Number Theoretic Flavor

Problem. Let $p > 3$ be a prime number and n be an integer such that p divides $n^3 - 1$ but doesn't divide $n - 1$, prove that p divides $n - \frac{n^2}{2} + \cdots - \frac{n^{p-1}}{p-1} \equiv 0 \pmod{p}$.

Problem. Prove that there are infinitely many n such that if a prime number divides $n(n+1)$, then p^2 divides $n(n+1)$.

Problem. Let $(1+\sqrt{33})^n = x_n + y_n\sqrt{33}$. Prove that for every prime number p, it divides at least one of y_{p-1}, y_p, y_{p+1}.

Problem. Find all positive integers n such that $\varphi(n)$ divides $n^2 + 3$.

Problem. We know about the sequence a_n such that $a_1 = 1$, $a_n = a_{\lfloor \frac{n}{2} \rfloor} + a_{\lfloor \frac{n}{3} \rfloor} + \cdots . + a_{\lfloor \frac{n}{n} \rfloor} + 1$, Prove that there exist infinitely many n such that $a_n \equiv n \pmod{2^{2020}}$.

Problem. Prove that there exist infinitely many composite positive integers n such that n divides $3^{\frac{n-1}{2}} + 1$.

Problem. Solve the equation $(2^n - 1)(3^n - 1) = m^2$.

Problem. Let $p > 5, H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$, prove that:

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2}.$$

Problem. Let $p \equiv 1 \pmod{4}$, be a prime number. Suppose that

$$\sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} = \frac{A}{B}, \gcd(A, B) = 1.$$

And

$$\sum_{k=1}^{\frac{p-1}{4}} \frac{1}{k} = \frac{C}{D}, \gcd(C, D) = 1.$$

Prove that p divides A if and only if p divides C. How about the case $p \equiv 3 \pmod{4}$.

Problem. Prove that there are infinitely positive integers x, y, z that are pair-wise coprime and $nx^2 + y^3 = z^4$, for some positive integer n.

Problem. Prove that the product of 2^{2018} numbers of the form $\pm\sqrt{1} \pm \sqrt{2} \pm \cdots \pm \sqrt{2018}$ is a perfect square.

Problem. Let p be a prime number, lets define $S = \left\{n | 0 \leq n \leq p-1, p \nmid \sum_{k=0}^{p-1} k!\, n^k\right\}$. Prove that $|S| \geq \frac{p+1}{2}$.

Problem. Let $n \geq 2$, and $P(n)$ be the greatest prime number divides n. Prove that there exists a sequence of positive integers $a_1, \ldots.$ such that:

i. $P(a_1) < P(a_2) < \cdots$.

ii. If $1 \leq i_1 < i_2 < \cdots . < i_k, k \geq 1$ then, $a_{i_1} + \cdots + a_{i_k}$ is not a perfect power.

iii. If $1 \leq i_1 < i_2 < \cdots . < i_k, k \geq 2$ then, $a_{i_1} a_{i_k}$ is a perfect power.

Problem. Let a be an odd prime. We call the positive integer n, *Olympicus* if it could be written as $n = a^q - 1$. For some prime number q. Let C be an arbitrary set of prime numbers with following properties:

i. Any prime divisor of an Olympicus number is in C.

ii. For any $p_1, \ldots, p_k$ in C, all primes dividing $p_1 + \cdots + p_k$ lie in C.

Prove that C is the set of prime numbers.

Problem. Prove that the sequence of the first digits(from left) of $a_n = 2^n + 3^n$ is not periodic.

Problem. Let k be a fixed positive integer and a_n be the $k-$th digit of 2^n for all $n = 1, 2, \ldots$ Prove that $0.a_1 a_2 \ldots.$ is irrational.

Problem. Find the least positive integer $n > 1$ for which, there exists a positive integer k such that n^k ends with at least 2012 number 1.

Problem. Prove that there exists a positive integer n such that 333333^{333333^n} ends with the digits of 333333^{333333}.

Problem. Prove that there are $x, y > 1$ such that for all positive integers m, n, we have: $\lfloor x^n \rfloor \neq \lfloor y^m \rfloor$.

Problem. Prove that for each integer t, there are integers x, y, z, r such that $t = x^2 + y^3 + z^3 + r^3$.

Problem. Solve the equation $2^x + 1 = 7^y + 3^z$.

Problem. Prove that there are 2018 irreducible fractions with positive integer nominators and denominators, which are distinct, such that, if we subtract any of them, we shall arrive at denominators less than our original nominators.

Problem. The fractions $\frac{1}{n}, \frac{2}{n-1}, \frac{3}{n-2}, \dots, \frac{n}{1}$ are written on the board. Misery Vasya calculates the absolute value of all differences between any two neighboring numbers. She finds 10000 fractions of the form $\frac{1}{k}$ (i. e, . Egyptian fractions) for some natural number k. Prove that She can at least find 5000 numbers more than what she finds.

Problem. Are there positive integer k, and a non-constant sequence $a_1, \dots$ such that for each positive integer $n \geq 0$;

$$a_n = \gcd(a_{n+k}, a_{n+k+1}).$$

?

Problem. Find all positive integers m, n for which, there exists and integer x such that, n divides $1 + x + \dots + x^{m-1}$ and m divides $1 + x + \dots + x^{n-1}$.

Problem. Let $p \geq 3$ be a prime number, solve the equation $2 + x + \dots + x^{p-1} = y^2$ in positive integers.

Problem. Are there positive integers $a, b, a \neq b$ such that for all positive integer n, $2^a n - 1$ is a prime number if and only if $2^b n - 1$ is a prime number?

Problem. Let $b > 1$ be a positive integer and S be a set of integers containing 0 and no two elements of S have the same remainders mod b, we select $s_1, s_2, \dots$ from the elements of S. It is known that:

$$\sum_{i=1}^{\infty} \frac{s_i}{b^i} = 0.$$

Prove that $s_i = 0$.

Problem. Prove that there are infinitely many positive integers a, b, c such that

$$a + b + c, ab + ac + bc, a^2 + b^2 + c^2, abc$$

Are all perfect square. (Hint: think about this example: $(a, b, c) = (136, 153, 72)$.)

Problem. Let $P(x)$ be a polynomial with integer coefficients such that the absolute values of its coefficients are less than 5.10^6. If the equations $P(x) = x, P(x) = 2x, \dots P(x) = 20x$, have at least one integer roots, prove that $P(0) = 0$.

Problem. Let a, b be odd integers, prove that there exists a positive integer k such that at least one of $b^k - a^2$ or $a^k - b^2$ is divisible by 2^{2019}.

Problem. Let p be a prime number. We call an integer x, good for p, if for all $i, j, k \in \mathbb{N} \cup \{0\}$, $x^i + x^j - x^k$ is not divisible by p. Prove that for each positive integer N, there exists a prime number p for which $\gcd(x, p) = 1$, x is good for p, and $\mathrm{ord}_p^x > N$.