# G.C.D and L.C.M

1. Let $a, b, c$ being positive rational numbers such that $a+b+c=a^3+b^2+c^2$ and being an integer , prove that there exist relatively prime positive integers $m, n$ such that $abc=\frac{n^2}{m^3}$. (Bulgarian Olympiads)

2. Let $a, b, c$ being positive integers prove that $\gcd(ab+1\ , bc+1, ca+1) \leq \frac{a+b+c}{3}$. (St-Petersburg)

3.Find all positive integers $a, b, c$ such that $\gcd(\ a, b, c)=\gcd(\frac{a+b}{2}, \frac{b+c}{2}, \frac{c+a}{2})$ (Ukrainian Training Camps)

4. Determine all integers can be expressed as the form $\gcd(a, b)+\gcd(a, c)+\gcd(c, b)$ , for some positive integers $a, b, c$. (St-Petersburg)

5. Are the inequality $\gcd(a, b)+\gcd(a, b+c) \leq a+c$ are always true for positive integers $a, b, c$. ? (St-Petersburg)

6. Find all sequences of natural numbers such that for all $i, j$ we have $\gcd(a_i, a_j)=\gcd(i, j)$ . (Russian Olympiads)

7. for the positive integer sequence $a_n$ , define $a_i=\frac{a_{n-1}+a_{n-2}}{\gcd(a_{n-1}, a_{n-2})}$ , is the sequence is bounded? (Russian Olympiads)

8. Prove that :

i. if $2n-1$ being prime , for all group of distinct integers $a_1, \ldots, a_n$ , there exist $i, j$ such that $\frac{a_i+a_j}{\gcd(a_i, a_j)} \geq 2n-1$.

ii. if $2n-1$ being composite , there exist group of distinct integers $a_1, \ldots, a_n$ such that for all $i, j$ we have $\frac{a_i+a_j}{\gcd(a_i, a_j)}<2n-1$. (Chinese Olympiads)

9. We know about the subset $S$ of Integers that for all $a, b$ in $S$ we have $\gcd(a, b)=\gcd(a-2, b-2)=1$ and if $x, y$ being in $S$, then $x^2-y$ so, prove that $S$ is indeed , Set of integer number. (USAMO)

10. Let $\gcd(a^2-1, b^2-1, c^2-1)=1$ , prove that $\gcd(ab+c\ , bc+a, ca+b)=\gcd(a, b, c)$ (Kvant Journal)

11. find all positive integers $a, b, c$ such that:

$$\gcd(a^2, b^2)+\gcd(a, bc)+\gcd(b, ac)+\gcd(c, ab)=239^2=ab+c$$

(St-Petersburg)

12.We know about the sequence $a_1, \dots, a_n$ of integers such that $\gcd(a_k, a_l) | a_{k+l}$ prove that for all $1 \le k < n$ , we have $a_1 \dots . a_k | a_{n-k+1} \dots . a_n$ (Kurchak Math Competitons)

13. We know about positive integers $a > b > 1$ such that $a + b | ab + 1$ , $a - b | ab - 1$ prove that $a < b\sqrt{3}$ . (Komal)

14. Prove that the number $100111000011111 \dots \dots .000 \dots 011111 \dots 11$, where number of last zeros is $2m$ and number of last ones is $2m + 1$. Is divisible by 111.....11, where number of ones is $2m + 1$.

15. Let $a, b, c$ being positive integers and $a(b^2 + c^2) = 2b^2c$ prove that $2b \le c + a\sqrt{a}$ . (St-Petersburg)

16. We know about positive integers $a, b, c$ such that none of them divides each other and $\gcd(a, b) + lcm(a, b) = \gcd(a, c) + lcm(a, c) + 1$ prove that $c < b \le \frac{3c}{2}$ (St-Petersburg)

17. There are 100 integers $a_1, \dots, a_{100}$ which $\gcd(a_1, \dots, a_{100}) = 1$, arranged around the circle ,and in each step we can add $gcd$ of two neighbor to the both of them , can in finite move , we make new numbers coprime?

(Russian Olympiads)

18. Let $m, n > 1$ and $a_1, \dots, a_m$ being integers such that , $a_i < n^m$, prove there exist $b_1, , , , b_m \le n$ such that $\gcd(a_1 + b_1 , \dots . . , a_m + b_m) < n$ (EGMO-2015)

19. Is there exist a sequence $a_1, \dots$ of natural numbers such that if $|m - n| = 1$ then $\gcd(a_m, a_n) = 1$ (Romanian Masters in Mathematics)

20. Let $p$ being prime prove that values of $n$ such that the expression $m^2 + n^2 + p^2 - 2mn - 2mp - 2np$, is square , not depends on $p$. (Bulgarian Olympiads 2014)

21. Let $x_1 = 3, y_1 = 2, x_{n+1} = 3x_n + 2y_n$ , $y_{n+1} = 4x_n + 3y_n$ , prove that both sequences doesn't contain third power. (Bulgarian Olympiads)

22. let $p$, being prime determine number of triples $a, b, c$ of the set $\{1,2, \dots . ,2p^2\}$ such that:

$$\frac{lcm\,(a,c) + lcm\,(b,c)}{a+b} = \frac{p^2+1}{p^2+2}$$

23. let we have $a_0 = 1, a_1 = 3, a_{n+1} = a_n + a_{n-1}$, prove that $\gcd(na_{n+1} + a_n, na_n + a_{n-1}) > 1$

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24. let $a, b, c$ being integers such that $\frac{bc}{b+c}, \frac{ac}{a+c}, \frac{ab}{a+b}$ being integers, prove that $\gcd(a, b, c) > 1$

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25. We know about positive integers $a, b$ that $a. gcd(a, b) + b. lcm(a, b) < \frac{5ab}{2}$ prove that $a$ is divisible by $b$

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26.are three different positive integer numbers, each of which is equal to the least common multiplier of differences from two others ?

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27. Find all positive integers $x, y$ such that $lcm(x^2, y) + lcm(y^2, x) = 1996$ Petersburg-95-37

28. let $n > m$ are positive integers, prove that $lcm(m, n) + lcm(m + 1, n + 1) > 2m\sqrt{n}$

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29. let $n > m$ are positive integers, prove that $lcm(m, n) + lcm(m + 1, n + 1) > \frac{2mn}{\sqrt{m-n}}$

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30. Let $a_1, \dots, a_n, \dots$ be a sequence of positive integers such that the greatest common divisor of any two consecutive terms is greater than the preceding term; in symbols, $\gcd(a_i, a_{i+1}) > a_{i-1}$. Prove that $a_n \geq 2^n$ for all n. (IMO-SL)