COUNTING CYCLES IN GRAPHS WITH STRUNIMA

Mladen Valkov

Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences, Bulgaria
mladen.vulkov@math.bas.bg; mlado1992@abv.bg; https://strunima.valkovbg.com/
WoS Researcher ID: KXS-1772-2024
ORCID iD:0009-0000-2536-2065

Abstract

Cycle count in graphs is a commonly investigated property in various types of graphs – in standard education of mathematics and informatics, competitive mathematics and open mathematical problems. The purpose of the developed digital tools in StruniMa is to systemize some of them into interactive educational tutorials and problems in different difficulty levels so they become available to wider range of students.

Keywords: *Graphs; Cycles in Graphs; Spanning Tree.*

INTRODUCTION

One of the main topics in StruniMa [1] is Graphs and chains and in particular non-oriented graphs, which contain given number of cycles with given length. Finding such graphs can be challenging which makes the topic suitable for digital resources and competitions in digital environment.

EXPOSITION

Identifying the number of cycles in a graph G(V,E) with given length k depends on both the manner that its edges are connected (connected components, the degrees of its vertices, the chromatic number of the graph and others), and the number k – common case is k=3 [2], 4 [3] and k=|V| [4], or many values grouped by divisibility by some number (for example for odd or even k). In StruniMa there are developed checks, which by a given parameter can find the number of cycles with given length, educational themes, which cover some well-known competitive problems and theorems[1] and competitive levels for time, the constructions for which are not trivial and there isn't a clear pattern between constructions with different number of vertices.

The used problems and themes can be a good introduction to a broader circle of students for getting familiar with strong mathematical methods like the method of the finite element. An example of this is to choose the longest path in the graph and if there are other edges from the ends of the path, they return to a vertex belonging to the path – so somewhere in the graph there is a cycle. There are attempts to introduce graphs in standard mathematical education [5], but most often this happens in classes with advanced mathematical classes in school or for preparation for competitions. In Bulgaria, graphs are more often covered in computer science classes, with an emphasis on searching them using algorithms, with the mathematical part left for specialized university courses. In StruniMa, the topic is used to prepare students

[1] StruniMa, Available at: https://strunima.valkovbg.com/GraphsWithoutCycles.html (last view: 12-09-2025).

for competitions, and together with topics such as Euler cycles, Steiner groups, Ramsey graphs, and planar graphs, it forms the topic Graphs and Chains.

The topic is based on well-known constructions, mathematical competition problems, and open problems. The most well-known construction of this type is a graph tree, in which the number of cycles of any length is 0 – it occupies the first part of the educational topic, which considers the inductive proof of the maximum number of edges in a tree (n-1 edges for n vertices), as well as concepts such as a connected graph and a spanning tree.

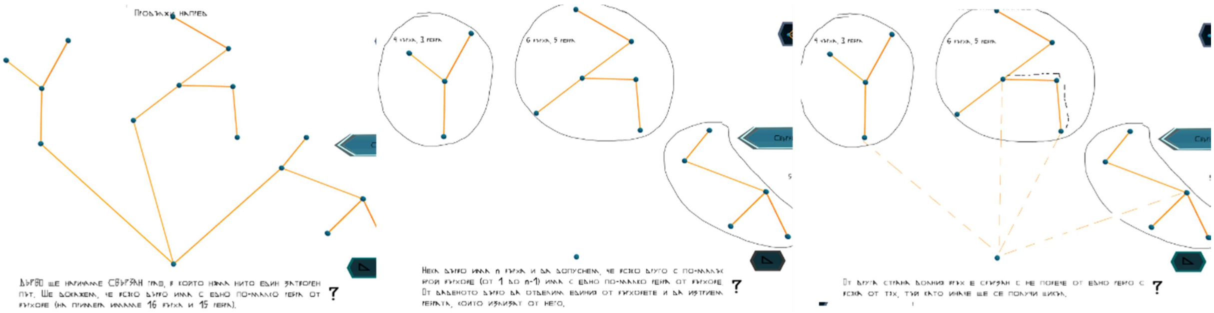

Figure 1. Interactive steps with that there are maximum n-1 edges in a tree with n vertices

The next part is based on the maximum number of edges in a graph without an even-length cycle. Here we use the basic fact (quite common in competition problems – e.g. BMO 2002) that if two cycles are "adjacent" next to each other (there are consecutive edges from both that are common) (Fig 2.), then somewhere there is a cycle of even length.

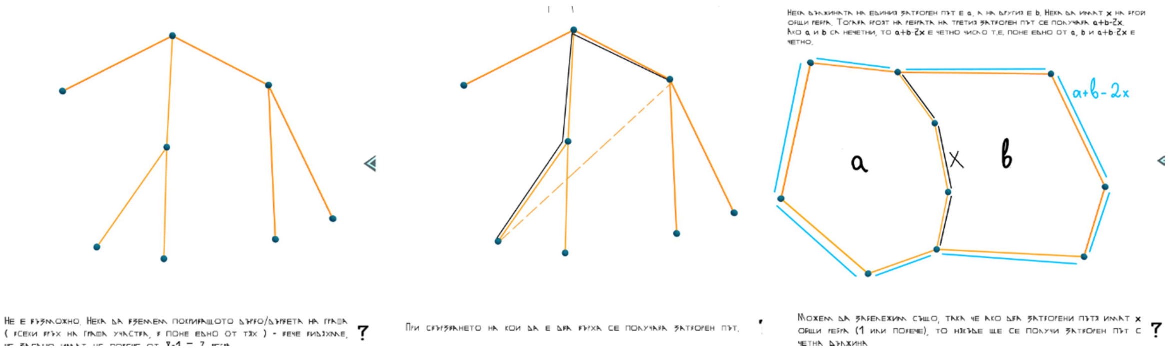

Figure 2. Parity of two "adjacent" cycles

From here, to the spanning trees of the sought maximal graph, we can add edges that form cycles with edges from the tree that have not been "used" until now – i.e. when the new edges form triangles, together with some two of the edges of the tree (Fig. 3). Which also gives the estimate that the number of edges of such a graph does not exceed $\left[\frac{3(n-1)}{2}\right]$. The construction follows trivially from here.

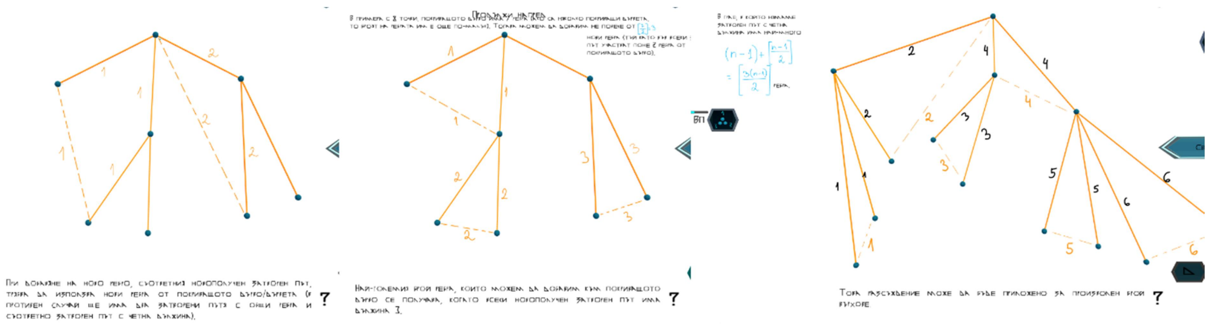

Figure 3. Maximum independent cycles with length 3

The upper construction is seen in the following mathematical competition problem:

(WMC2013, 10.4, prof. Ivan Landjev) In a country with n (n is at least 4) cities, more than 3(n-1)/2 direct flights are maintained between pairs of cities. A route is a sequence of cities between any two consecutive cities from which there is a direct flight. Two routes are independent if they have no cities in common except for the first and last cities. Prove that there are at least three independent routes between some two cities.

Having more than $\left[\frac{3(n-1)}{2}\right]$ edges means that there are two "adjacent" cycles, i.e. there are three independent routes between them. Here we do not necessarily have a spanning tree of the graph, but we can take a set of such and apply the above reasoning to each of them. This type of problem can be addressed with many built-in checks in the software that check whether there is a cycle of a certain length:

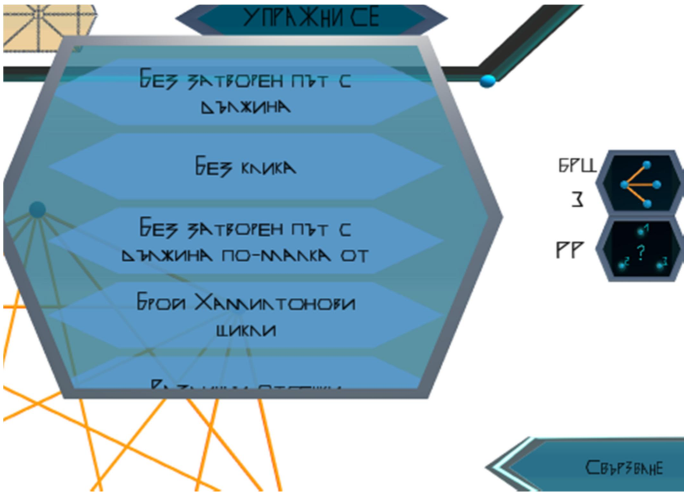

Figure 4. Different built-in checks which can be done in StruniMa while examining graphs

Another interesting type of problem that can also be considered by students in a wider spectrum is a graph with a certain number of Hamiltonian cycles. It is well known that check whether a graph is Hamiltonian is NP-complete. In StruniMa, functionality has been added to detect the number of Hamiltonian cycles of sufficiently small graphs. The following problem was created using it:

There are 11 points, located at equal distances on a circle. A closed broken line is called Hamiltonian if it passes through each of the points exactly once. Also, a closed Hamiltonian broken line will be called "interesting" if one of the bounded polygons into which the plane is broken contains the center of the circle. Prove that the number of interesting Hamiltonian closed broken lines is odd.

Firstly, we can count all possible Hamiltonian closed broken lines – for n points, each such line corresponds to exactly 2n permutations of the numbers from 1 to n (each of the points can be considered the leftmost in the permutation and the others can be ordered in either of the two possible directions), i.e. their number is $\frac{1}{2n}n! = \frac{1}{2}(n-1)!$

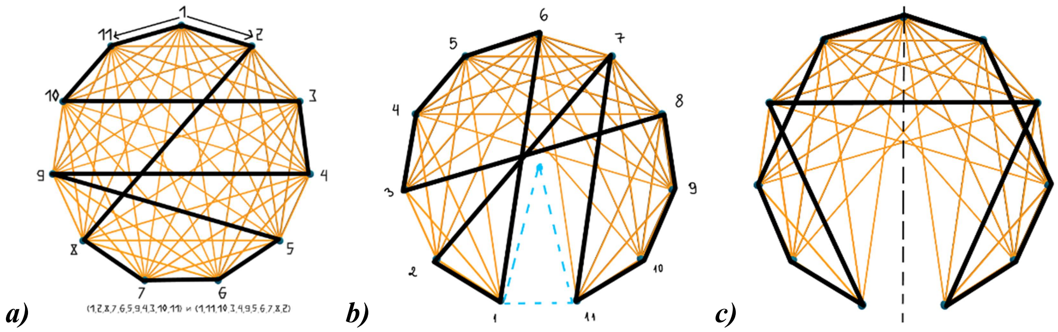

Figure 5. a) Hamiltonian cycle and permutations, b) Hamiltonian broken line which does not contain the center, c) Symmetrical Hamiltonian broken line

From them we can extract all Hamiltonian broken lines that do not contain the center of the circle, and for each such we can notice that there is exactly one isosceles triangle with vertices the center of the circle and two adjacent points on the circle, which lies entirely in the unbounded region. Since the number of these triangles is odd, the parity of the sought lines depends on the parity of the lines that do not contain a selected isosceles triangle constructed in the upper manner (in figure 5b the base of it is 1-11). Moreover, we can only consider the Hamiltonian closed broken lines that are symmetric with respect to the vertical axis of symmetry (fig. 5c).

Every "symmetric" broken line is defined by the set of its "bridges" - the segments that connect points from the set {1,2,3,4,5,6} with points from {6,7,8,9,10,11}. It turns out that for every symmetric set of bridges, the Hamiltonian closed broken line can be "completed" in an even number of ways (so that it remains symmetric) except for one (Fig. 6), which means that the number of broken lines that do NOT contain the center is odd.

Figure 6. „Interesting" Hamiltonian broken line, in which the edges that are not bridges, cannot be made in another way in order to preserve the degrees of the vertices in the path

With using the built-in check in the software, it turns out that for 13, 15 and 17 points, again the number of lines that do not contain the center is odd - but here the proof that for a given symmetric set of bridges we can complete the line in an even number of ways (except for one of the broken lines) becomes more complicated.

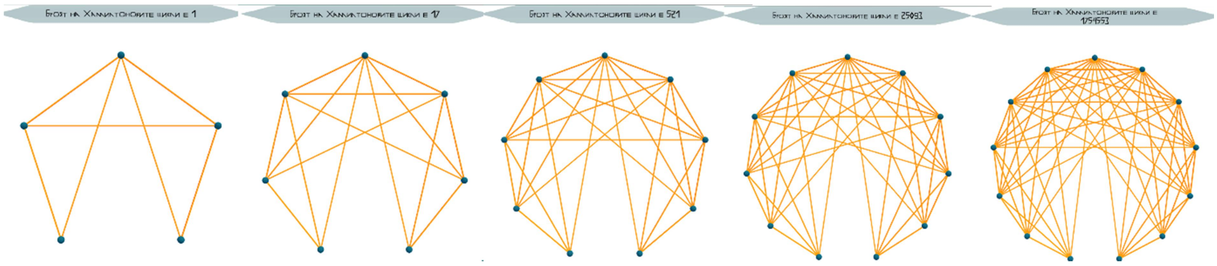

Figure 7. Subgraphs in which each Hamiltonian broken line does not contain the center of the circle

There are competition levels designed for these topics and problems and for most of them the difficulty of finding a construction increases with the number of vertices in the graph. The topics include:

- finding a graph with the maximum number of edges without cycles of even length, which we considered above;
- finding a graph with maximum number of edges without cycles of length less than a given number (based on problem 9.6, NMO 2008, Regional Round, Ivan Landjev) – here the player must build a spanning tree and consider at which levels of the tree he can add more edges;
- finding a graph with maximum number of edges without cycles of given length k. When k is 3, then Turan's theorem is used here (for which another training topic has been created), and when k is 4, then the maximum number of edges without K2,2 is sought[2]. The special case k=4 is quite common in various forms and includes knowledge such as counting pairs of rows in a table (in a table representation of the graph), Cauchy-Bunyakovsky inequality, etc.
- finding a graph with a given number of Hamiltonian cycles;
- 2-colored graphs with given number of monochromatic triangles (Ramsey problems)

Figure 8. Competition levels from the different topics

SUMMARY

In PPMG „Nancho Popovich", Shumen two classes were held with students of 6th grade in which the upper digital instruments of the system were used. The first part of the lesson proceeded by presenting a graph, types of graphs (complete and bipartite), finding the number of edges in a complete graph, counting triangles in a complete graph and one in which the maximum number of edges is reached without a triangle appearing. The students in the first

[2] A006855 OEIS, Available at: https://oeis.org/A006855 (last view: 12-09-2025).

class were subsequently divided into 3 groups and were provided with pre-made accounts, through which a time competition was organized. Although there were ideas for finding a construction for the given tasks, the students had difficulty. In the second group, the topics in the competition were reduced, focusing on finding constructions with a given number of triangles (21 triangles and 0 triangles), and here the students did better. From this it can be concluded that for a lasting understanding of the topics of graphs and closed paths, long-term preparation with the software and targeted preparation for each of the types of tasks related to cycles of a certain length are required, as well as careful discussion of the auxiliary knowledge necessary for them - in this case, counting edges in a complete graph, counting triangles in different ways, using the fact that the sum of the degrees of the vertices must be an even number, etc. Although this was the students' first encounter with graphs, some of them spontaneously suggested different methods for counting - for example, for counting triangles, it was suggested in a complete graph to count the angles formed by the pairs of edges coming out of the vertices and to divide the result by 3.

Graph theory can build connections between different areas of mathematics for example between combinatorics and algebra. Ways to find closed paths in a graph have wide application in finite systems, geometrically to visualize algebraic problems (like the permutations of the numbers from 1 to n) and to be a learning ground for introducing mathematical methods, like the finite element method. Having digital tools for this can be beneficial to accelerate this process.

References

1. Valkov, M. (2022)., Synchronous remote learning with the educational game "StruniMa", E-Magazine "Pedagogical Forum", Year Tenth (2022), Issue First // [Вълков М. (2022) Синхронно дистанционно обучение в образователната игра "СтруниМа"E списание "Педагогически форум", брой 1, 2022], DOI: https://doi.org/10.15547/pf.2022.005, ISSN: 1314 7986
2. H. Liu, O. Pikhurko, K.Staden, The exact minimum number of triangles in graphs with given order and size, Forum of mathematics, Pi (2020), DOI: https://doi.org/10.1017/fmp.2020.7
3. F.A.Firke, P.M.Kosek, E.D.Nash, J.Williford, Extremal graphs without 4-cycles, (2013), Journal of Combinatorial Theory, Series B, Volume 103, Issue 3, 327-336
4. G.A.Dirac, Some theorems on abstract graphs, Proceedings of the London Mathematics Society, Third Series 2, (1951), 69-81
5. N. Asghari, A. Shahvarani, A. R. Haghighi, Graph theory as a Tool for teaching Mathematical Processes, (2012), International Journal for Cross-Disciplinary Subjects in Education (IJCDSE), Volume 3, Issue 2, DOI: https://doi.org/10.20533/ijcdse.2042.6364.2012.0104

Received: 19-09-2025   Accepted: 16-12-2025   Published: 31-12-2025

Cite as:
Valkov, M. (2025). “Counting Cycles in Graphs with Strunima”, Science Series “Innovative STEM Education”, volume 07, ISSN: 2683-1333, pp. 348-353, 2025. DOI: https://doi.org/10.55630/STEM.2025.0732