Selected topics in Functional Equations

Additivity and Boundedness

In this lecture we present a method , for solving functional equations problems . this method is based on use of Cauchy functional equations and boundedness of the functions. Which leads to solve some Olympiad caliber problems. First of all we present a theoretical preliminary which forms the baseline of our arguments.

Lemma: if the function $f:\mathbb{R}\to\mathbb{R}$.satisfy the equation $f(x+y)=f(x)+f(y)$and be bounded above or below on some interval (a,b) then it has the form cx for some real number c.

Proof: first we prove if the function be bounded above in the interval (a,b). It will be bounded on every interval $(-\epsilon,\varepsilon)$ where $\varepsilon>0$ for this reason we define the function $g(x)=f(x)-f(1)x$and clearly g satisfies the Cauchy equation and for all rational number r, we have $g(r)=0$. Set $x\in(-\epsilon,\varepsilon)$.then there exist rational number r such that $x+r\in(a,b)$ then we have $g(x)=g(x)+g(r)=g(x+r)=f(x+r)-f(1)(x+r)$which implies that g is bounded . and so f is bounded on$(-\epsilon,\varepsilon)$ (since f is odd) now we can prove that f is continuous on zero. Let x_n be a sequence converges to zero and $r_n$be sequence of rational numbers diverges to $+\infty$such that $r_nx_n\to 0$then $|f(r_nx_n)|$is bounded above by some real number M. and $|f(x_n)|=\left|f(\frac{1}{r_n}.r_nx_n)\right|=\frac{1}{r_n}|f(r_nx_n)|\le\frac{M}{r_n}$thus $\lim_{x\to 0}f(x)=f(0)=0$. As f be continuous its obvious that is continuous on the whole of real line (since: $\lim_{h\to 0}f(x+h)=f(x)+\lim_{h\to 0}f(h)=f(x)$).and the continuous solution of the Cauchy equation is of the form $f(x)=f(1)x$.

Note: if the function f be monotone and additive for all $x\in(\frac{-1}{n},\frac{1}{n})$ $(n\in\mathbb{N})$ we have$-\frac{f(1)}{n}\le f(x)\le\frac{f(1)}{n}$. No take n as large as we find that f is continuous at zero. and do the same as above.

Problem-1: find all function $f:\mathbb{R}\to\mathbb{R}$ which is bounded on (0,1) and satisfy the equation: $x^2f(x)-y^2f(y)=(x^2-y^2)f(x+y)-xyf(x-y)$

Bulgarian Olympiads

Solution: if $x>y+\frac{1}{2}$and $x,y\in(0,n)$then $x+y\in(0,\frac{2n-1}{2})$by induction we can see that f is bounded on $(0,\frac{2^k+1}{2})$and if $0<x<y$ then f is bounded on $(-2^k,0)$repectively. Thus we conclude that f is bounded on finite subset of $\mathbb{R}$. Now set $y=-x$recieving that $f(x)-f(-x)=f(2x)$and analogously $f(-x)-f(x)=$

$f(-2x)$. this shows that $-f(-2x) = f(2x) = 2f(x)$. inductively we can prove $f(nx) = nf(x)$(put x=(n-1)y)and also for all rational number $f(r) = rf(1)$now by defining the function $g(x) = f(x) - f(1)x$we see that g satisfies the problem conditions and id bounded. Thus for all real number x we have $g(x) < M$for some real number M. then take arbitrary number x and set the equation for all positive integers n.

$$g(x) = \frac{1}{n}g(nx) < \frac{M}{n}$$
$$g(x) = \frac{-1}{n}g(-nx) < \frac{M}{n}$$

Take n as large as possible we find that g is equal to zero.

Comment-1:Lets complete the proof by use of the method we used in proving the lemma.

Comment-2: one can prove that f is continuous and derivable indeed pick any nonzero x and take $y \to 0$ then $\lim_{y\to 0} f(x+y) = f(x)$ which prove the continuity of the function. and lets rewrite the equation as

$$xf(x-y) + y(f(x+y) - f(y)) = x^2\frac{f(x+y)-f(x)}{y}$$

Take $y \to 0$ then the limit $\lim\limits_{y\to 0}\frac{f(x+y)-f(x)}{y}$ exists and then equals to $f'(x)$ then we have: $xf(x) = x^2f'(x)$ or $\frac{d}{dx}lnf(x) = \frac{1}{x}$ thus $f(x) = cx$.

Problem-2: Find all function $f: \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that:

$$f(x+y) + f(x-y) - 2f(x) = 2y^2 \qquad \text{Korean Olympiads}$$

Solution: we can see that the function $f(x) = x^2$satisfies the condititien now define $g(x) = f(x) - x^2$ thus we have :

$$g(x+y) + g(x-y) = 2g(x)$$

By setting $xx = y$then we have $g(2x) + g(0) = 2g(x)$then we can see that

$$g(x+y) + g(x-y) = g(2x) + g(0)$$

Which leads to additivity of the function $g(x) - g(0)$.(set $x+y = a, x-y = b$)
Now we can use the lemma , since the function $g(x) - g(0) = f(x) - f(0) - x^2$ is bounded in some interval and additive we conclude that $g(x) - g(0) = f(x) - f(0) - x^2 = cx$. Thus $f(x) = f(0) + cx + x^2$

Problem-3: find all function $f: \mathbb{R}^+ \to \mathbb{R}$such that $f(1) = 2008$and for all positive real number we have $|f(x)| \le x^2 + 1004^2$ and for all positive reals x,y we have:

$$f\left(x+y+\frac{1}{x}+\frac{1}{y}\right)=f\left(x+\frac{1}{y}\right)+f(y+\frac{1}{x})$$

Korean Olympiads

Solution: first of all we choose our old trick, and create a system

$$x+\frac{1}{y}=a, y+\frac{1}{x}=b$$

After solving the system we receive the second-degree equation $ay^2-aby+b=0$which has solution iff $ab\geq 4$. Thus for all real numbers a,b such that $ab\geq 4$ we have $f(a)+f(b)=f(a+b)$. Then we must generalize. For this we insert new variable c, such that for all real numbers a,b the following equations holds true:

$$f(a+b+c)=f(c)+f(a+b)=f(a)+f(b+c)=f(a)+f(b)+f(c)$$

Which reduces to this inequalities: $c(a+b), a(b+c), bc\geq 4$which all sounds true for sufficiently large enough c. thus f is additive function. Now we need the lemma of boundedness , for this define the function

$$g(x)=f(x)-f(1)x=f(x)-2008x$$

Thus by the problem criteria we find the inequality $-(x+1004)^2\leq g(x)\leq (x-1004)^2$ which prove the boundedness of the function. Thus by use of the lemma the problem is solved.

Problem-4: find all function $f:\mathbb{R}\rightarrow\mathbb{R}$ such that:

$$f^2(x+y)=f^2(x)+2f(xy)+f^2(y)$$ Ukrainian TST

Solution: first put $(-x,x)$to receiving the equation $f^2(0)=f^2(x)+f^2(-x)+2f(-x^2)$. Then put $(x+y,-x)$receiving that $f^2(y)=f^2(x+y)+2f(-x^2-xy)+f^2(-x)$by comparing this with the first equation and the problem statement we receive the following equation:

$$f(-x^2)=f(xy)+f(-x^2-xy)+\frac{1}{2}f^2(0)$$

Now define the function $g(x)=f(x)+\frac{1}{2}f^2(0)$then receive the following equality:

$$g(-x^2)=g(xy)+g(-x^2-xy)$$

Now set the system $xy=a, x^2+xy=-b$thus $x=\pm\sqrt{-(a+b)}$and $y=\pm\frac{a}{\sqrt{-(a+b)}}$. Indeed the system has solutions for all $a+b\leq 0$ thus for all values of reals for which $a+b \leq 0$ we have $g(a+b) = g(a) + g(b)$. Now if $a+b \geq 0$ there exist real number c such that$c+a+b\,, c+a \leq 0$. Then we have:

$$g(c) + g(a+b) = g(a+b+c) = g(b) + g(a+c) = g(b) + g(a) + g(c)$$

Thus g is additive for all reals a,b. now we need to prove its boundedness for this just regard the relation:$f(-x^2) = \frac{f^2(0)-f^2(x)-f^2(-x)}{3} \leq \frac{1}{2}f^2(0)$ thus we have $g(-x^2) \leq f^2(0)$ which implies that g is bounded from above on the left side of real line , thus we have $g(x) = cx$then $f(x) = cx + d$by checking we find that there exist four functions$f(x) = 0, -2, x, x-2$.

Problem-5:About the monotone function $f: \mathbb{R}^{+} \cup \{0\} \to \mathbb{R}$ we know: $f(0) + 3f(2) = 3f(1) + f(3)$and:

$$f(x+y) - f(x) - f(y) = f(1+xy) - f(xy) - f(1)$$

Find the function. Ukrainian Olympiads

Solution: first we insert new variable then we have the following equalities:

$$\begin{aligned} f(x+y+z) - f(x) - f(y) - f(z) &= f(x+y+z) - f(x+y) - f(z) + (x+y) - f(x) - f(y) \\ &= f(xz+yz+1) - f(xz+yz) - f(1) + f(xy+1) - f(xy) - f(1) \\ &= f(1+xy+zx) - f(xy+xz) - f(1) + f(1+yz) - f(yz) - f(1) \end{aligned}$$

The last equality is obtained by switching z,x. then if we set$yz = a, zx = b, xy = c$. We have for positive reals a,b,c:

$$f(a+b+1) - f(a+b) + f(1+c) - f(c) = f(1+c+b) - f(c+b) + f(1+a) - f(a)$$

Lets define the function $g: \mathbb{R}^{+} \cup \{0\} \to \mathbb{R}$ such that$g(x) = f(x+2) - f(x+1) + f(1) - f(2)$ without less of generality assume f is strictly increasing then by the monotonicity of f, is greater than $f(1) - f(2)$. then set $a = 1, b = u, c = 1 + v$ we have $g(u) + g(v) = g(u+v)$and by the boundedness of g we have $g(x) = cx$then set if we set $a = u, b = c = 1$and use the equality $f(0) + 3f(2) = 3f(1) + f(3)$and the fact that $g(x) = cx$. We find that:

$$f(1+u) - f(u) = cu + f(1) - f(0)$$

Set $u = xy$then by use of problem statement we receive that:

$$f(x+y) - f(x) - f(y) = cxy - f(0)$$

By defining function: $h(x) = f(x) - f(0) - \frac{c}{2}x^2$ one can see that , h is additive and bounded from below in any interval (since $f(x) \geq f(0)$)then $f(x) - f(0) - \frac{c}{2}x^2 = ax$and $f(x) = f(0) + ax + \frac{c}{2}x^2$.

Problem-6: find all functions f: $\mathbb{R} \to \mathbb{R}$ which is strictly increasing for all positive real numbers and satisfy the following equation:

$$f(x+y) + f(xy) = f(x) + f(y) + f(xy+1)$$

Solution: lets define the function $g(x) = f(x) - f(x-1)$then:

$$g(xy) = f(x+y) - f(x) - f(y)$$

Thus $f(x+y+z) = f(x) + f(y+z) + g(xy+zx) = f(x) + f(y) + f(z) + g(yz) + g(xy+zx)$ now by changing x,z we receive the following equality

$$g(yz) + g(xy+zx) = g(yx) + g(zy+zx)$$

Now set $yz = a, zx = b, xy = c$ where $abc = (xyz)^2 \geq 0$ and

$$g(a) + g(b+c) = g(c) + g(a+b)$$

Set $b = 1$then $g(a+1) - g(a) = g(c+1) - g(c) = k =$ Cte. Then for all $c \geq b$ take $a = c - b$. Then $g(c-b) + g(c+b) = 2g(c)$ take $c = b+1$ then we have:

$$g(1) + g(2c-1) = g(1) + g(2c) - k = 2g(c)$$

Thus $g(c-b) + g(c+b) = g(2c) + l$ for some real number l. then the function $g(x) - l$is additive on $\mathbb{R}^+$and then in $\mathbb{R}$ (by the same trick of problem-4) thus $g(x) = ax + b$(since f is strictly increasing on right ray)now $f(x+1) - f(x) = ax + b$ or $f(1+xy) - f(xy) = axy + b$thus we have:

$$f(x+y) - f(x) - f(y) = axy + b$$

Then the function: $f(x) - b - \frac{a}{2}x^2$is additive and bounded from below.and we are done.

Comment: for proving the addictiveness of the function g please regard this insight: for $c > b$ and $bc > 0$ we have $g(c+b) + g(c-b) = 2g(c)$ thus if we take $c + b = y, c - b = x$ we have for $x > 0$ and $bc = y^2 - x^2 > 0$ we have $g(x) + g(y) = g(x+y)$. Thus for all $x > 0, y^2 > x^2$ we have $g(x) + g(y) = g(x+y)$

Problem-7: we know about the function $f: \mathbb{R} \to \mathbb{R}$such that :

i. For all real number $f(x) \geq 0$
ii. $\forall a, b, c, d \in \mathbb{R}$. $ab + bc + cd = 0$ we have:
$$f(a-b) + f(c-d) = f(a) + f(d) + f(b+c)$$ Korean Olympiads

Solution: if we take $a = c = 0, b = d$ we receive $f(-b) = f(b)$and if we take all variables equal to zero then $f(0) = 0$. Rewrite the problem assumption as $(a + c)(b + d) = ad$. Then take $a = x, d = y, c = 1 - x, b = xy - y$now we have :

$$f(x + y - xy) + f(-x - y + 1) = f(x) + f(y) + f(1 - x - y + xy)$$

As the function is even we have the following equation:

$$f(x + y - xy) + f(x + y - 1) = f(x) + f(y) + f((x - 1)(y - 1))$$

Now set $(1 + x, 1 + y)$instead of then we receive the equality :

$$f(1 - xy) + f(x + y + 1) = f(x + 1) + f(y + 1) + f(xy)$$

And as the function f is even we can see that:

$$f(xy - 1) + f(x + y + 1) = f(x + 1) + f(y + 1) + f(xy)$$

Thus $f(1 + x + y) = f(x + 1) + f(y + 1) + f(xy) - f(xy - 1)$ now insert new variable z to find that:

$$\begin{aligned} f(x + y + z + 1) &= f(x + 1) + f(y + z + 1) + f(xy + xz) - f(xy + xz - 1) \\ &= f(x + 1) + f(y + 1) + f(z + 1) + f(yz) - f(yz - 1) + f(xy + xz) - f(xy + xz - 1) \end{aligned}$$

Now by switching x,z we have the identity:

$$f(yz) - f(yz - 1) + f(xy + xz) - f(xy + xz - 1) = f(yx) - f(yx - 1) + f(zy + zx) - f(zy + zx - 1)$$

Now take $xy = a, yz = b, zx = c$ where $abc > 0$ we have :

$$f(b) - f(b - 1) + f(a + c) - f(a + c - 1) = f(a) - f(a - 1) + f(b + c) - f(b + c - 1)$$

Define $f(x) - f(x - 1) = g(x)$now we have: $g(xy) = f(x + y + 1) - f(x + 1) - f(y + 1)$ take $y = -x$then we have $g(-x^2) = f(1) - f(x + 1) - f(-x + 1) \leq f(1)$(since f is positive) thus g is bounded above on the interval $(-\infty, 0]$ and we have:

$$g(b) + g(a + c) = g(a) + g(b + c) = g(c) + g(a + b) \qquad abc > 0$$

Now for $a > c, ac > 0$take$b = a - c$recieving$g(a - c) + g(a + c) = 2g(a)$ and for $c > a, ac > 0$ take $b = c - a$receiving$g(c - a) + g(c + a) = 2g(c)$ assume the first then take $c = a - 1$ thus for all $a < 0$ we have $g(1) + g(2a - 1) = 2g(a)$by taking $b = 1$ in first equation we find that for all $ac > 0$ $g(a + 1) - g(a) = \text{constant}$ thus we have $g(2a) + k = 2g(a)$ thus we have for all $a, c < 0, a > c$ $g(a - c) + g(a + c) = g(2a) + k$ take $a - c = x, a + c = y$we have:

$$g(x) + g(y) = g(x + y) + k$$

For all x, y which $x > 0, y < 0, ac = \frac{y^2 - x^2}{4} > 0$thus $|y| > |x|$or $-y > x$which implies that:

$g(x) + g(y) = g(x + y) + k$ for all $x > 0, x + y < 0$ now take $x, y < 0$ then as: $x + y + (-x), x + y + (-y) < 0$ we have the following equations:

$$g(x+y)+g(-x)=g(y)+k\ ,\ g(x+y)+g(-y)=g(x)+k$$

Thus we have :$g(x)+g(-x)=g(y)+g(-y)=l=\text{constant}(*)$ then for all $x,y<0$ we have : $g(x+y)=g(x)+g(y)+k-l$ thus the function $g(x)+k-l$ is bounded from above in left side of real line an is additive , thus it is of the form cx (and by the (*)relation it has the same equation for complete real line) thus we have : $f(x+y+1)-f(x+1)-f(y+1)=cxy+d$ and we find that $f(x)=a_2x^2+a_1x+a_0$then by checking the even ness of the function we find that $a_1,a_0=0$ then $f(x)=a_2x^2$.

Second Solution: After this long run solution we present a concise solution based on a tricky lemma: indeed if we prove the function f satisfies the following lemma then the final part of solution is too easy.

Lemma: if $x^2+y^2=z^2$ then $f(x)+f(y)=f(z)$.

Proof: set $a=\frac{x-y+z}{2}$,$b=\frac{x-y-z}{2}$,$c=\frac{x+y+z}{2}$,$d=y$then $ab+bc+cd=\frac{1}{2}(x^2+y^2-z^2)=0$ now by substituting the values of a,b,c,d we have:

$$f(z)+f\left(\frac{x-y+z}{2}\right)=f\left(\frac{x-y+z}{2}\right)+f(x)+f(y)$$

Now define the function $g:\mathbb{R}^+\cup\{0\}\to\mathbb{R}^+\cup\{0\}$such that $g(x)=f(\sqrt{x})$ then $g(a+b)=g(a)+g(b)$ then by its boundedness or (for $a\geq b\geq 0$ we have $g(a)=g(a-b)+g(b)\geq g(a)$)we can see that $g(x)=cx$and etc…

Problem-8: let $a>b>c>d>0$and $ad=bc$ find all function$f:\mathbb{R}^+\to\mathbb{R}^+$such that:

$$f(a+d)+f(b-c)=f(a-d)+f(b+c) \qquad \text{ELMO}$$

Solution: We start to solve the problem by a method of the second solution we present for the problem-7. Thus set $a=\frac{x+y}{2}$,$b=\frac{z+t}{2}$,$c=\frac{z-t}{2}$,$d=\frac{x-y}{2}$ where $x>y>z>t$ and $z+y>x+t$ then we have $x^2+t^2=z^2+y^2$ and $f(x)+f(t)=f(y)+f(z)$ now we define the function $g(x)=f(\sqrt{x})$ and we have for all $a,b,c,d>0$where $a+b=c+d$ we have$g(a)+g(b)=g(c)+g(d)$ it is obvious that $g(a+1)-g(a)$ is constant and then $g(a)+g(b)=g(a+b-1)+g(1)=g(a+b)+k$ for some constant k. ensure that the function $g(x)-k$ is additive and bounded from below . thus the old trick works and $f(x)=cx^2+d$.

Problem-9: find all function $f:\mathbb{R}^+\to\mathbb{R}^+$such that for all positive real numbers x,y,z such that x+y>z we have:

$$f(x+y-z)+f\left(2\sqrt{xz}\right)+f\left(2\sqrt{yz}\right)=f(x+y+z) \qquad \text{Mongolian Olympiad}$$

Solution: first define a function $g(x) = f(\sqrt{x})$ then we have the following identity:

$$g((x+y-z)^2) + g(4xz) + g(4yz) = g((x+y+z)^2) \ (*)$$

Now set the system $(x+y-z)^2) = a$, $4xz = b$, $4yz = c$ then after solving the system we find that$z = \frac{\sqrt{a+b+c}-\sqrt{a}}{2}$, $y = \frac{c}{4z}$, $x = \frac{b}{4z}$ which holds for all positive reals a,b,c and we have:

$$g(a) + g(b) + g(c) = g(a+b+c) \qquad \forall a,b,c > 0$$

Now first of all set $(x, \frac{x}{2}, \frac{x}{2})$in the (*) receiving that:

$$2g(x^2) + g(2x^2) = g(4x^2) \ (**)$$

Now set $b = c$ to have: $g(a) + 2g(b) = g(a+2b)$then set $2a$ instead of a receiving that $g(2a) + 2g(b) = g(2a+2b)$swich a, bwe have $g(2a) + 2g(b) = g(2b) + 2g(a)$ implies that $g(2a) = 2g(a) + C$ by substituting this at the (**)we have $C = 0$ thus $g(a) + g(2b) = g(a+2b)$which us equivalent to $g(a) + g(b) = g(a+b)$implies that g is additive and bounded thus $g(x) = ax$ and the problem solved.

Problem-10: find all injective function $f: \mathbb{R} - \{0\} \to \mathbb{R} - \{0\}$ such that:

$$\forall x, y, x+y \neq 0 \ \ f(x+y)\big(f(x) + f(y)\big) = f(xy) \quad \text{Brazilian Olympiads}$$

Solution: define the function $g(x) = \frac{1}{f(x)}$ then we have $\frac{g(x)+g(y)}{g(x)g(y)} = \frac{g(x+y)}{g(xy)}$ or more simply $\frac{g(x)+g(y)}{g(x+y)} = \frac{g(x)g(y)}{g(xy)}$. Put $y = 1$ then we have $g(x+1) = cg(x) + 1$ where $c = \frac{1}{g(1)}$. By setting $x: x+1, y = 1$ we have $g(x+2) = c^2 g(x) + c + 1$ and inductively $g(x+n) = c^n g(x) + c^{n-1} + \cdots + c + 1$ then $g(n+1) = 2c^{n-1} + c^{n-2} + \cdots + c + 1$if $|c| > 1$ then $\frac{g(n+1)}{2c^{n-1}} \to 1$ thus by the relation $2g(n^2) = g(n).g(2n)$ $(*)$as we have $g(n^2) \sim \frac{1}{2c^{n^2-2}}, g(n) \sim \frac{1}{2c^{n-2}}, g(2n) \sim \frac{1}{2c^{2n-2}}$. Which leads to contradiction .

If we have $|c| < 1$ then $g(n) \sim \frac{1}{1-c}$ by substituting in the (*) we have $\frac{2}{1-c} = \frac{1}{(1-c)^2}$ then $c = \frac{1}{2}$. If we accept this we will have $\forall n \in \mathbb{N}: g(n+1) = \frac{1}{2^{n-2}} + \frac{1}{2^{n-2}} + \frac{1}{2^{n-3}} + \cdots + \frac{1}{2} + 1 = 2$ which contradicts the injectivity of f. then we conclude that $|c| = 1$. if c=-1 we have $g(x+2) = g(x)$ contradiction. Thus $c = 1$ and we have $g(1) = 1, g(x+1) = 1 + g(x)$ thus $g(2) = 2$then set y=2. We receive that:

$$\frac{2+g(x)}{2+g(x)} = \frac{2g(x)}{g(2x)}$$

Thus $g(2x) = 2g(x)$ then set $y = x$we hae:$g(x^2) = g^2(x)$. now we have a function satisfy the equations: $g(x+1) = 1 + g(x), g(x^2) = g^2(x)$ its obvious that this function is identity and then $f(x) = \frac{1}{x}$.

Problem-11: for the function f: $\mathbb{R} \to \mathbb{R}$ we have $f\big(x + y + f(y)\big) = f(x) + 2f(y)$ prove that$f\big(f(x)\big) = f(x)$ and $f(x+y) = f(x) + f(y)$ The College Mathematics Journal

Solution: first note that$f(x) = f\big(x + f(x) - f(x)\big) = f\big(-f(x)\big) + 2f(x)$ thus $f\big(-f(x)\big) = -f(x)$or $-f(x)$is fixed point of function f. now regard the following works:

$$\begin{aligned} f\big(x + 0 + f(0) - f(0)\big) &= f\big(x - f(0)\big) + 2f(0) = f\Big(x + f\big(-f(0)\big) + 0 + f(0)\Big) \\ &= f\Big(x + f\big(-f(0)\big)\Big) + 2f(0) = f\Big(x + f(0) - f(0) + f\big(-f(0)\big)\Big) + 2f(0) \\ &= f\big(x + f(0)\big) + 2f\big(-f(0)\big) + 2f(0) = f(x + f(0)) \end{aligned}$$

Thus $f(x) = f(x + f(0))$then $f(0) = f(f(0))$and if we set $x = y = 0$ we have $f\big(f(0)\big) = 3f(0)$results that $f(0) = 0$.

Now regard that

$$f\big(x - f(x)\big) = f\Big(x + f\big(-f(x)\big)\Big) = f\Big(x + f(x) - f(x) + f\big(-f(x)\big)\Big) = f\big(x + f(x)\big) + 2f(-f(x))$$

And then $f\big(x + f(x)\big) = f\big(0 + x + f(x)\big) = 0 + 2f(x)$. thus we have $f\big(x - f(x)\big) = 0$. Now consider the relation:

$$f(x+y) = f\big(f(x) + x + y - f(x)\big) = f\big(y + f(x)\big) + 2f(x - f(x))$$

Thus $f\big(x + f(y)\big) = f(x+y) = f(y + f(x))$ now put $y = 0$and the first statement proved.

Now set $f(2x) = f\big(0 + x + f(x)\big) = 0 + 2f(x)$ then $f(2x) = 2f(x)$.now write:

$$f(x+y) = f\left(x + \frac{y}{2} + \frac{y}{2}\right) =$$

Then set $\left(x, f\left(\frac{y}{2}\right)\right)$in the original equation we have:

$$f\left(x + f\left(\tfrac{y}{2}\right) + f\left(f\left(\tfrac{y}{2}\right)\right)\right) = f\left(x + 2f\left(\tfrac{y}{2}\right)\right) = f\big(x + f(y)\big) = f(x+y) = f(x) + 2f\left(\tfrac{y}{2}\right) = f(x) + f(y)$$

And we are done.

Problem-12: for the function $f:\mathbb{R}\to\mathbb{R}$ we have $f\big(x+2f(y)\big)=f(x)+y+f(y)$ prove that f-is additive. AMM

Solution: at first we calculate the value of $f(0)$ set $f(0)=a$ then if we set $x=y=0$then $f(2a)=2a$, if we set $x=0,y=2a$ then $f(4a)=5a$and if we set $x=2a,y=0$ then $f(4a)=3a$ then $a=0$. Put $x=0$then $f\big(2f(y)\big)=y+f(y)$ and then: $f\big(2f(x)+2f(y)\big)=f\big(2f(x)\big)+y+f(y)=x+y+f(x)+f(y)$

Then set $y=-x$, then $f(2\big(f(x)+f(-x)\big)=f(x)+f(-x)$ fix the value of x then $f(x)+f(-x)=b$and now $f(2b)=b$then $2f(2b)=2b$and therefore $f\big(2f(2b)\big)=f(2b)=2b+f(2b)$ thus $b=0$.implies that f is odd function.

Now define $C(x,y)=f(x+y)-f(x)-f(y)$ thus :

$$f\big(2c(x,y)\big)=c(x,y)+f\big(c(x,y)\big)=c(x,y)+f(f(x+y)-f(x)-f(y))$$

Now we count $f\big(2c(x,y)\big)$ in another way , since $f\big(2c(x,y)\big)=f(-2\big(f(x)+f(y)\big)+2f(x+y))$we have $f\left(-2\big(f(x)+f(y)\big)+2f(x+y)\right)=f\left(-2\big(f(x)+f(y)\big)\right)+x+y+f(x+y)$ the last equality is equal to

$$-f\big(2f(x)+2f(y)\big)+x+y+f(x+y)=-\big(x+y+f(x)+f(y)\big)+x+y+f(x+y)=C(x,y)$$

Thus $f\big(2C(x,y)\big)=C(x,y)$thus as we proved the only number for which $f(2b)=b$ is zero , the proof is complete.

Problem-13:find all function $f:\mathbb{R}\to\mathbb{R}$ such that for all nonzero values of x,y we have: $f(x+y)=x^2f\left(\frac{1}{x}\right)+y^2f(\frac{1}{y})$ Mathematics and youth

Solution: if we set $x=y$ then $f(2x)=2x^2f\left(\frac{1}{x}\right)$now we have $f(x+y)=\frac{f(2x)+f(2y)}{2}$ insert new variable z then we have:

$$f(x+y+z)=\frac{f(4x)+f(4y)+2f(2z)}{2}=\frac{f(4x)+f(4z)+2f(2y)}{2}$$

We have $f(4y)-2f(2y)=f(4x)-2f(2y)=C=\text{constant}$ then $f(2x)=C+f(x)$. as $x=y=1$we have $f(2)=2f(1)$thus $C=0$anf now $f(2x)=2f(x)$.now we must find a function f which satisfy the following conditions:

$$f(x+y)=f(x)+f(y)\ \text{ and }\ f(x)=x^2f\left(\frac{1}{x}\right)$$

Then note that $f\left(\frac{1}{x(x+y)}+\frac{1}{y(x+y)}\right)=f\left(\frac{1}{xy}\right)=f\left(\frac{1}{x(x+y)}\right)+f\left(\frac{1}{y(x+y)}\right)=\frac{f(x^2+xy)}{x^2(x+y)^2}+\frac{f(y^2+xy)}{y^2(x+y)^2}$ by the additivity of f we have: $f\left(\frac{1}{xy}\right)=\frac{y^2\big(f(x^2)+f(xy)\big)+x^2(f(y^2)+f(xy))}{x^2y^2(x+y)^2}=\frac{f(xy)}{x^2y^2}$ now we must have: $2xyf(xy) = y^2f(x^2) + x^2f(y^2)(*)$ set $y = 1$ then $f(x^2) = 2xf(x) - x^2f(1)$ thus if we substitute this in the (*) we have

$$2xyf(xy) = 2xy\big(yf(x) + xf(y)\big) - 2x^2y^2f(1)$$

Or $f(xy) = yf(x) + xf(y) - xyf(1)$ set $xy = 1$ then $f(1) = \frac{f(x)}{x} + xf\left(\frac{1}{x}\right) - f(1) = \frac{2f(x)}{x} - f(1)$. We conclude that $f(x) = f(1)x$

Problem-14: find all function $f: \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and for all nonzero reals x,y we have: $f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f^2(x)+f^2(y)}{2f(x)f(y)}$

Solution: first set $x = y = 1$ then $f(1) = 1$. Now set $y = zx$ to find that

$$f\left(\frac{1+z^2}{2z}\right) = \frac{1+f^2(z)}{2f(z)} = \frac{f^2(x)+f^2(xz)}{2f(x)f(xz)}$$

Now we have $f(xz) = \frac{f(x)}{f(z)}$ or $f(xz) = f(x)f(z)$ assume the first condition occur, switch x,z having $\frac{f(x)}{f(z)} = \frac{f(z)}{f(x)}$ now $f^2(x) = f^2(z)$ and $f(xz) = \pm 1$, set $x = z$ then $f(x^2) = 1$ and $f(-x) = \frac{f(x)}{f(-1)}$. Then we find the following functions:

$$f_1(x) = f(x) = \begin{cases} 0, & x = 0 \\ 1, & x \neq 0 \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} -1, & x < 0 \\ 0 & x = 0 \\ 1, & x > 0 \end{cases}$$

Now assume the second case then $f(xz) = f(x)f(z)$ and then $f(x)f\left(\frac{1}{x}\right) = 1$ now $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}$ then we have $f\left(\frac{x^2+y^2}{2xy}\right) = \frac{f(x^2+y^2)}{f(2)f(x)f(y)}$ then we receive the following equation:

$$f(x^2 + y^2) = \frac{f(2)}{2}(f^2(x) + f^2(y))$$

We need to find the value of $f(2)$ now pass the following procedure:

$$x = 2, y = 1 \text{ then } f(5) = \frac{f(2)}{2}(f^2(2) + 1)$$
$$x = 3, y = 1 \text{ then } f(10) = \frac{f(2)}{2}(f^2(3) + 1) \text{ then } f(5) = \frac{1}{2}(f^2(3) + 1)$$
$$x = 3, y = 4 \text{ then } f(25) = \frac{f(2)}{2}\big(f^2(3) + f^2(4)\big) \text{ then } f^2(5) = \frac{f(2)}{2}\big(f^2(3) + f^4(2)\big)$$

Solving the equation for $f(2)$ we find that: $f^5(2) - 2f^4(2) - f(2) + 2 = 0$ then $(f^4(2) - 1)(f(2) - 2) = 0$.

We study every case part by part. If $f(2) = 2$ then

$$f(x^2 + y^2) = f^2(x) + f^2(y) = f(x^2) + f(y^2)$$

Now for all positive real x we have $f(x) = x$ and then by the equation $f(-x) = f(-1)f(x)$ we find that $f(-1) = \pm 1$(set x=-1)thus we find the functions: $f_3(x) = x, f_4(x) = |x|$.

If $f(2) = 1$ we receive the equation $f(x^2 + y^2) = \frac{1}{2}\left(f^2(x) + f^2(y)\right) = \frac{f\left(x^2\right)+f(y^2)}{2}$ now for all positive reals x,y we have $f(x + y) = \frac{f(x)+f(y)}{2}$. Now insert new variable z then we find that:

$$f(x + y + z) = f\left((x + y) + z\right) = \frac{f(x+y)+f(z)}{2} = \frac{f(x)}{4} + \frac{f(y)}{4} + \frac{f(z)}{2} = \frac{f(x)}{2} + \frac{f(y)}{4} + \frac{f(z)}{4} \text{ (switch!)}$$

Now we have $f(x) = f(z)$and then f must be constant function.

If $f(2) = -1$ then $f^2\left(\sqrt{2}\right) = -1$. Contradiction.