Verification Methods Using Taylor Models

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For more than forty years, interval arithmetic has been used in enclosure methods for the calculation of rigorous error bounds for many problems in numerical analysis, such as the computation of range bounds for real functions, the solution of linear and nonlinear equations, or the integration of initial value problems for ODEs. These topics were already addressed in the pioneering work of R. E. Moore in 1966.

Interest in interval arithmetic has been primarily aroused by its ability for controlling roundoff errors on digital computers. Interval analysis is also used for bounding all kinds of truncation errors: the truncation error of an infinite iteration, the remainder term of a convergent series, discretization errors in the numerical solution of differential equations, etc.

Unfortunately, interval arithmetic is sometimes affected by overestimation, such that computed error bounds are over-pessimistic. Overestimation is often caused by the *dependency problem*, which is the lack of interval arithmetic to identify different occurrences of the same variable, or by the *wrapping effect*, which appears when intermediate results of a computation are enclosed into intervals. Overestimations due to wrapping are one of the major problems in the interval arithmetic treatment of ODEs.

For reducing both the dependency problem and the wrapping effect, interval arithmetic has been endorsed with symbolic extensions. A rigorous multivariate Taylor arithmetic has been developed by M. Berz and his group since the 1990s. A *Taylor model* of a function f on some interval X consists of the Taylor polynomial p_n of order n of f and an interval remainder term I_n, which encloses the approximation error $|f - p_n|$ on X. In computations that involve f, the function is replaced by $p_n + I_n$. The polynomial part is propagated by symbolic calculations where possible. The interval remainder term is processed according to the rules of interval arithmetic. All truncation and roundoff errors in intermediate operations are also enclosed into the remainder interval of the final result.

A software implementation of Taylor model arithmetic has been given by Berz and his co-workers in the COSY Infinity package. Using COSY Infinity, Taylor models have been applied with great success to a large variety of problems, such as global optimization, validated multidimensional integration, or the validated solution of ODEs and DAEs.

In our talk, we review recent advances in Taylor model methods for validated global optimization and for the validated solution of ODEs.