**A SELECTION OF PROBLEMS**

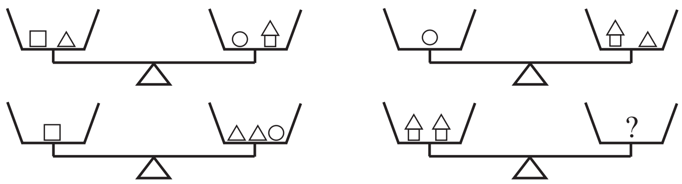
**FROM THE EIGHT MATHEMATICAL COMPETITION – SOFIA**

## Type One – Multiple Choice Answer

### Fifth Grade

<u>Problem</u>

The four scales are in balance. What CANNOT be put in the place with the question mark?




**A)** △△△△△△ **B)** ⌂△△△ **C)** □ **D)** △○○

<u>Solution</u>

From balance № 2 we can conclude that △○○ weigh as much as △⌂△⌂△.

Therefore △○○ is heavier than ⌂⌂.

<u>Conclusion</u>

The answer is D.

### Sixth grade

<u>Problem</u>

Some Knights and Liars are living on the Here-and-There Island.

The Knights always tell the truth. The Liars always lie.

Six islanders met on the Here-and-There Island. Everyone told the others: “You are ALL liars!”

How many are the Knights among those who met there?

**A)** 1; **B)** 3; **C)** 5; **D)** 6.

Solution

If ALL six islanders were Liars, then the statement “You are ALL liars!” would be TRUE. But this would mean that the Liars spoke the truth, which is IMPOSSIBLE $\Rightarrow$ answer D is IMPOSSIBLE.

If at least TWO of those six islanders were Knights, then NONE of them would say “You are ALL liars!” because that is NOT TRUE $\Rightarrow$ answers B and C are IMPOSSIBLE.

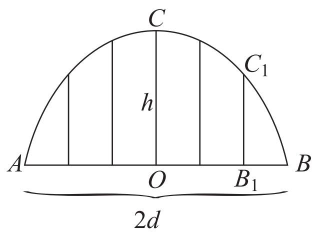
Conclusion

The answer is A.

**Eleventh – Twelfth Grade**

Problem

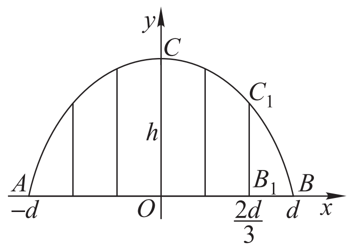
The arch *AB* is part of a bridge. It is also part of a parabola with an apex *C* which is the midpoint of the arch *AB*. There are five vertical trusses. They divide the segment *AB* into six equal parts. The main truss *OC* has a height of *h*. $AB = 2d$. Find $B_1C_1 : OC = ?$




**A)** $5:9$**;** **B)** $5:d$**;** **C)** $\frac{2}{3}:d$**;** **D)** $1:3$.

Solution

We choose a co-ordinate system with *OC* and *AB* axes. In this system $A(-d;\,0)$, $B(d;\,0)$ and $C(0;\,h)$. Since *OC* is the axis of symmetry of the parabola with $C(0;\,h)$ as its apex, then the parabola has an equation $y = f(x) = ax^2 + h$.




If $x = d$, $ad^2 + h = 0$, then $a = -\frac{h}{d^2}$, and so the arch *AB* has an equation $y = f(x) = -\frac{h}{d^2}x^2 + h = h\left[1 - \left(\frac{x}{d}\right)^2\right]$ when $x \in [-d; d]$.

Point $B_1$ has co-ordinates $\left(\frac{2d}{3}; 0\right)$ and $B_1C_1 = f\left(\frac{2d}{3}\right) = h\left[1 - \left(\frac{2}{3}\right)^2\right] = \frac{5}{9}h$.

Therefore $B_1C_1 : OC = \frac{5}{9}h : h = 5 : 9$.

<u>Conclusion</u>

The answer is **A**.

## Type Two – Free Answer

**Third Grade**

<u>Problem</u>

We assume that % = 10 and %:Q = 2.

Find %.Q = ?

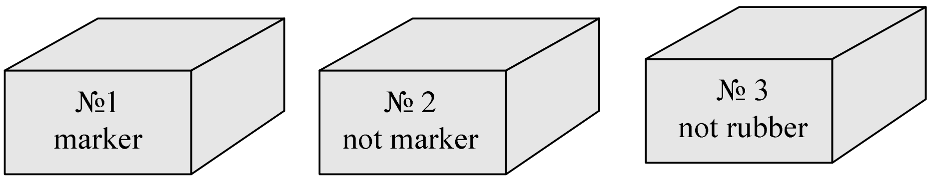
<u>Solution</u>

We know that % = 10. Then, from %:Q = 2, we find10:Q = 2 $\Rightarrow$ Q = 5.

Therefore %.Q = 10.5 = 50.

**Seventh grade**

<u>Problem</u>




In each of these three Boxes (1, 2, 3), there is just ONE object – a marker, a rubber, a pen.

Box № 1 is labelled “marker”, Box № 2 – “not marker”, and Box № 3 – “not rubber”. You know that only ONE of the labels is true. Find out what is the object in Box № 1.

Solution

We will solve the problem by a table, writing plus (+) if the label is TRUE, and minus (-) if it is NOT TRUE.

| **Box № 1** | | **Box № 2** | | **Box № 3** | |
|---|---|---|---|---|---|
| marker | + | pen | + | rubber | – |
| marker | + | rubber | + | pen | + |
| pen | – | marker | – | rubber | – |
| pen | – | rubber | + | marker | + |
| rubber | – | pen | + | marker | + |
| rubber | – | marker | – | pen | + |

The solution is in the line with ONE plus and TWO minuses.

In our case this is line 6 and the object in Box № 1 is a RUBBER.

**Ninth – Tenth grade**

Problem

***x*** and ***y*** are positive numbers. We have the equation $\frac{x}{y}+\frac{x+1}{y+1}+...+\frac{x+2006}{y+2006}=2007$.

The numerical value of the quotient $\frac{x^{2007}+xy^{2007}}{y^{2007}}$ is equal to …………

Solution

$$\frac{x}{y}+\frac{x+1}{y+1}+...+\frac{x+2006}{y+2006}=2007 \Leftrightarrow \left(\frac{x}{y}-1\right)+\left(\frac{x+1}{y+1}-1\right)+...+\left(\frac{x+2006}{y+2006}-1\right)=0 \Leftrightarrow$$

$$\frac{x-y}{y}+\frac{x+1-y-1}{y+1}+...+\frac{x+2006-y-2006}{y+2006}=0 \Leftrightarrow (x-y)\left(\frac{1}{y}+\frac{1}{y+1}+...+\frac{1}{y+2006}\right)=0 \Leftrightarrow$$

$$\Leftrightarrow x-y=0 \text{ or } \frac{1}{y}+\frac{1}{y+1}+...+\frac{1}{y+2006}=0.$$

Since $y>0$ then $\frac{1}{y}+\frac{1}{y+1}+...+\frac{1}{y+2006}>0$.

When $x>0,\ y>0$ then $\frac{x}{y}+\frac{x+1}{y+1}+...+\frac{x+2006}{y+2006}=2007 \ \Leftrightarrow \ x=y$.

Therefore if $x=y,\ y>0$, $\frac{x^{2007}+xy^{2007}}{y^{2007}}=\frac{x^{2007}+xx^{2006}}{x^{2007}}=\frac{2x^{2007}}{x^{2007}}=2$.

The answer is 2.

Type Three – fully-developed solution required

**Ninth – tenth grade**

Problem

The sequence 4, 7, 1, 8, 9, 7… has general term $a_n (n\in N, n>2)$.This general term $a_n$ is equal to the last digit of the sum $a_{n-1}+a_{n-2}$.

Find $\boldsymbol{n}$ if the partial sum of the first $\boldsymbol{n}$ terms is 12 345.

Solution

Finding out more terms of the sequence $\underbrace{4,\ 7,\ 1,\ 8,\ 9,\ 7,\ 6,\ 3,\ 9,\ 2,\ 1,\ 3}_{12\ terms},\ 4,\ 7,\ ...$ we see that it is periodic with a period of 12.

We assume that $S_n$ is the partial sum of the first $\boldsymbol{n}$ terms

$\left(n=12k+m;\ k=0,\ 1,\ 2,\ 3,\ ...;\ m=0,\ 1,\ 2,\ 3,\ ...,\ 11\right)$.

We find $S_{12}=4+7+1+8+9+7+6+3+9+2+1+3=60$.

Hence $S_n=k.S_{12}+S_m=60k+S_m\ \left(0\le S_m<60\right)$.

Since $S_n=12345=60.205+45$, then $k=205$ and $S_m=45$ e.g. $m=8$.

Therefore $n=12.205+8=2468$.