Geometry of the twistor spaces

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Sofia, 20th of May 2026

COMPATIBLE COMPLEX STRUCTURES ON A 4-DIMENSIONAL VECTOR SPACE

Let $V, <, >$ be a 4-dimensional real vector space with a scalar product and fixed orientation. $V^c = V \otimes \mathbb{C}$ is the complexification and $<, >$ is extended by complex linearity.

Definition

A compatible complex structure (CCS) on V is a splitting

$$V^c = V^{(1,0)} \bigoplus V^{(0,1)}$$

with $V^{(1,0)}$ maximal isotropic and $V^{(0,1)} = \overline{V^{(1,0)}}$

Definition

A compatible complex structure (CCS) on V is a map $I : V \to V$, such that $I^2 = -Id, < IX, IY > = < X, Y >$

Additionally we require I to induce the given orientation.

Aim: Describe the set of all compatible complex structures on V geometrically.

Isotropic vector in V^c → isotropic line. $<,>$ on V^c → quadratic form Q Q descends to $\mathbb{P}(V^c) = \mathbb{CP}^3$ The set

$q = \{[v] \in \mathbb{P}(V^c) | Q([v]) = 0\}$ is $\mathbb{CP}^1 \times \mathbb{CP}^1$ Suppose κ is the

complex conjugation of V^c. Then

$$\kappa(V^{(1,0)}) = V^{(0,1)} \rightarrow \kappa(I) = -I$$

Note: In real basis,

$$Q([\overline{v}]) = \overline{Q([v])}$$

Since

$$\mathbb{P}(V^{(1,0)}) \subset q$$

$$\{\text{Max isotropic subspaces of} V^c\} \leftrightarrow \{\text{Projective lines in } \mathbb{CP}^3 \text{entirely in q}\}$$

And

$$\{\text{Projective lines in } \mathbb{CP}^3 \text{entirely in q}\} = \mathbb{CP}^1 \cup \mathbb{CP}^1$$

One copy for each orientation. However, the CCS depend on κ and:

1. If $<,>$ is positive definite, then both copies of $\mathbb{CP}^1$ are preserved.
2. If $<,>$ has signature (3,1), $\kappa(\mathbb{CP}^1) \cap \mathbb{CP}^1 = \emptyset$
3. If $<,>$ has signature (2,2), $\kappa(\mathbb{CP}^1) \cap \mathbb{CP}^1$ is the union of the two open hemispheres

LINEAR ALGEBRA OF CCS IN DIMENSION 4

For $(V, <,>)$ equipped with an orientation consider

$$* : \Lambda^2 V^* \to \Lambda^2 V^*$$

called Hodge star. Since $*^2 = Id$

$$\Lambda^2 V^* = \Lambda^+ \oplus \Lambda^-$$

$<,>$ defines a norm in $\Lambda^\pm$

$\{\omega \in \Lambda^+, ||\omega||^2 = 1\} \leftrightarrow \{I - \text{CCS on V with positive orientation}\}$

$\omega(X, Y) = < IX, Y >$

So $\mathbb{CP}^1 \equiv S^2$ when $<,>$ is positive definite.

When $<,>$ has signature (2,2), $||\omega||^2 = 1$ is 2-sheeted hyperboloid.

CURVATURE AND TWISTOR SPACE

Let (M, g) is oriented Riemannian manifold. $\Lambda^2 M = \Lambda^2(T^*M)$ - bundle of 2-forms

$$\Lambda^2 M = \Lambda^+ M \oplus \Lambda^- M$$

Definition

Twistor space Z of M is the 2-sphere bundle $\pi : Z = S(\Lambda^+ M) \to M$ of unit spheres in $\Lambda^+ M$.

Let ∇ is the Levi-Civita connection on M, R - its curvature and $\mathcal{R} : \Lambda^2 M \to \Lambda^2 M$ is the operator $g(\mathcal{R}(X \wedge Y), Z \wedge W) = g(R(X, Y)Z, W)$ then

$$\mathcal{R} = \begin{pmatrix} W^+ + \frac{scal}{6} Id & B \\ B^t & W^- + \frac{scal}{6} Id \end{pmatrix}$$

Where *scal* is the scalar curvature, $trW^+ = trW^- = 0$. Moreover $B = 0$ iff g Einstein. When $W^\pm = 0$, g is called (anti-)selfdual (ASD or SD).

ALMOST COMPLEX STRUCTURES ON THE TWISTOR SPACE

Consider g - positive definite. Levi-Civita connection ∇ defines $TZ = \mathcal{H} \oplus \mathcal{V}$. Since $z \in Z$ is a complex structure I_z on $TM = \mathcal{H}$, define

$$\mathcal{I}_{1,2} = I_z|_{\mathcal{H}} \pm I_{CP^1}|_{\mathcal{V}}$$

$\mathcal{I}_1$ is defined by Atiyah-Hitchin-Singer, Penrose

Used for ASD solutions of Yang-Mills equations

$\mathcal{I}_2$: Eels-Salamon

Used for harmonic maps.

Theorem (Atiyah-Hitchin-Singer, Penrose; Eels-Salamon)

$\mathcal{I}_1$ is integrable $\leftrightarrow$ (M,g) is ASD
$\mathcal{I}_2$ is never integrable.

We focus on $\mathcal{I}_1$ unless otherwise stated.
Examples:
$M = S^4$, $Z = \mathbb{CP}^3$

$$\pi : [z_0, z_1, z_2, z_3] = [z_0 + jz_1, z_2 + iz_3] \in \mathbb{HP}^1 = S^4$$

$M = \overline{\mathbb{CP}^2}$, $Z = Fl_{1,2}$.

$M = \mathbb{R}^4$, $Z = \mathcal{O}(1) + \mathcal{O}(1)$

M-hyperkahler, $Z \equiv M \times S^2$ but the complex structure is not a product.

ALMOST HERMITIAN GEOMETRY OF Z

There is $g_t = \pi^*(g_M) + tg_{S^2}$ on Z,

Compact Z admits Kähler metric iff M is ASD and Einstein with *scal* > 0 iff (M, g) is S^4 or $\overline{\mathbb{CP}^2}$ (Hitchin). Then Z is Fano and has Kähler - Einstein metric.

Basic properties of $(g_t, \mathcal{I}_{1,2})$:

When $(Z, \mathcal{I}_1)$ is complex, g_t is *balanced* (Michelsohn, Mushkarov).

When M is ASD and Einstein with *scal* < 0, $(g_t, \mathcal{I}_2)$ is symplectic for a particular t (Mushkarov).

CURVATURE PROPERTIES OF Z

Calculation of sectional and holomorphic sectional curvature of $(Z, \mathcal{I}_{1,2}, g_t)$ (Davidov-Mushkarov)

There is t for which $(Z, \mathcal{I}_{1,2}, g_t)$ has Hermitian Ricci tensor iff (*i*) M is ASD and Einstein or (*ii*) M is ASD and its Ricci tensor has 3 equal eigenvalues (Davidov-Mushkarov).

There is t for which $(Z, \mathcal{I}_{1,2}, g_t)$ is $*$-Einstein iff M is ASD and Einstein with $scal \neq 0$. (Davidov-Grantcharov-Mushkarov). Here $*$-Einstein means that the fundamental form is eigenvector of $\mathcal{R}$

COMPLEX GEOMETRY OF Z AND COMPATIBLE COMPLEX STRUCTURES ON M

Z has foliation of rational curves with normal bundle $\mathcal{O}(1) + \mathcal{O}(1)$, but may have few or no divisors. It also has an anti-holomorphic involution defined by κ fiberwise. The Penrose correspondence reverses the construction: Z with such properties is a twistor space of a 4-dimensional manifold.

A compatible almost complex structure on (M, g) corresponds to a section $\sigma : M \to Z$. Main property is:

Theorem

The section σ defines an integrable almost complex structure $I = I_\sigma$ iff $\sigma(M) \subset Z$ is a complex submanifold.

In such case there is a divisor $D = [\sigma(M)] \cup [-\sigma(M)]$ which differs from the $K_Z^{-1/2}$ by an element in $Pic_0(Z)$.

(M, g, I) - Kähler : g is ASD $\iff$ $scal = 0$ $\iff$ $\sigma(M)$ is parallel.

(M, g, I_1, I_2) with $I_1 \neq \pm I_2$ is called *bihermitian*. Then $D_1 \cap D_2$ is anticanonical divisor for both I_1 and I_2 (up to a $Pic_0(M)$)

In particular bihermitian ⇝ holomorphic Poisson.

Extension to higher dimensional generalized geometry (Hitchin, Gualtieri)

Similar result for indefinite g (Davidov - Granctharov - Mushkarov - Yotov)

Remark

For $E \to M$ a vector bundle, ∇ - connection is ASD (instanton) if $\star F^{\nabla} = -F^{\nabla}$

(E, ∇) - instanton $\iff$ $F^{\nabla} \wedge \omega_I = 0$ for every I in Z. Hence $\pi^*(E)$ is holomorphic vector bundle on Z.

Another properties concern the algebraic dimension of Z;

Theorem (Campana, LeBrun-Poon)

For compact M, Z is Moichezon iff Z is in Fujiki's class $\mathcal{C}$. When M is simply-connected, M is homeomorphic to S^4 or $\#k\overline{\mathbb{CP}^2}$.

Moichezon space satisfies $\partial\overline{\partial}$-lemma.

Theorem (Fino-Grantcharov-Tardini-Tomassini-Vezzoni)

If M is simply-connected and compact and Z satisfies $\partial\overline{\partial}$-lemma, then M is homeomorphic to S^4 or $\# k\overline{\mathbb{CP}^2}$.

THANK YOU FOR YOUR ATTENTION!