Selected topics in Functional Equations

Inserting New Variable

In this lecture we try to establish the way of solving functional equations with inserting new variable. Indeed this method can be viewed as a discovering way. Which can reduce complexities of the problem.

Problem-1: lets $f:\mathbb{R}^{+}\to\mathbb{R}$ such that $f(x+y)=f\left(\frac{x+y}{xy}\right)+f(xy)$ prove that for all positive real number x,y we have $f(xy)=f(x)+f(y)$. Taiwanese Olympiads

Solutions: if we regard the system: $\frac{x+y}{xy}=a\,,xy=b$. And solve it we receive that $x=\frac{a+\sqrt{a^2-\frac{4}{b}}}{\frac{2}{b}}\,,y=\frac{a-\sqrt{a^2-\frac{4}{b}}}{\frac{2}{b}}$ thus for all positive real number a,b such that $a^2b\geq 4$we have $f(ab)=f(x+y)=f\left(\frac{x+y}{xy}\right)+f(xy)=f(a)+f(b)$. Now we must generalize it. For this mission we insert new variable z, to count $f(xyz)$ twice. implies that we must find a variable z, for which: $f(xyz)=f(z)+f(xy)=f(x)+f(yz)=f(x)+f(y)+f(z)$. Or the following equalities holds simultaneously: $f(xyz)=f(z)+f(xy)$, $f(xyz)=f(x)+f(yz)$ and $f(yz)=f(y)+f(z)$ ensure that we need a number z such that $x^2y^2z\,,x^2yz\,,y^2z\geq 4$ thus holds for all sufficiently large z. then we have: $f(z)+f(xy)=f(x)+f(y)+f(z)$ and we are done.

Problem-2:Let n be a nonzero integer such that for function $f:\mathbb{Z}\to\mathbb{Z}$ we have:

$$f\big(x+y+f(y)\big)=f(x)+ny$$

Find the function f.

Solution: first of all we will prove f must be injective. Assume the contrary thus there exist distinct integers y,z such that $f(y)=f(z)=a$.then we can see that: $f(y)=f\big(-a+y+f(y)\big)=f(-a)+ny\neq f(-a)+nz=f\big(-a+z+f(z)\big)=f(z)$. Contradiction. And then we have:

$$f\big(x+z+f(y)+y+f(y)\big)=f\big(x+z+f(z)\big)+ny=f(x)+n(y+z)=f(x+y+z+f(y+z))$$

Then we have: $x+z+f(y)+y+f(y)=x+y+z+f(y+z)$.implies that f is additive. And we are fone.

Second Solution: define the function g as: $g(y)=y+f(y)$ thus we have $f\big(x+g(y)\big)=f(x)+ny$ set $x{:}\,x+g(y),x-g(y)$ we can deduce that for all integer k. we have: $f\big(x+kg(y)\big)=f(x)+nky$. Now take two different integers y_1,y_2 and use the result for $k=g(y_1),g(y_2)$ and y=y_2 , $y_1$we have:

$$f\big(x+g(y_1)g(y_2)\big)=f(x)+ng(y_1)y_2$$
$$f\big(x+g(y_1)g(y_2)\big)=f(x)+ng(y_2)y_1$$

Comparing this equalities we must have: $g(y_1)y_2=g(y_2)y_1$ implies that g is ,linear function . and then f is so…

Comment: this problem for the case n=2. Was proposed in Saint-Petersburg Olympiads-2002. And the problem adding up the case n=0. Proposed in Chinese selection test 2012. Now try The case n=0.

Problem-3: Find all functions f from positive real numbers to reals such that: $f\left(\sqrt{\frac{x^2+xy+y^2}{3}}\right)=\frac{f(x)+f(y)}{2}$ Poolish Olympiads

Solutions: define the *-operator as: $x*y=\sqrt{\frac{x^2+xy+y^2}{3}}$ thus we have: $f(x*y)=\frac{f(x)+f(y)}{2}$. Now we have $f\big((x*y)*(z*t)\big)=\frac{f(x)+f(y)+f(z)+f(t)}{4}$the last equality is also equal to $f((x*t)*(z*y))$. Now define $g(x)=(x^2*x)*(x*1)$and $h(x)=(x^2*1)*(x*x)$.also by the homogeneity of the operator we have $ta*tb=t(a*b)$ we have . thus for all positive reals x,t we have: $f\big(tg(x)\big)=f(th(x))$. easy calculations show us:$g(1)=h(1)=1$ and $g(2)=\frac{7}{3}$,$h(2)=\sqrt{\frac{11+2\sqrt{7}}{3}}$implies that $g(2)>h(2)$and then take: $\lambda=\frac{g(2)}{h(2)}>1$now by the continuity of the function $\frac{g(x)}{h(x)}$we deduce that this function is surjective on the interval $(1,\lambda)$for all x belong to $(1,2)$.(this trivially deduced from the intermediate value property) now choose a,b such that $\frac{b}{a}\in(1,\lambda)$ thus there exist number z belongs to $(1,2)$.such that $\frac{b}{a}=\frac{g(z)}{h(z)}$ now set $t=\frac{a}{h(z)},x=z$. We have:

$$f(b)=f\left(a.\frac{b}{a}\right)=f\left(a.\frac{g(z)}{h(z)}\right)=f\left(\frac{a}{h(z)}.g(z)\right)=f\left(\frac{a}{h(z)}.h(z)\right)=f(a)$$

Thus f is constant on the interval $(a,\lambda a)$ now take $\tau=\sqrt[n]{\frac{b}{a}}\le\lambda$ then f is constant on these intervals: $(a,\tau a),(\tau a,\tau^2 a),(\tau^2 a,\tau^3 a),\dots(\tau^{n-1}a,\tau^n a=b)$ and we are done…

Problem-4: for functions $f:\mathbb{R}^+\to\mathbb{R}$ we have:

$$f(x+y)=f(x)+f(y)+f(\frac{1}{x}+\frac{1}{y})$$

Prove that: $f(xy)=f(x)+f(y)$

Solutions: define $F(x,y)=f(x+y)-f(x)-f(y)$thus we have

$$F(x+y,z)+F(x,y)=F(y+z,x)+F(y,z)$$

Now by use of the problem assumption we have :

$$f\left(\frac{1}{x+y}+\frac{1}{z}\right)+f\left(\frac{1}{x}+\frac{1}{y}\right)=f\left(\frac{1}{y+z}+\frac{1}{x}\right)+f\left(\frac{1}{y}+\frac{1}{z}\right)$$

From the problem statement we can easily deduce that $f(1)=0, f(x)+f\left(\frac{1}{x}\right)=0$ now set y,z such that $\frac{1}{y}+\frac{1}{z}=1$then we have $y,z>1$and $z=\frac{y}{y-1}$ then we have the following relations:

$$\frac{1}{x+y}+\frac{1}{z}=\frac{y^2+xy-x}{y(x+y)} \quad \text{and} \quad \frac{1}{y+z}+\frac{1}{x}=\frac{y^2+xy-x}{xy^2}$$

Now we have: $f\left(\frac{y^2+xy-x}{y(x+y)}\right)+f\left(\frac{1}{x}+\frac{1}{y}\right)=f\left(\frac{y^2+xy-x}{xy^2}\right)$. Then regard the system $u=\frac{1}{x}+\frac{1}{y}, v=\frac{y^2+xy-x}{y(x+y)}$ then $f(uv)=f(u)+f(v)$. Now we must solve the system and finf the domain of its validity. Lets regard that$v=1-\frac{x}{y(x+y)}$ thus we have $v\in(0,1)$ and also $v=1-\frac{1}{uy^2}$or $uv=u-\frac{1}{y^2}$ now we can represent x,y explicitly:

$$y=\frac{1}{\sqrt{u(1-v)}} \quad ,x=\frac{u+\sqrt{u(1-v)}}{u(u+v-1)}$$

Since$u+v-1$ must be positive and $0<v<1$ we must have $u>1$. Thus we conclude that: $\forall u>1, 0<v<1$ $f(uv)=f(u)+f(v)$.

Now its obvious that for all reals u,v there exist number z such that $zu>1, \frac{v}{z}<1$(take z sufficiently large). Then we have:

$$f(u)=f\left(zu.\frac{1}{z}\right)=f(zu)+f\left(\frac{1}{z}\right)$$
$$f(v)=f\left(\frac{v}{z}.z\right)=f\left(\frac{v}{z}\right)+f(z)$$

Add up the following equations we have $f(u)+f(v)=f(zu)+f\left(\frac{v}{z}\right)$and as $zu>1, \frac{v}{z}<1$ we have:$f(zu)+f\left(\frac{v}{z}\right)=f\left(zu.\frac{v}{z}\right)=f(uv)$ and our proof is complete.

Problem-5: . Prove that if the function f is defined on the set of positive real numbers, its values are real, and f satisfies the equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{2xy}{x+y}\right)=f(x)+f(y)$$

for all positive x,y, then

$$2f\left(\sqrt{xy}\right) = f(x) + f(y)$$

for every positive number pair x, y. Miklows Shwitzer

Solutions: Let a,b,c,d be positive real numbers. By applying the functional equation several times, we have

$$f(a)+f(b)+f(c)+f(d)=$$

$$= f\left(\frac{a+b}{2}\right) + f\left(\frac{2ab}{a+b}\right) + f\left(\frac{c+d}{2}\right) + f\left(\frac{2cd}{c+d}\right) =$$

$$= f\left(\frac{\frac{a+b}{2}+\frac{c+d}{2}}{2}\right) + f\left(\frac{2\frac{a+b}{2}\cdot\frac{c+d}{2}}{\frac{a+b}{2}+\frac{c+d}{2}}\right) +$$

$$+ f\left(\frac{\frac{2ab}{a+b}+\frac{2cd}{c+d}}{2}\right) + f\left(\frac{2\frac{2ab}{a+b}\cdot\frac{2cd}{c+d}}{\frac{2ab}{a+b}+\frac{2cd}{c+d}}\right) =$$

$$f\left(\frac{a+b+c+d}{4}\right) + f\left(\frac{(a+b)(c+d)}{a+b+c+d}\right) +$$

$$+ f\left(\frac{abc+abd+acd+bcd}{(a+b)(c+d)}\right) + f\left(\frac{4abcd}{abc+abd+acd+bcd}\right).$$

By repeating the above procedure with b and c interchanged, we get

(1) $$f\left(\frac{(a+b)(c+d)}{a+b+c+d}\right) + f\left(\frac{abc+abd+acd+bcd}{(a+b)(c+d)}\right) =$$

$$= f\left(\frac{(a+c)(b+d)}{a+b+c+d}\right) + f\left(\frac{abc+abd+acd+bcd}{(a+c)(b+d)}\right).$$

Let $a=c, b=\frac{a^2}{d}$and$t=\frac{a}{b}+\frac{b}{a}$. It is easy to check that

$$\frac{(a+b)(c+d)}{a+b+c+d} = \frac{abc+abd+acd+bcd}{(a+b)(c+d)} = a,$$

$$\frac{(a+c)(b+d)}{a+b+c+d} = a\cdot\frac{2t}{2+t},$$

and finally

$$\frac{abc+abd+acd+bcd}{(a+c)(b+d)}=a\cdot\frac{2+t}{2t}.$$

Substitution of the results into (1) gives

(2) $$2f(a)=f\left(a\cdot\frac{2t}{2+t}\right)+f\left(a\cdot\frac{2+t}{2t}\right)$$

t can be any number not smaller than 2, and $\frac{2t}{2+t}$ can be any number not smaller than 1. Thus for every number pair $x\geq y$ there exist the numbers a and t, such that $\frac{2at}{2+t}=x, \frac{a(2+t)}{2t}=y, a=\sqrt{xy}$.

Problem-6:Let $\mathbb{N}_m$ be set of positive integers greater and equal to m find all functions $f:\mathbb{N}_m\to\mathbb{N}_m$ such that: $f\big(x^2+f(y)\big)=y+f^2(x)$ Mathematics and youth

Solution: its obvious that f is one-to-one. let $k\in\mathbb{N}_m$ thus $k\geq m$ and take $f(k)=l$.set x=k. then $f\big(k^2+f(y)\big)=y+l^2$ (1).then set y=k.to receive $f(x^2+l)=k+f^2(x)$(2(.and also we have $f(k^2+l)=k+l^2$ (3) .now set $y=k^2+l$in (1)recieng :$f\big(k^2+f(k^2+l)\big)=f(k^2+k+l^2)=k^2+l+l^2$. Now we want to prove the following statement by induction:

$$\forall n\in\mathbb{N}: f\big(\, l-l^2+n(k^2+l^2)\big)=n(k^2+l^2)+k-k^2$$

The statement is obvious for n=1, assume it holds true for n, and we must prove its truth for n+1. For this we may prove the maxim statement such that:

$$f\big(k+n(k^2+l^2)\big)=l+n(k^2+l^2)$$

Which may be proven by induction assume it holds for n. then Set $y=k+n(k^2+l^2)$in(1) to see that:

$$f\left(k^2+f\big(k+n(k^2+l^2)\big)\right)=k+n(k^2+l^2)+l^2$$

Thus $f\big(k^2+l+n(k^2+l^2)\big)=f\big(l-l^2+(n+1)(k^2+l^2)\big)=(n+1)(k^2+l^2)+k-k^2$ now we must prove the truth of the maxim statement for n+1. Indeed if we set y= $l-l^2+(n+1)(k^2+l^2)$in(1) we have

$$f\left(k^2+f\big(l-l^2+(n+1)(k^2+l^2)\big)\right)=f(k+(n+1)(k^2+l^2))=l-l^2+(n+1)(k^2+l^2)+l^2$$
$$=l+(n+1)(k^2+l^2)$$

And it proved.

Now if we set $x = k^2 + k + l^2$and y=k. we can see that:

$$f(l + (k^2 + k + l^2)^2) = k - k^2 + (k^2 + l^2)\underbrace{(k^2 + (l+1)^2)}_{n}$$

We can see that: $f(l - l^2 + \underbrace{(k^2 + (l+1)^2)}_{n}(k^2 + l^2)) = \ k - k^2 + \underbrace{(k^2 + (l+1)^2)}_{n}(k^2 + l^2)$

Now by the injectivity of f we must have:

$$l + (k^2 + k + l^2)^2 = l - l^2 + (k^2 + l^2)(k^2 + (l+1)^2)$$

Which implies that k=l. thus $f(x) = x$ is the only solution.

Comment: in the inductive step first we prove that if the maxim statement holds true for n, then the original statement will be true for n+1. And then prove that if the original statement holds true for n+1 then the maxim statement holds true for n+1.and this procedure will works inductively.

Problem-7: find all functions f:$\mathbb{R} \to \mathbb{R}$ which is positive for all positive real numbers and satisfy the following equation:

$$f(x + y + xy) = f(xy) + f(x) + f(y)$$

Solution: its obvious that $f(0) = 0$ and $f(-x) = -f(x)$,now by putting $(-x, -y)$instead of (x, y)we have $f(xy - x - y) = f(xy) - f(x) - f(y)$add up to the problem statement we have $f(xy + x + y) + f(xy - x - y) = 2f(xy)$. Then set $xy = a$ if $a \leq 0$ then $x + y$ covers $\mathbb{R}$(since the inequality $(x + y)^2 \geq 4xy$) now if we set $xy = u, x + y = v$thwn for all $u \leq 0, v \in \mathbb{R}$ we have $f(u + v) + f(u - v) = 2f(u)$. Now by the oddness of the function it holds true for all reals u,v. now set $u = v$ then $f(2u) = 2f(u)$.and then take $u + v = a, u - v = b$ then we have $f(a) + f(b) = f(a + b)$.and f is additive. By the positivity of f on the right side of real line we can conclude that f is linear function passing through origin.

Problem-8: find all functions f:$\mathbb{R} \to \mathbb{R}$ which is positive for all positive real numbers and satisfy the following equation:

$$f(x + y + 2xy) + f(-2xy) = f(x) + f(y)$$

Solution: Define a function $G: \mathbb{R}^2 \to \mathbb{R}$ such that $G(x, y) = f(x) + f(y) - f(-2xy)$ thus we can see that $G\left(2u, \frac{-v}{2}\right) + G(2uv, w) = G(2u, vw) + G(\frac{-V}{2}, w)$ which leads to the equation:

$$f\left(2u - \frac{v}{2} - 2uv\right) + f(2uv + w + 4uvw) = f(2u + uw + 4uvw) + f(\frac{-v}{2} + w - uw)$$

Set $2vw = -1$then we must have

$$f\left(2u - \frac{v}{2} - 2uv\right) + f\left(2uv - \frac{1}{2v} - 2u\right) = f\left(\frac{-1}{2}\right) + f\left(\frac{-v}{2} + \frac{1}{2} - \frac{1}{2v}\right)$$

Now choose arbitrary reals x,y such that

$$2u - \frac{v}{2} - 2uv = \frac{y+1}{2}$$
$$2uv - \frac{1}{2v} - 2u = \frac{x+1}{2}$$

Adding up the equations we can see that $-\left(v + \frac{1}{v} + 2\right) = x + y$ thus for negative values of $x + y$ we have $f\left(\frac{y+1}{2}\right) + f\left(\frac{x+1}{2}\right) = f\left(\frac{-1}{2}\right) + f\left(\frac{x+y+3}{2}\right)$define $g(x) = f\left(\frac{x+1}{2}\right) - f\left(\frac{-1}{2}\right)$.receive that $g(x + y + 2) = g(x) + g(y)$ set $y = 0$we see that $g(x + 2) = g(x) + g(0)$thus $g(x + y + 2) = g(x + y) + g(0) = g(x) + g(y)$then set the function $h(x) = g(x) - g(0)$we have $h(x + y) = h(x) + h(y)$its obvious that if $x + y \leq 0$ the equality holds now we must prove it for positive values of $x + y$. For positive value of $x + y$ there exist real number z such that $z + x + y$ and $z + x$ be negative. Thus we have: $h(z) + h(x + y) = h(z + x + y) = h(z + x) + h(y) = h(z) + h(x) + h(y)$implies that h is additive. And as h is bounded from below(by the positivity of f in the right side of real line) we must have $h(x) = cx$for some real c.

Problem-9:find all functions f:$\mathbb{R}^+ \to \mathbb{R}^+$such that:

$$f(xy) + f(x + y) = f(x + xy) + f(y)$$

Solution: first of all set $y + 1$instead of$y$. To see that:

$$f(xy + x) + f(x + y + 1) = f(xy + 2x) + f(y + 1)$$

Thus we have $f(xy) + f(x + y) + f(x + y + 1) = f(y) + f(xy + 2x) + f(y + 1)$ now put $(\frac{x}{2}, 2y)$instead of (x, y) in the above relation we have:

$$f(xy) + f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + 1\right) = f(2y) + f(xy + x) + f(2y + 1)$$

Now putting $f(x + y) - f(y)$instead of $f(xy + x) - f(xy)$ we receive the equation:

$$f\left(\frac{x}{2} + 2y\right) + f\left(\frac{x}{2} + 2y + 1\right) - f(x + y) = f(2y) + f(2y + 1) - f(y)$$

Put $x - y$instead of $x$we receive that:

$$f\left(\frac{x}{2} + \frac{3y}{2}\right) + f\left(\frac{x}{2} + \frac{3y}{2} + 1\right) = f(2y) + f(2y + 1) - f(y) + f(x)$$

Now set $\frac{y}{3}$instead of $y$we receive the following equation:

$$f\left(\frac{x}{2}+\frac{y}{2}\right)+f\left(\frac{x}{2}+\frac{y}{2}+1\right)=f\left(\frac{2y}{3}\right)+f\left(\frac{2y}{3}+1\right)-f\left(\frac{y}{3}\right)+f(x)$$

Then define $f_1(x)=f\left(\frac{2x}{3}\right)+f\left(\frac{2x}{3}+1\right)-f\left(\frac{x}{3}\right)$, $f_2(x)=f(x)$and $f_3(x)=f\left(\frac{x}{2}\right)+f\left(\frac{x}{2}+1\right)$

Then we have $f_1(y)+f_2(x)=f_3(x+y)$. Then we need the following lemma:

Lemma: the functions f,g,h satisfying the equation $f(x+y)=g(x)+h(y)$ is shift of additive function.

Proof: set $y=0$and $h(0)=\alpha$then $f(x)=g(x)+\alpha$then set $x=0$and $g(0)=\beta$ and then $h(y)=f(y)-\beta$. Thus we have $f(x+y)=f(x)+f(y)-\alpha-\beta$

Problem-10: let αbe positive real number find all functions $f:\mathbb{N}\rightarrow\mathbb{R}$ such that for all positive integers k,m such that $\alpha m\leq k<(\alpha+1)m$ we have

$$f(k+m)=f(k)+f(m)$$ Chinese TST

Solution: first of all we prove the following lemma:

Lemma: for sufficiently large enough n there exist real number u such that

$$\alpha(n+1)\leq u<(\alpha+1)(n+1) \text{ and } \alpha n\leq u+1<(\alpha+1)n$$

Proof : it suffices to prove that :

$$\alpha n<\alpha(n+1)\leq u<u+1<(\alpha+1)n<(\alpha+1)n+1$$

This is obvious for large enough n $\alpha n\leq u<u+1<(\alpha+1)(n-1)$ indeed for this u. we have :

$$(n-1)\alpha<\alpha(n+1)\leq u<u+1<(\alpha+1)(n-1)$$

Now for the existence of such u, we must have: $(\alpha+1)(n-1)-\alpha(n+1)\geq 2$ or $n\geq 2\alpha+3$. Take n larger than this quantity we are done.

Now we have the following equations which leads to solving the problem:

$$f(n-1)+f(u+1)=f(n+u)=f(n)+f(u)$$

thus$f(n)-f(n-1)=f(u+1)-f(u)$. We also have:

$$f(n+1)+f(u)=f(n+u+1)=f(u+1)+f(n)$$

Thus $f(n+1)-f(n)=f(u+1)-f(u)$. Indeed foe all $n\geq 2\alpha+3$ the quantity $f(n+1)-f(n)$ remains constant(i.e. $f(n+1)-f(n)=f(n)-f(n-1)$)

Now set $n_0 = 1 + \lfloor 2\alpha + 3 \rfloor$(its obvious that $n_0 \geq 3$) thus for all $n \geq n_0 - 1$ we have

$$f(n+1) - f(n) = f(n_0) - f(n_0 - 1)$$

Then we can easily find that:

$$f(n) = (n - n_0 + 1)\big(f(n_0) - f(n_0 - 1)\big) + f(n_0 - 1)$$

Then set k,m such that $m \geq n_0$,$\alpha m \geq n_0$ and $\alpha m \leq k < (\alpha + 1)m$ which leads to :$k \geq n_0$ then:

$$\begin{aligned} f(k+m) = f(k) + f(m) &= (k + m - n_0 + 1)\big(f(n_0) - f(n_0 - 1)\big) + f(n_0 - 1) \\ &= (k - n_0 + 1)\big(f(n_0) - f(n_0 - 1)\big) + f(n_0 - 1) \\ &+ (m - n_0 + 1)\big(f(n_0) - f(n_0 - 1)\big) + f(n_0 - 1) \end{aligned}$$

Comparing the equalities we receive that: $(n_0 - 1)f(n_0) = n_0 f(n_0 - 1)$. Thus we can see that $f(n_0) = an_0$ for some real number a. then for all $n \geq n_0 - 1$ we have $f(n) = an$. now assume there exist positive integer n_1 such that $f(n_1) \neq an_1$. Now assume $\alpha > 1$ then there exist positive integer k such that $\alpha n_1 \leq k < (\alpha + 1)n_1$ then we have $k > n_1$,$k + n_1 > n_1$. This implies that: $f(k + n_1) = f(k) + f(n_1)$ or $f(n_1) = f(k + n_1) - f(k)$ now if we setn_1 minimal number which the inequality holds the problem was killed.

Now if $\alpha \leq 1$ then $\alpha n_1 \leq n_1 < n_1(\alpha + 1)$ then $f(2n_1) = f(n_1) + f(n_1)$ now if we set n_1 be the maximal value which the inequality holds then $f(2n_1) = 2an_1$ and we are done.