+1}}$and by adding this summations we obtain that: $\sum_{m=0}^{\infty}(\frac{f(2^{m+1}x_0)}{2^{m+1}} - \frac{f(2^m x_0)}{2^m}) = \lim_{n\to\infty} \frac{f(2^{n+1}x_0)}{2^{n+1}}$. Now by the comparison test of convergence we know that $\sum_{m=0}^{\infty}\left|\frac{f(2^{m+1}x_0)}{2^{m+1}} - \frac{f(2^m x_0)}{2^m}\right|$converges. Then wee need some lemma.

Lemma-1: a sequence a_nConverges iff it was Cauchy(i.e. for any $\varepsilon > 0$ there is N,such that for any $m, n > N$ we have $|a_m - a_n| < \varepsilon$.)

Proof. If a sequence converges there exist N such that for all $n > N$ we have $|a_n - L| < \frac{\varepsilon}{2}$. Thus $|a_m - a_n| < |a_n - L| + |a_m - L| < \varepsilon$for all m>n. for the other side first fix n and see that the sequence must be bounded . then we can find that unique numbers $\bar{a}, \underline{a}$such that $a_n < \bar{a} + \varepsilon$for large n and $a_n > \bar{a} - \varepsilon$for infinitely many n. and by the same definition $a_n > \underline{a} - \varepsilon$ for large n and $a_n < \underline{a} + \varepsilon$for infinitely many n. for justifying this we can find some number α, βsuch that $\alpha < a_n < \beta$(since the sequence is bounded) if we define $M_k = \mathrm{Sup}\{a_k, a_{k+1}, \dots\}$ then $\alpha \leq M_k \leq \beta$then the sequence $\{M_k\}$is bounded and we can see that the sequenceM_k is non increasing then we find that$\{M_k\}$ converges, set $\lim_{k\to\infty} M_k = \bar{a}$.if $\varepsilon > 0$ We have $M_k < \bar{a} + \varepsilon$ for large k.and since $a_n \le M_k$ for all $n \ge k$. Then the quantity $\bar{a}$satisfies our claim. And if the inequality $a_n > \bar{a} - \varepsilon$ not holds for infinitely many n then for all large n we have $a_n \le \bar{a} - \varepsilon$. However this leads to $M_k \le \bar{a} - \varepsilon$ which is false. The uniqueness proof is trivial. Then by our claim we have $|a_n - \bar{a}| < \varepsilon$for some n>N and $\left|a_m - \underline{a}\right| < \varepsilon$ for some m>N. then $\left|\bar{a} - \underline{a}\right| \le |a_n - \bar{a}| + |a_m - a_n| + \left|a_m - \underline{a}\right| < 3\varepsilon$. Imply that $\bar{a} = \underline{a} = a$.thus for large n we have $a - \varepsilon < a_n < a + \varepsilon$ then a_n converges.

Lemma-2: if $\sum |a_n|$ converges then $\sum a_n$ so,

Proof: by the lemma-1 we have $\sum_n^m |a_k| < \varepsilon$, then by triangle inequality we have: $|\sum_n^m a_k| \le \sum_n^m |a_k| < \varepsilon$ this leads to convergence of $\sum a_n$.

By lemma1,2. We find that $\lim_{n\to\infty} \frac{f(2^{n+1}x_0)}{2^{n+1}}$exists and define it as $g(x_0)$. It is clear that g must be additive.

Problem-13: find all function $f: \mathbb{R}^+ \to \mathbb{R}^+$ satisfying this equations:

$$f(x+1) = f(x) + 1$$
$$f\left(\frac{1}{f(x)}\right) = \frac{1}{x}$$

Tuymada Olympiads

Solutions: First of all we will prove that $f\big((0,1)\big) = (0,1)$for sake of this reason by use of the first equation we find that if $x > 1$then $f(x) > 1$assume there exist number x_0 in the interval(0,1) such that $f(x_0) > 1$ by the surjectivity of f, we can find y such that $f(y) = f(x_0) - 1$ then $f(y+1) = f(x_0)$implies that $x_0 > 1$.then we can see that if $x \le 1$then $f(x) \le 1$and if $x > 1$then $f(x) > 1$ .and we can easily prove that $f(1) = 1$. Now consider the interval Isuch that $f(I) = I$, if we define $I + n = \{x + n | x \in I, n \in \mathbb{N}\}$and $\frac{1}{I} = \left\{\frac{1}{X}\middle| x \in I\right\}$. By use of first and second conditions we find that $f(I+n) = I + n$and $f\left(\frac{1}{I}\right) = \frac{1}{I}$. We can easily check then the intervals $(n, n+1]$and $[n, n+1)$satisfied in the condition of interval I. lets define a continued fraction of an irrational number x. which is of the form $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots}}}$where$a_1, \dots$ are positive integers and $a_0$is an integer , this representation is unique , and by use of this we can see that $f(a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{I}}}) = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{I}}}$ Indeed as we depict the interval I, as (0,1) or (n,n+1)and continuing this work k-steps to receiving k-th convergent of the continued fraction of x (a_k, a_{k+1})we receive a new interval J_k such that$f(J_k) = J_k$and we can obviously find that $x \in J_k$and then $f(x) \in J_k$as k be large enough we find that $f(x) = x$. if x be a rational number this fractions stops at k steps for some k. let J_k be an open interval which exclude x, and $H_k$be closed interval include x, we find that $f(J_k) = J_k$, $f(H_k) = H_k$and $f(H_k - J_k) = H_k - J_k$. We know that $H_k - J_k = \{x\}$and we are done.