All About Generalized Interval Distributive Relations.
I. Complete Proof of the Relations.

Evgenija D. Popova

Institute of Mathematics & Informatics, Bulgarian Academy of Sciences
Acad. G. Bonchev Str., block 8, BG-1113 Sofia, Bulgaria
e-mail: epopova@iph.bio.bas.bg, epopova@argo.bas.bg

Abstract

The arithmetic on an extended set of proper and improper intervals presents algebraic completion of the conventional interval arithmetic allowing thus efficient solution of interval algebraic problems. This paper generalizes the distributive relations, known by now, on multiplication and addition of proper and improper intervals. A complete proof of the main results is presented, demonstrating an original technique based on functional notations and transition formulae between different interval structures. A variety of equivalent forms and different representations are discussed together with some examples.

This paper is an extraction from [19] and will be updated permanently to include current improvements, generalizations and applications of the conditionally distributive relations. The second part of the paper is scheduled for the end of 2000 and will include several directions for the application of the generalized distributive relations.

Keywords: Interval Arithmetic, Distributive Law, Generalized Intervals, Applications

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Contents

1 Introduction

Among several extensions of the classical interval arithmetic [2, 12, 23] that have been proposed, we consider that one aiming at an algebraic completion of interval arithmetic. First developed by H.-J. Ortolf [13] and E. Kaucher [6, 8], further investigated by E. Gardeñes et al. [4, 5] and others, it is obtained as an extension of the set of normal (proper) intervals by improper intervals and a corresponding extension of the definitions of the interval arithmetic operations and functions. The generalized interval arithmetic structure, thus obtained, possesses group properties with respect to addition and multiplication operations. Lattice operations are closed with respect to the inclusion order relation. Handling of norm and metric are very similar to norm and metric in linear spaces [8]. In order to emphasize that a generalized interval can be considered as a pair of a proper interval (in set-theoretical sense) and a "direction", sometimes the algebraic extension of the conventional interval arithmetic is called directed interval arithmetic [9]. The term "modal interval analysis" [5, 11] reflects another interpretation of generalized intervals in terms of modal logic.

The algebraic properties of generalized interval arithmetic make it a suitable environment for solving interval algebraic problems, e. g. some interval algebraic equations, which are not linear in general, can be solved explicitly just by applying elementary algebraic transformations due to the existence of inverse elements with respect to addition and multiplication operations [17]. However, the efficient solution of some interval algebraic problems is hampered by the lack of well studied distributive relations between generalized (proper and improper) intervals.

While the existence of inverse elements with respect to addition and multiplication follows from the isomorphic embedding of the set of conventional intervals into a group [6], the validity of certain distributive relations is not straightforward. The well-known subdistributivity property of normal intervals is extended in [6] for improper intervals and some special cases of distributivity for degenerated (point) intervals are discussed there. Gardeñes et al. [4] define distributive domains where a distributive relation involving generalized intervals of certain types is valid. Much later, in [3] a more general conditionally distributive relation for three generalized intervals not involving zero is formulated in a concise form.

Here we present a complete characterization of the conditionally distributive relations on multiplication and addition of generalized intervals. This paper expands, generalizes and clarifies the conditionally distributive relations, discussed in [15, 16, 18]. Section 1.1 provides some basic concepts of the arithmetic on proper and improper intervals and introduces special functional notations which are essential for an efficient (both analytic and computer) handling of generalized intervals. Section 3.2 contains a detailed proof of the basic conditionally distributive relations on multiplication and addition of generalized (proper and improper) intervals. Some auxiliary assertions, used by the main proof and helpful for the application of the distributive relations, are presented and proven in Section 3.1. The specific of the generalized conditionally distributive relations involve rules and conditions for taking a common multiplier out of brackets, and rules and conditions for disclosing brackets when multiplying out a sum of intervals. How to disclose brackets when multiplying a sum of intervals in the general case if there is no distributivity is presented in Section 2 and further specified in Section 4. Many sufficient conditions for the distributivity in multiplication of a finite interval sum are also given in Section 4. Section 5 discusses a technique for obtaining various equivalent forms of the distributive relations. Some references to papers, containing illustrative examples for the application of the generalized interval distributive relations, are given in Section 6.

1.1 The arithmetic on proper and improper intervals

The set of conventional (proper) intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^- \leq a^+, a^-, a^+ \in \mathbb{R}\}$ is extended by the set $\overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^- \geq a^+, a^-, a^+ \in \mathbb{R}\}$ of improper intervals obtaining thus the set $\mathbb{D} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\} \cong \mathbb{R}^2$ of all ordered couples of real numbers called here generalized intervals. Denote the set of zero involving generalized intervals by $\mathbb{T} = \{A \in \mathbb{D} \mid a^-a^+ \leq 0\}$.

Dual is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{D}$. For $A = [a^-, a^+] \in \mathbb{D}$, its "dual" is defined by[1]

$$\mathtt{Dual}[A] = A_- = [a^+, a^-].$$

Very often, consideration of one or another interval end-point or the dualization of a generalized interval depends on some binary valued variables. To avoid long branching formulae and to simplify the proofs, in our investigations we use functional notations for the interval end-points and for the dualization of intervals.

Define $\Lambda = \{+, -\}$. For $\mu, \nu \in \Lambda$, define product $\lambda = \mu\nu \in \Lambda$ by

$$\lambda = \begin{cases} + & \text{if } \mu = \nu, \\ - & \text{if } \mu \neq \nu. \end{cases}$$

This product is commutative: $\mu\nu = \nu\mu$ for $\mu, \nu \in \Lambda$.

For $\lambda \in \Lambda$, define $a^\lambda = \begin{cases} a^+ & \text{if } \lambda = +, \\ a^- & \text{if } \lambda = - \end{cases}$ and $A_\lambda = \begin{cases} A & \text{if } \lambda = +, \\ A_- & \text{if } \lambda = -. \end{cases}$

Next, we define some interval functionals, useful for describing certain classes of generalized intervals. Denote $\mathcal{L} = \{\{+\}, \{-\}, \{+, -\}\}$. For an interval $A \in \mathbb{D}$, define a functional, called "*direction set*", $\mathcal{T} : \mathbb{D} \to \mathcal{L}$ by

$$\mathcal{T}(A) = \begin{cases} \{+\} & \text{if } a^- < a^+, \\ \{-\} & \text{if } a^- > a^+, \\ \{+, -\} & \text{if } a^- = a^+, \end{cases}$$

and a functional, called "*direction*", $\tau : \mathbb{D} \to \Lambda$ by $\tau(A) \in \mathcal{T}(A)$. A generalized interval A is called *proper* if $\tau(A) = +$ and *improper* if $\tau(A) = -$. For degenerate (point) intervals $A \in \mathbb{R}$, the direction set is $\mathcal{T}(A) = \{+, -\}$. Therefore these intervals belong to both sets: the set of proper intervals and the set of improper intervals. We shall see in Section 3.2 that the freedom to choose the direction of a point interval arbitrary from its direction set is essential for obtaining all possible distributive relations. Degenerate intervals $A \in \mathbb{R}$ will be also denoted by $A = a$. (Sometimes instead of $[a, a]$ we shall simply write a, e.g. $[0, 0] = 0$.) For $A = a \in \mathbb{R}$ and $\lambda \in \Lambda$, $a_\lambda = a$.

For an interval $A \in \mathbb{D}\backslash\mathbb{T}$, define "*sign*" $\sigma : \mathbb{D}\backslash\mathbb{T} \to \Lambda$ by

$$\sigma(A) \;=\; \begin{cases} + & \text{if } a^{-\tau(A)} > 0, \\ - & \text{if } a^{\tau(A)} < 0. \end{cases}$$

In particular, σ is well defined over $\mathbb{R} \setminus \{0\}$ and $\sigma(A) = \sigma(a)$ for $A = a \in \mathbb{R} \setminus \{0\}$.

[1] In some papers $\mathtt{Dual}[A]$ is denoted by $\overline{A}$.

With every interval $A \in \mathbb{D}$ we can associate a proper interval $\mathrm{pro}(A) = A_{\tau(A)} = [a^{-\tau(A)}, a^{\tau(A)}]$ wherein $a^{-\tau(A)} \le a^{\tau(A)}$. For $A \in \mathbb{D}$, $\mathrm{pro}(A)$ is a projection of the generalized interval A onto the conventional interval space $I\!\!R$.

The definition of the well-known χ-functional, introduced by H. Ratschek in [20], is extended for generalized intervals, $\chi : \mathbb{D} \to [-1, 1]$ by $\chi([0, 0]) = -1$ and

$$\chi(A) = \begin{cases} a^-/a^+ & \text{if } |a^-| \le |a^+|, \\ a^+/a^- & \text{if } |a^-| \ge |a^+|. \end{cases}$$

To provide a convenient manipulation of interval formulae involving χ-functionals, we define a functional $\mathcal{N} : \mathbb{D} \to \mathcal{L}$ by

$$\mathcal{N}(A) = \begin{cases} \{+\} & \text{if } |a^+| > |a^-|, \\ \{-\} & \text{if } |a^+| < |a^-|, \\ \{+, -\} & \text{if } |a^+| = |a^-|, \end{cases}$$

and a functional $\nu : \mathbb{D} \to \Lambda$ by $\nu(A) \in \mathcal{N}(A)$. Thus, the definition of χ for $A \in \mathbb{D} \setminus \{0\}$ becomes

$$\chi(A) = a^{-\nu(A)}/a^{\nu(A)}.$$

Intervals A, such that $\chi(A) = -1$ are called zero-symmetric (symmetric, for simplicity). For the symmetric intervals, $A \in \mathbb{T}$ with $\chi(A) = -1$, we have $\mathcal{N}(A) = \{+, -\}$. Hence, for a symmetric interval A we have the freedom to choose either $\nu(A) = +$ or $\nu(A) = -$ which is also essential for the distributive relations.

The inclusion order relation between normal intervals is extended for $A, B \in \mathbb{D}$ by

$$A \subseteq B \iff (b^- \le a^-) \text{ and } (a^+ \le b^+).$$

The arithmetic operations $+$ and $\times$ are extended from the familiar set $I\!\!R$ of proper intervals to $\mathbb{D}$. In [6], [4] and [8] the definition of $\times$ is given in a table form, while using the functional "$\pm$" notations we gain a concise presentation of the interval arithmetic formulae facilitating their manipulation.

$$A + B = [a^- + b^-,\ a^+ + b^+] \qquad \text{for } A, B \in \mathbb{D}; \tag{1}$$

$$A \times B = \begin{cases} [a^{-\sigma(B)} b^{-\sigma(A)},\ a^{\sigma(B)} b^{\sigma(A)}], & A, B \in \mathbb{D} \setminus \mathbb{T}; \\ [a^{\sigma(A)\tau(B)} b^{-\sigma(A)},\ a^{\sigma(A)\tau(B)} b^{\sigma(A)}], & A \in \mathbb{D} \setminus \mathbb{T},\ B \in \mathbb{T}; \\ [a^{-\sigma(B)} b^{\sigma(B)\tau(A)},\ a^{\sigma(B)} b^{\sigma(B)\tau(A)}], & A \in \mathbb{T},\ B \in \mathbb{D} \setminus \mathbb{T}; \\ [\min\{a^- b^+, a^+ b^-\},\ \max\{a^- b^-, a^+ b^+\}]_{\tau(A)}, & A, B \in \mathbb{T}, \tau(A) = \tau(B); \\ 0, & A, B \in \mathbb{T}, \tau(A) = -\tau(B). \end{cases} \tag{2}$$

The occurrence of min and max functions at the end-points of the result on multiplication of two zero-involving intervals hampers the analytical derivations in interval analysis and affects the performance of corresponding computer operation. It was proven in [14] (see also [15]) that for $A, B \in \mathbb{T}$, such that $\tau(A) = \tau(B) = \tau$

$$A \times B = \begin{cases} [a^{-\nu(B)\tau} b^{\nu(B)},\ a^{\nu(B)\tau} b^{\nu(B)}] = A \times b^{\nu(B)}, & \mathit{if}\ \chi(A) \le \chi(B); \\ [a^{\nu(A)} b^{-\nu(A)\tau},\ a^{\nu(A)} b^{\nu(A)\tau}] = a^{\nu(A)} \times B, & \mathit{if}\ \chi(A) \ge \chi(B). \end{cases} \tag{3}$$

For $A, B \in \mathbb{T}$, such that $\tau(A) = \tau(B)$ and $\chi(A) = \chi(B)$, we have $A \times b^{\nu(B)} = a^{\nu(A)} \times B$.

Interval subtraction and division can be expressed as composite operations $A - B = A + (-1) \times B$ and $A/B = A \times (1/B)$, where $1/B = [1/b^+, 1/b^-]$ if $B \in \mathbb{D}\backslash\mathbb{T}$. End-pointwise:

$$
\begin{aligned}
A - B &= [a^- - b^+, a^+ - b^-], \quad A, B \in \mathbb{D}; \\
A/B &= \begin{cases} [a^{-\sigma(B)}/b^{\sigma(A)}, a^{\sigma(B)}/b^{-\sigma(A)}], & A, B \in \mathbb{D}\backslash\mathbb{T}; \\ [a^{-\sigma(B)}/b^{-\sigma(B)\tau(A)}, a^{\sigma(B)}/b^{-\sigma(B)\tau(A)}], & A \in \mathbb{T},\ B \in \mathbb{D}\backslash\mathbb{T}. \end{cases}
\end{aligned}
$$

The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space. [2, 12, 23].

Addition and multiplication operations are commutative and associative. For $A, B, C \in \mathbb{D}$ and $\circ \in \{+, \times\}$

$$
\begin{aligned}
A \circ B &= B \circ A \\
(A \circ B) \circ C &= A \circ (B \circ C).
\end{aligned}
$$

The systems $(\mathbb{D}, +, \subseteq)$ and $(\mathbb{D}\backslash\mathbb{T}, \times, \subseteq)$ are isotone groups. Hence, there exist unique inverse elements $-A_-$ and $1/(B_-)$ with respect to the operations $+$ and $\times$ such that

$$
A - A_- = 0 \quad \text{and} \quad B/(B_-) = 1 \ \text{ for } A \in \mathbb{D},\ B \in \mathbb{D}\backslash\mathbb{T}. \tag{4}
$$

The definition of norm, metric, many topological and lattice properties of $(\mathbb{D}, +, \times, \subseteq)$ are given in [6, 8]. Some other properties and applications of the arithmetic on proper and improper intervals can be found in [4] and [11].

An important property of the dual operator is its distributivity with respect to the arithmetic and lattice operations:

$$
(A_1 \circ \ldots \circ A_n)_- = (A_1)_- \circ \ldots \circ (A_n)_-, \qquad A_i \in \mathbb{D},\ i = 1, \ldots, n; \quad \circ \in \{+, -, \times, /\}. \tag{5}
$$

In [6, 7] the so-called hyperbolic product is introduced by

$$
A \times_h B = [a^- b^-,\ a^+ b^+], \qquad A, B \in \mathbb{D}. \tag{6}
$$

The inverse elements $-A_-$, $1/A_-$ generate operations

$$
\begin{aligned}
A -_h B &= A + (-B_-) \quad = [a^- - b^-,\ a^+ - b^+], \quad A, B \in \mathbb{D}, \\
A\ /_h\ B &= A \times_h (1/B_-) = [a^-/b^-,\ a^+/b^+], \quad A \in \mathbb{D},\ B \in \mathbb{D}\backslash\mathbb{T},
\end{aligned}
$$

called hyperbolic subtraction, resp. hyperbolic division. The interval arithmetic addition (1) together with the hyperbolic multiplication (6) form a field $\{\mathbb{D}, +, \times_h\}$ [7], where a complete distributive law

$$
A \times_h C + B \times_h C = (A + B) \times_h C
$$

holds true for $A, B, C \in \mathbb{D}$. Due to the associativity of addition operation, the distributive law can be generalized for any finite number of generalized intervals $A_i, C \in \mathbb{D},\ i = 1, \ldots, n$

$$
A_1 \times_h C + \ldots + A_n \times_h C = (A_1 + \ldots + A_n) \times_h C. \tag{7}
$$

In the next sections we shall use the unconditional distributivity (7) of the hyperbolic product to prove the conditionally distributive relations on multiplication and addition of generalized intervals. To this end, we apply an original technique for projecting generalized interval arithmetic formulae onto the hyperbolic space and vice verse, using essentially the following formula for transition between interval and hyperbolic multiplication.

$$A \times B = \begin{cases} A_{\sigma(B)} \times_h B_{\sigma(A)}, & \text{for } A, B \in \mathbb{D}\backslash\mathbb{T}; & (8.1)\\ a^{\sigma(A)\tau(B)} \times_h B_{\sigma(A)}, & \text{for } A \in \mathbb{D}\backslash\mathbb{T},\ B \in \mathbb{T}; & (8.2)\\ A_{\sigma(B)} \times_h b^{\tau(A)\sigma(B)}, & \text{for } A \in \mathbb{T},\ B \in \mathbb{D}\backslash\mathbb{T}; & (8.3)\\ A_{\nu(B)\tau} \times_h b^{\nu(B)} & \text{for } A, B \in \mathbb{T},\ \tau(A) = \tau(B) = \tau,\ \chi(A) \le \chi(B); & (8.4)\\ a^{\nu(A)} \times_h B_{\nu(A)\tau}, & \text{for } A, B \in \mathbb{T},\ \tau(A) = \tau(B) = \tau,\ \chi(A) \ge \chi(B); & (8.5)\\ 0, & \text{for } A, B \in \mathbb{T},\ \tau(A) \neq \tau(B). & (8.6) \end{cases}$$

For $A \in \mathbb{T}$, $b \in \mathbb{R}$ by (8.1) and (8.3) we get

$$A_{\sigma(b)} \times_h b \;=\; A \times b \qquad \text{and} \qquad A_{\sigma(b)} \times b \;=\; A \times_h b \tag{9}$$

As early as in [6], E. Kaucher represents in table form the result of interval multiplication (2) by the hyperbolic product of the operand end-points. The considerable progress, we achieved using the projecting technique, is due to the functional "±" notations and the representation (3).

1.2 Historical remarks

While most algebraic properties of the generalized interval arithmetic follow straightforward from the isomorphic embedding of the set $\mathbb{IR}$ of normal intervals into a group, the validity of some distributive relations is obvious only for certain special cases of generalized intervals. Two very special cases of the distributive law (for proper and degenerate intervals) are given in [8]:

$$c \times (A + B) \;=\; c \times A + c \times B, \qquad A, B \in \mathbb{D};\ \tau(A) = \tau(B) = +;\ c \in \mathbb{R}; \tag{10}$$

$$C \times (a + b) \;=\; C \times a + C \times b, \qquad C \in \mathbb{D};\ \tau(C) = +;\ a, b \in \mathbb{R};\ ab \ge 0. \tag{11}$$

Gardeñes et al., in [4], define distributivity domains by the following assertion:

Every $C \in \mathbb{D}$ determines on $\mathbb{D}$ the C-distributive domains $D_r(C)$, $r = 1, 2, 3, 4$, such that if $(\exists r)$ $(\forall i = 1, \ldots, n)$ $A_i \in D_r(C)$, then

$$C \times (A_1 + \ldots + A_n) = C \times A_1 + \ldots + C \times A_n. \tag{12}$$

The $D_r(C)$ domains are defined as follows:

$$D_1(C) \;:=\; \{A \mid \sigma(A) = +\} \cup \{B \mid b^{\sigma(\nu_B)} \ge 0,\ \tau(B) = \tau(C),\ \chi(C) \le \chi(B) \le 0\};$$

$$D_2(C) \;:=\; \{A \mid \sigma(A) = -\} \cup \{B \mid b^{\sigma(\nu_B)} \le 0,\ \tau(B) = \tau(C),\ \chi(C) \le \chi(B) \le 0\};$$

$$D_3(C) \;:=\; \begin{cases} \text{no distributivity} & \text{if } \chi(C) = -1, \tau(C) = +,\\ \{A \mid \tau(A) = +,\ -1 \le \chi(A) \le \min\{0,\ \chi(C)\}\}, & \text{if } \tau(C) = +,\\ \{A \mid \tau(A) = +,\ -1 \le \chi(A) \le 0\}, & \text{if } \tau(C) = -; \end{cases}$$

$$D_4(C) \;:=\; \begin{cases} \text{no distributivity} & \text{if } \chi(C) = -1, \tau(C) = -,\\ \{A \mid \tau(A) = -,\ -1 \le \chi(A) \le \min\{0, \chi(C)\}\}, & \text{if } \tau(C) = -,\\ \{A \mid \tau(A) = -,\ -1 \le \chi(A) \le 0\}, & \text{if } \tau(C) = +. \end{cases}$$

The above distributivity domains generalize conventional interval distributive relation (12) (studied in [21, 22] for normal intervals), for generalized (proper and improper) intervals. Although these distributive domains tell us much more than (10) and (11), they do not cover all possible cases of distributive relations between generalized intervals.

In [3] a more general conditionally distributive law was formulated for intervals from $\mathbb{D}\backslash\mathbb{T}$.

Proposition 1.1. *[3] For $A, B, C, A+B \in \mathbb{D}\backslash\mathbb{T}$ we have*

$$(A+B)\times C_{\sigma(A+B)} \quad = \quad A\times C_{\sigma(A)} + B\times C_{\sigma(B)}.$$

Corollary 1.2. *[10] In the special case when $C = c\in\mathbb{R}$ is degenerate*

$$c\times(A+B) = c\times A + c\times B.$$

In the special case when the intervals $A = a\in\mathbb{R}$ and $B = b\in\mathbb{R}$ are degenerate, we have

$$(a+b)\times C_{\sigma(a+b)} = a\times C_{\sigma(a)} + b\times C_{\sigma(b)}.$$

A generalization of Proposition 1.1 for an arbitrary number of additive terms and generalized intervals involving zero was first done in [15] by studying the conditionally distributive relations

$$\left(\sum_{i=1}^{n} A_i\right)\times C \quad = \quad \sum_{i=1}^{n}\left(A_i\times C_{\widetilde{\mu}(A_i)\widetilde{\mu}(S)}\right) \tag{13}$$

and

$$\left(\sum_{i=1}^{n} A_i\right)\times C_{\widetilde{\mu}(S)} \quad = \quad \sum_{i=1}^{n}\left(A_i\times C_{\widetilde{\mu}(A_i)}\right) \tag{14}$$

wherein $S=\sum_{i=1}^{n} A_i$ and

$$\widetilde{\mu}(A) = \begin{cases} \sigma(A), & \text{if } A\in\mathbb{D}\backslash\mathbb{T};\\ \tau(A), & \text{if } A\in\mathbb{T}.\end{cases}$$

A number of conditions, under which these relations hold true, have been found. It was proven in [15] that equalities (13) and (14) are two equivalent forms of a conditionally distributive relation but the conditions under which each relation holds true differ slightly.

Latter on, in [18] the following conditionally distributive relations

$$\left(\sum_{i=1}^{n} A_i\right)\times C \quad = \quad \sum_{i=1}^{n}\left(A_i\times C_{\widehat{\mu}(A_i)\widehat{\mu}(S)}\right) \tag{15}$$

and

$$\left(\sum_{i=1}^{n} A_i\right)\times C_{\widehat{\mu}(S)} \quad = \quad \sum_{i=1}^{n}\left(A_i\times C_{\widehat{\mu}(A_i)}\right) \tag{16}$$

were studied wherein $S=\sum_{i=1}^{n} A_i$ and

$$\widehat{\mu}(A) = \begin{cases} \sigma(A), & \text{if } A\in\mathbb{D}\backslash\mathbb{T};\\ \nu(A)\tau(A), & \text{if } A\in\mathbb{T}.\end{cases}$$

It was shown by examples that relation (15) is not more general than the relation (13), that is there exist some cases for which relation (15) holds true but the relation (13) does not and vice verse: there exist cases for which relation (13) holds true but the relation (15) does not. In the next section we shall give the theoretical background for this phenomenon and shall describe precisely in what cases which relation holds true.

2 Disclosing Brackets in Multiplication of Interval Sums

Next Theorem shows how, in the general case, to disclose brackets multiplying a sum of generalized intervals. This Theorem covers all possible cases for the participating intervals.

Theorem 2.1. *For* $A_i, C \in \mathbb{D}$ *and* $S = \sum_{i=1}^{n} A_i$

$$\left(\sum_{i=1}^{n} A_i\right) \times C = \begin{cases} \sum\limits_{A_i \in \mathbb{D}\setminus\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum\limits_{A_i \in \mathbb{T}} [a_i^{-\sigma(C)} c^{-\sigma(S)},\ a_i^{\sigma(C)} c^{\sigma(S)}] & \text{if } S \in \mathbb{D}\setminus\mathbb{T},\ C \in \mathbb{D}\setminus\mathbb{T}; \\ \sum\limits_{A_i \in \mathbb{D}\setminus\mathbb{T}} \left(A_i \times c^{\sigma(C)\tau(S)}\right) + \sum\limits_{A_i \in \mathbb{T}} \left(A_i \times C_{\tau(A_i)\tau(S)}\right) & \text{if } S \in \mathbb{T},\ C \in \mathbb{D}\setminus\mathbb{T}; \\ \sum\limits_{A_i \in \mathbb{D}\setminus\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum\limits_{A_i \in \mathbb{T}} \left(a_i^{\sigma(S)\tau(C)} \times C_{\tau(C)\tau(A_i)}\right) & \text{if } S \in \mathbb{D}\setminus\mathbb{T},\ C \in \mathbb{T}; \\ 0 & \text{if } S, C \in \mathbb{T},\ \tau(C) \neq \tau(S); \\ \sum\limits_{i=1}^{n} \left(a_i^{\nu(S)} \times C_{\sigma(a_i^{\nu(S)})\nu(S)\tau(S)}\right) & \text{if } S, C \in \mathbb{T},\ \tau(C) = \tau(S),\ \chi(C) \leq \chi(S); \\ \sum\limits_{i=1}^{n} \left(A_i \times c^{\nu(C)}\right) & \text{if } S, C \in \mathbb{T},\ \tau(C) = \tau(S),\ \chi(C) \geq \chi(S). \end{cases}$$

Proof.

For $S, C \in \mathbb{D}\setminus\mathbb{T}$

$$\begin{aligned} \left(\sum_{i=1}^{n} A_i\right) \times C &\overset{(8.1)}{=} \left(\sum_{i=1}^{n} A_i\right)_{\sigma(C)} \times_h C_{\sigma(S)} \overset{(5),(7)}{=} \sum_{i=1}^{n} \left(A_{i\sigma(C)} \times_h C_{\sigma(S)}\right) \\ &= \sum_{\substack{A_i \in \mathbb{D}\setminus\mathbb{T} \\ \sigma(A_i)=\sigma(S)}} \left(A_{i\sigma(C)} \times_h C_{\sigma(S)}\right) + \sum_{\substack{A_i \in \mathbb{D}\setminus\mathbb{T} \\ \sigma(A_i)\neq\sigma(S)}} \left(A_{i\sigma(C)} \times_h C_{\sigma(S)}\right) \\ &\quad + \sum_{A_i \in \mathbb{T}} \left(A_{i\sigma(C)} \times_h C_{\sigma(S)}\right) \\ &= \sum_{\substack{A_i \in \mathbb{D}\setminus\mathbb{T} \\ \sigma(A_i)=\sigma(S)}} (A_i \times C) + \sum_{\substack{A_i \in \mathbb{D}\setminus\mathbb{T} \\ \sigma(A_i)\neq\sigma(S)}} (A_i \times C_-) + \sum_{A_i \in \mathbb{T}} \left(A_{i\sigma(C)} \times_h C_{\sigma(S)}\right) \\ &= \sum_{A_i \in \mathbb{D}\setminus\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{A_i \in \mathbb{T}} [a_i^{-\sigma(C)} c^{-\sigma(S)},\ a_i^{\sigma(C)} c^{\sigma(S)}]. \end{aligned}$$

For $S \in \mathbb{D}\backslash\mathbb{T}$ and $C \in \mathbb{T}$

$$
\begin{aligned}
\left(\sum_{i=1}^{n} A_i\right) \times C \quad &\overset{(8.2)}{=} \left(\sum_{i=1}^{n} A_i\right)^{\sigma(S)\tau(C)} \times_h C_{\sigma(S)} \overset{(5),\,(7)}{=} \sum_{i=1}^{n} \left(A_i^{\sigma(S)\tau(C)} \times_h C_{\sigma(S)}\right) \\
&= \sum_{\substack{A_i \in \mathbb{D}\backslash\mathbb{T} \\ \sigma(A_i)=\sigma(S)}} \left(A_i^{\sigma(A_i)\tau(C)} \times_h C_{\sigma(A_i)}\right) + \sum_{\substack{A_i \in \mathbb{D}\backslash\mathbb{T} \\ \sigma(A_i)\neq\sigma(S)}} \left(A_i^{\sigma(A_i)\tau(C_-)} \times_h C_{-\sigma(A_i)}\right) \\
&\quad + \sum_{A_i \in \mathbb{T}} \left(A_i^{\sigma(S)\tau(C)} \times_h C_{\sigma(S)}\right) \\
&\overset{(8.2),(9)}{=} \sum_{A_i \in \mathbb{D}\backslash\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{A_i \in \mathbb{T}} \left(a_i^{\sigma(S)\tau(C)} \times C_{\tau(C)\tau(A_i)}\right).
\end{aligned}
$$

For $S \in \mathbb{T}$ and $C \in \mathbb{D}\backslash\mathbb{T}$

$$
\begin{aligned}
\left(\sum_{i=1}^{n} A_i\right) \times C \quad &\overset{(8.3)}{=} \left(\sum_{i=1}^{n} A_i\right)_{\sigma(C)} \times_h C^{\sigma(C)\tau(S)} \\
&\overset{(5),\,(7)}{=} \sum_{i=1}^{n} \left((A_i)_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)}\right) \\
&= \sum_{A_i \in \mathbb{D}\backslash\mathbb{T}} \left((A_i)_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)}\right) + \sum_{A_i \in \mathbb{T}} \left(A_i \times C_{\tau(A_i)\tau(S)}\right) \\
&\overset{(9)}{=} \sum_{A_i \in \mathbb{D}\backslash\mathbb{T}} \left(A_i \times c^{\sigma(C)\tau(S)}\right) + \sum_{A_i \in \mathbb{T}} \left(A_i \times C_{\tau(A_i)\tau(S)}\right).
\end{aligned}
$$

It follows from (8.6) that

$$
\left(\sum_{i=1}^{n} A_i\right) \times C = 0 \qquad \text{for } \left(\sum_{i=1}^{n} A_i\right), C \in \mathbb{T};\ \tau(C) \neq \tau(S).
$$

For $S, C \in \mathbb{T},\ \tau(C) = \tau(S) = \tau,\ \chi(C) \leq \chi(S)$

$$
\begin{aligned}
\left(\sum_{i=1}^{n} A_i\right) \times C \quad &\overset{(8.4)}{=} \left(\sum_{i=1}^{n} A_i\right)^{\nu(S)} \times_h C_{\nu(S)\tau} \overset{(5),\,(7)}{=} \sum_{i=1}^{n} \left(a_i^{\nu(S)} \times_h C_{\nu(S)\tau}\right) \\
&\overset{(9)}{=} \sum_{i=1}^{n} \left(a_i^{\nu(S)} \times C_{\sigma(a_i^{\nu(S)})\nu(S)\tau}\right).
\end{aligned}
$$

For $S, C \in \mathbb{T},\ \tau(C) = \tau(S) = \tau,\ \chi(C) \geq \chi(S)$

$$
\begin{aligned}
\left(\sum_{i=1}^{n} A_i\right) \times C \quad &\overset{(8.5)}{=} \left(\sum_{i=1}^{n} A_i\right)_{\nu(C)\tau} \times_h c^{\nu(C)} \overset{(5),\,(7)}{=} \sum_{i=1}^{n} \left((A_i)_{\nu(C)\tau} \times_h c^{\nu(C)}\right) \\
&\overset{(9)}{=} \sum_{i=1}^{n} \left(A_i \times c^{\nu(C)}\right).
\end{aligned}
$$

□

Theorem 2.1 shows that disclosing brackets when multiplying a sum of generalized intervals is always possible but some particular operand end-points may participate in the resulting expression. That is why, except for some very special cases for the participating intervals – e.g. when all intervals do not involve zero, Theorem 2.1 cannot be used as a generalization of the distributive relations on multiplication and addition of generalized intervals. Theorem 2.1 will be further precised in Section 4.

3 Multiplication Distributivity for Two Additive Terms

3.1 Auxiliary results

Proposition 3.1. *For a generalized interval* $A \in \mathbb{D}$ *and* $\lambda \in \Lambda$

1. $\sigma(A_\lambda) = \sigma(A),\ A \in \mathbb{D}\backslash\mathbb{T}$;	2. $\tau(A_\lambda) = \lambda\tau(A)$;	3. $\chi(A_\lambda) = \chi(A)$;
4. $\nu(A_\lambda) = \lambda\nu(A)$;	5. $\sigma(a^{\nu(A)}) = \nu(A)\tau(A)$;	6. $(C_\lambda)^{\nu(C_\lambda)} = c^{\nu(C)}$.

Proof. It is straightforward. □

For $B \in \mathbb{T}$, the following properties are equivalent:

$$\chi(B) = 0 \quad \textit{and} \quad b^{-\nu(B)} = 0;$$
$$B = 0 \quad \textit{and} \quad b^{\nu(B)} = 0.$$

Proposition 3.2. *For* $A, B \in \mathbb{T}$*, such that* $\tau(A) = \tau(B),\ \chi(A) = \chi(B) = -1$ *and for* $\lambda \in \Lambda$

$$A \times B \;=\; a^{\lambda\tau(A)} \times_h B_\lambda \;=\; A_\lambda \times_h b^{\lambda\tau(A)}.$$

Proof. Follows from (3), if we assume $\nu(A) = \nu(B) \in \{+, -\}$ and denote $\lambda = \nu(A)\tau(A)$. □

Lemma 3.3. *For* $U \in \mathbb{T},\ V \in \mathbb{D}\backslash\mathbb{T}$ *and* $\lambda \in \Lambda$

$$U_{\sigma(V)} \times_h v^{\nu(U)\sigma(V)\lambda} \;=\; U_{\sigma(V)} \times_h V_\lambda \tag{17}$$

iff either $V = v \in \mathbb{R}$*, or* $U = 0$*, or* $u^{-\nu(U)} = 0$*.*

Proof. It is obvious that equality (17) holds true iff either $V = v \in \mathbb{R}$, or $U = 0$.

Let $V \notin \mathbb{R},\ U \neq 0$ and equality (17) holds true.

For $\nu(U) = \sigma(V)$ the equality is equivalent to

$$\left| \begin{array}{lcl} u^{-\nu(U)}v^{\lambda} &=& u^{-\nu(U)}v^{-\lambda} \\ u^{\nu(U)}v^{\lambda} &=& u^{\nu(U)}v^{\lambda} \end{array} \right. \quad \Longrightarrow \; u^{-\nu(U)} = 0.$$

For $\nu(U) \neq \sigma(V)$ the equality is equivalent to

$$\left| \begin{array}{lcl} u^{\nu(U)}v^{-\lambda} &=& u^{\nu(U)}v^{-\lambda} \\ u^{-\nu(U)}v^{-\lambda} &=& u^{-\nu(U)}v^{\lambda} \end{array} \right. \quad \Longrightarrow \; u^{-\nu(U)} = 0.$$

Let $V \notin \mathbb{R},\ U \neq 0$ and $u^{-\nu(U)} = 0$. For $A \in \mathbb{T} \setminus \{0\}$, such that $a^{-\nu(A)} = 0$, we have the representation $A = [0, a]_{\nu(A)}$, wherein $a \in \mathbb{R} \setminus \{0\}$.

If $\nu(U) = +$

$$U_{\sigma(V)} \times_h V_\lambda = [0, u]_{\sigma(V)} \times_h [v^{-\lambda}, v^\lambda] = [0, uv^{\sigma(V)\lambda}]_{\sigma(V)}$$
$$= [0, u]_{\sigma(V)} \times_h v^{\sigma(V)\lambda} = U_{\sigma(V)} \times_h v^{\nu(U)\sigma(V)\lambda}.$$

If $\nu(U) = -$

$$U_{\sigma(V)} \times_h V_\lambda = [u, 0]_{\sigma(V)} \times_h [v^{-\lambda}, v^\lambda] = [uv^{-\sigma(V)\lambda}, 0]_{\sigma(V)}$$
$$= [u, 0]_{\sigma(V)} \times_h v^{-\sigma(V)\lambda} = U_{\sigma(V)} \times_h v^{\nu(U)\sigma(V)\lambda}.$$

□

Lemma 3.4. *For $U, V \in \mathbb{T} \setminus \{0\}$, such that $\tau(V) = \tau$, the equality*

$$U_{\nu(V)\tau} \times_h v^{\nu(V)} = u^{\nu(U)} \times_h V_{\nu(U)\tau} \tag{18}$$

holds true iff $\chi(U) = \chi(V)$.

Proof. Let equality (18) holds true. For $\nu(U)\tau = \nu(V)$ the equality (18) is equivalent to

$$\left| \begin{array}{lcl} u^{-\nu(U)}v^{\nu(V)} & = & u^{\nu(U)}v^{-\nu(V)} \\ u^{\nu(U)}v^{\nu(V)} & = & u^{\nu(U)}v^{\nu(V)} \end{array} \right. \quad \Longrightarrow \chi(U) = \chi(V)$$

For $\nu(U)\tau \neq \nu(V)$ the equality (18) is equivalent to

$$\left| \begin{array}{lcl} u^{\nu(U)}v^{\nu(V)} & = & u^{\nu(U)}v^{\nu(V)} \\ u^{-\nu(U)}v^{\nu(V)} & = & u^{\nu(U)}v^{-\nu(V)} \end{array} \right. \quad \Longrightarrow \chi(U) = \chi(V).$$

Let $\tau(U) = \tau(V)$ and $\chi(U) = \chi(V)$. The equality (18) follows from the equality of the representations (8.4) and (8.5) for $U \times V$. If $\tau(U) \neq \tau(V)$ and $\chi(U) = \chi(V)$, the equality (18) follows from the equality of the representations (8.4) and (8.5) for $U \times V_-$. □

Proposition 3.5. *For $A, B \in \mathbb{T} \setminus \{0\}$, $A + B = S \in \mathbb{D} \backslash \mathbb{T}$ and $\chi(A) = \chi(B) = 0$, we have*

$$\nu(A) \neq \nu(B).$$

Proof. $\chi(A) = 0 \iff a^{-\nu(A)} = 0$ and $\chi(B) = 0 \iff b^{-\nu(B)} = 0$.
Assume that $\nu(A) = \nu(B) = \nu$. Then $s^{-\nu} = a^{-\nu(A)} + b^{-\nu(B)} = 0$, which contradicts the initial assumption that $S \in \mathbb{D} \backslash \mathbb{T}$. □

Proposition 3.6. *The following two requirements are equivalent.*

- *$A, B \in \mathbb{T} \setminus \{0\}$, $\tau(A) \neq \tau(B)$, $\nu(A) \neq \nu(B)$, $\chi(A) = \chi(B) = 0$;*
- *$A, B \in \mathbb{T} \setminus \{0\}$, $A + B = S \in \mathbb{D} \backslash \mathbb{T}$, $\chi(A) = \chi(B) = 0$.*

The proof is trivial.

Lemma 3.7. *For $U, V \in \mathbb{T}$, such that $\tau(U) = \tau(V) = \tau$,*

$$U_{\nu(V)\tau} \times_h v^{\nu(V)} = u^{-\nu(U)} \times_h V_{-\nu(U)\tau} \qquad \textit{iff } \chi(U) = \chi(V) = -1. \tag{19}$$

Proof. Let the equality (19) holds true.

For $\nu(U)\tau = \nu(V)$ the equality (19) is equivalent to

$$\left| \begin{array}{lcl} u^{-\nu(U)}v^{\nu(V)} & = & u^{-\nu(U)}v^{\nu(V)} \\ u^{\nu(U)}v^{\nu(V)} & = & u^{-\nu(U)}v^{-\nu(V)} \end{array} \right. \qquad \Longrightarrow \chi(U) = \chi(V) = -1.$$

For $\nu(U)\tau \neq \nu(V)$ the equality (19) is equivalent to

$$\left| \begin{array}{lcl} u^{\nu(U)}v^{\nu(V)} & = & u^{-\nu(U)}v^{-\nu(V)} \\ u^{-\nu(U)}v^{\nu(V)} & = & u^{-\nu(U)}v^{\nu(V)}. \end{array} \right. \qquad \Longrightarrow \chi(U) = \chi(V) = -1$$

Let $\chi(U) = \chi(V) = -1$. The equality (19) follows from the equality of the representations (8.4) and (8.5), and Proposition 3.2. □

Proposition 3.8. *For $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T} \setminus \{0\}$ and $A + B = S \in \mathbb{T}$, such that $\nu(B) \neq \nu(S)$ and $(\chi(S) = 0 \vee \chi(B) = 0)$*

$$\sigma(A) = \nu(S)\tau(S).$$

Proof.

$$\chi(S) = 0 \Longrightarrow s^{-\nu(S)} = 0 \Longrightarrow a^{-\nu(S)} = -b^{-\nu(S)} \Longrightarrow a^{\nu(B)} = -b^{\nu(B)}$$
$$a^{\nu(B)} = -b^{\nu(B)} \Longrightarrow \nu(B)\tau(B) = \sigma(b^{\nu(B)}) = -\sigma(a^{\nu(B)}) = -\sigma(A)$$
$$\sigma(A) = -\nu(B)\tau(B) = -\nu(B)\tau(S) = \nu(S)\tau(S).$$

$$\chi(B) = 0 \Longrightarrow b^{-\nu(B)} = 0 \Longrightarrow b^{\nu(S)} = 0 \Longrightarrow s^{\nu(S)} = a^{\nu(S)}$$
$$s^{\nu(S)} = a^{\nu(S)},\ s^{-\nu(S)} = a^{-\nu(S)} + b^{-\nu(S)} \Longrightarrow \sigma(a^{-\nu(S)}) \neq \sigma(b^{-\nu(S)}) = \sigma(b^{\nu(B)}) = \nu(B)\tau(B)$$
$$\sigma(A) = \sigma(a^{-\nu(S)}) = -\nu(B)\tau(B) = -\nu(B)\tau(S) = \nu(S)\tau(S).$$

□

Proposition 3.9. *$A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T} \setminus \{0\}$, $A + B = S \in \mathbb{T}$ and $\chi(B) = 0 = \chi(S)$ imply*

$$\nu(S) \neq \nu(B).$$

Proof.

$$\chi(S) = 0 \iff a^{-\nu(S)} = -b^{-\nu(S)} \qquad \text{and} \qquad \chi(B) = 0 \iff b^{-\nu(B)} = 0$$

Assume that $\nu(B) = \nu(S)$. Then

$$a^{-\nu(S)} = -b^{-\nu(S)} = -b^{-\nu(B)} = 0$$

contradicts to $A \in \mathbb{D}\backslash\mathbb{T}$. □

3.2 Proof of the main result

In this section we prove the multiplication conditionally distributive relations involving two additive terms. To find the type of the relation and under what conditions it is valid, we study the following two distributive equalities:

$$(A+B)\times C = A\times C_{\widehat{\mu}(A)\widehat{\mu}(S)} + B\times C_{\widehat{\mu}(B)\widehat{\mu}(S)} \tag{20}$$

and

$$(A+B)\times C = A\times C_{\widetilde{\mu}(A)\widetilde{\mu}(S)} + B\times C_{\widetilde{\mu}(B)\widetilde{\mu}(S)}, \tag{21}$$

wherein

$$\widehat{\mu}(I) = \begin{cases} \sigma(I), & \text{if } I\in \mathbb{D}\backslash\mathbb{T}; \\ \nu(I)\tau(I), & \text{if } I\in\mathbb{T} \end{cases} \quad \text{and} \quad \widetilde{\mu}(I) = \begin{cases} \sigma(I), & \text{if } I\in \mathbb{D}\backslash\mathbb{T}; \\ \tau(I), & \text{if } I\in\mathbb{T}. \end{cases}$$

We shall consider 18 cases for the interval entities $A, B\in\mathbb{D},\ A+B=S\in\mathbb{D}$ and $C\in\mathbb{D}$.

It will be evident from the poof below, or can be easily proved by the same projecting technique, that for $C=c\in\mathbb{R}$

$$\left(\sum_{i=1}^{n} A_i\right)\times c \;=\; \sum_{i=1}^{n}(A_i\times c)\,.$$

Therefore, all considerations below will be for $C\in\mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R})$, whenever we have a nonzero common multiplier.

- Consider $A, B\in\mathbb{D}\backslash\mathbb{T}$ and $A+B=S\in\mathbb{D}\backslash\mathbb{T}$.

Because in this case $\widetilde{\mu}(\cdot)=\widehat{\mu}(\cdot)=\sigma(\cdot)$, we have to study only one distributive relation. The same way, as in the proof ot Theorem 2.1, we obtain:

For $A, B\in\mathbb{D}\backslash\mathbb{T},\ A+B=S\in\mathbb{D}\backslash\mathbb{T}$ (22)

$$(A+B)\times C \;=\; A\times C_{\sigma(A)\sigma(S)} + B\times C_{\sigma(B)\sigma(S)}.$$

- Consider $A, B\in\mathbb{D}\backslash\mathbb{T}$ and $A+B=S\in\mathbb{T},\ C\in\mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R})$.

Applying (8.1), Proposition 3.1, (8.3), (7) and (5), we obtain that equality (20) is equivalent to the equalities

$$(A+B)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)}\times_h C_{\nu(S)\tau(S)} + B_{\sigma(C)}\times_h C_{\nu(S)\tau(S)}$$
$$(A+B)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)} = (A+B)_{\sigma(C)}\times_h C_{\nu(S)\tau(S)}.$$

Last equality holds true iff either $S=0$, or $s^{-\nu(S)}=0$ (according to Lemma 3.3).
It is obvious from the above proof that relation (21) holds true iff either $S=0$, or $\nu(S)=+$, $s^{-\nu(S)}=0$. Therefore the relation (21) is a special case of the relation (20) for $\nu(S)=+$.

Note 1. *The relation (20) holds true for* $A, B \in \mathbb{D}\backslash\mathbb{T}$ *and* $S = 0$.
From $A, B \in \mathbb{D}\backslash\mathbb{T}$ *and* $S = 0$ *it follows that* $\sigma(A) \neq \sigma(B)$ *(that is* $\sigma(B) = -\sigma(A)$*). But*

$$S = 0 \implies \left| \begin{array}{l} \chi(S) = -1 \\ \mathcal{T}(S) = \{+,-\} \end{array} \right. \implies \left| \begin{array}{l} \mathcal{N}(S) = \{+,-\} \\ \mathcal{T}(S) = \{+,-\} \end{array} \right.$$

imply two distributive representations since we can choose an arbitrary $\nu(S) \in \mathcal{N}(S)$ *and an arbitrary* $\tau(S) \in \mathcal{T}(S)$.

$$(A+B) \times C \ = \ 0 \ = \ A \times C_{\sigma(A)} + B \times C_{-\sigma(A)} \ = \ A \times C_{-\sigma(A)} + B \times C_{\sigma(A)}.$$

Example 1.

$$\begin{array}{rcl} ([3,2]+[-3,-2]) \times [4,7] & = 0 \times [4,7] & = \ 0 \\ {[3,2]} \times [4,7]_- + [-3,-2] \times [4,7] & = & [21,8]+[-21,8] \ = 0 \\ {[3,2]} \times [4,7] + [-3,-2] \times [4,7]_- & = & [12,14]+[-12,-14] \ = 0 \end{array}$$

Summarizing, we have:

For $A, B \in \mathbb{D}\backslash\mathbb{T}$, $A+B = S \in \mathbb{T}$, $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$ and $\lambda \in \Lambda$ (23)

$$(A+B) \times C = \begin{cases} A \times C_\lambda + B \times C_{-\lambda}, & \text{iff } S = 0; \\ A \times C_{\sigma(A)\nu(S)\tau(S)} + B \times C_{\sigma(B)\nu(S)\tau(S)}, & \text{iff } \chi(S) = 0. \end{cases}$$

• Consider $A, B \in \mathbb{D}\backslash\mathbb{T}$ and $A+B = S \in \mathbb{T}$, $C \in \mathbb{T} \setminus 0$, such that $\tau(S) \neq \tau(C)$.

Applying (8.6), (8.3), Proposition 3.1, (7) and (5), we obtain that the equality (20) is equivalent to the equality

$$0 = (a+b)^{-\nu(S)} \times_h C_{\nu(S)\tau(S)},$$

which holds true iff $S = 0$, or $s^{-\nu(S)} = 0$.

It is obvious from the above proof (or can be easily verified) that relation (21) holds true iff $S = 0$, or $\nu(S) = +$, $s^{-\nu(S)} = 0$. Thus, the relation (21) is a special case of the relation (20). Note 1 also takes place in the case we consider now.

Summarizing, we have:

For $A, B \in \mathbb{D}\backslash\mathbb{T}$, $A+B = S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, $\tau(C) \neq \tau(S)$ and $\lambda \in \Lambda$ (24)

$$(A+B) \times C = \begin{cases} A \times C_\lambda + B \times C_{-\lambda}, & \text{iff } S = 0; \\ A \times C_{\sigma(A)\nu(S)\tau(S)} + B \times C_{\sigma(B)\nu(S)\tau(S)}, & \text{iff } \chi(S) = 0. \end{cases}$$

• Consider $A, B \in \mathbb{D}\backslash\mathbb{T}$ and $A+B = S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$ such that $\tau(S) = \tau(C)$.

For $\chi(C) \leq \chi(S)$, applying (8.5), (8.2), Proposition 3.1, (7) and (5), we obtain that equality (20) is equivalent to the equalities

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = a^{\sigma(A)\tau(C)\sigma(A)\nu(S)\tau(S)} \times_h C_{\sigma(A)\sigma(A)\nu(S)\tau(S)} + b^{\nu(S)\tau(C)\tau(S)} \times_h C_{\nu(S)\tau(S)}$$
$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = (a+b)^{\nu(S)\tau(C)\tau(S)} \times_h C_{\nu(S)\tau(S)}.$$

The latter holds true under the conditions it was derived.
For $\chi(C) \geq \chi(S)$, equality (20) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = (a+b)^{\nu(S)\tau(C)\tau(S)} \times_h C_{\nu(S)\tau(S)},$$

which holds true iff either $(S = 0 \ \wedge\ \chi(C) = -1)$, or $\chi(C) = \chi(S)$ (according to Lemma 3.4). That is, the case $\chi(C) \geq \chi(S)$ is a special case of $\chi(C) \leq \chi(S)$.

For $\chi(C) \leq \chi(S)$, equality (21) is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} \ = \ (a^+ + b^+) \times_h C_{\tau(S)},$$

which obviously holds true iff either $(S = 0 \ \wedge\ \chi(C) = -1)$, or $\nu(S) = +$. According to Proposition 3.2, the equality holds true also iff $(\nu(S) = - \ \wedge\ \chi(C) = \chi(S) = -1)$.
For $\chi(C) \geq \chi(S)$, the equality (21) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ (a^+ + b^+) \times_h C_{\tau(S)},$$

which holds true iff either $(S = 0 \ \wedge\ \chi(C) = -1)$, or $(\nu(S) = + \ \wedge\ \chi(C) = \chi(S))$ (according to Lemma 3.4) and iff $(\nu(S) = - \ \wedge\ \chi(C) = \chi(S) = -1)$ (according to Proposition 3.2 and Lemma 3.4).
Note 1 takes place in the general case we consider now.

Note 2. *When $\chi(S) = -1$ we also have two equal distributive representations:*

$$(A+B) \times C \ = \ A \times C_{\sigma(A)\tau(S)} + B \times C_{-\sigma(A)\tau(S)} \ = \ A \times C_{-\sigma(A)\tau(S)} + B \times C_{\sigma(A)\tau(S)}$$

because $\mathcal{N}(S) = \{+, -\}$.

Example 2.

$$([3,2] + [-1,-4]) \times [7,-7] \ = \ [14,-14]$$
$$[3,2] \times [7,-7]_- + [-1,-4] \times [7,-7] \ = \ [-14,14] + [28,-28] \ = \ [14,-14]$$
$$[3,2] \times [7,-7] + [-1,-4] \times [7,-7]_- \ = \ [21,-21] + [-7,7] \ = \ [14,-14]$$

Summarizing, we get:

For $A, B \in \mathbb{D}\backslash\mathbb{T}$, $A + B = S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, $\tau(C) = \tau(S)$ and $\lambda \in \Lambda$ (25)

$$(A+B) \times C = \begin{cases} A \times C_\lambda + B \times C_{-\lambda}, & \text{iff } \chi(C) = \chi(S) = -1; \\ A \times C_{\sigma(A)\nu(S)\tau(S)} + B \times C_{\sigma(B)\nu(S)\tau(S)}, & \text{iff } \chi(C) \leq \chi(S) \neq 0. \end{cases}$$

- Consider $A, B \in \mathbb{T} \setminus \{0\}$ and $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$.

For $S \in \mathbb{D}\backslash\mathbb{T}$, the equality (21) is equivalent (due to (8.1), (8.3), (7), (5)) to the equalities

$$\begin{aligned} S_{\sigma(C)} \times_h C_{\sigma(S)} &= A_{\sigma(C)} \times_h c^{\tau(A)\sigma(C)\tau(A)\sigma(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)} \\ S_{\sigma(C)} \times_h C_{\sigma(S)} &= (A+B)_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)}, \end{aligned}$$

which do not hold true for $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$.

For $S \in \mathbb{D}\backslash\mathbb{T}$, the equality (20) is equivalent to the equality

$$S_{\sigma(C)} \times_h C_{\sigma(S)} = A_{\sigma(C)} \times_h c^{\nu(A)\sigma(C)\sigma(S)} + B_{\sigma(C)} \times_h c^{\nu(B)\sigma(C)\sigma(S)}.$$

For $\nu(A) = \nu(B)$, the equality (20) holds true iff $C = c \in \mathbb{R}$.
For $\nu(A) \neq \nu(B)$, the last equality is equivalent to

$$\left| \begin{array}{lcl} a^{-\sigma(C)} c^{\nu(A)\sigma(C)\sigma(S)} + b^{-\sigma(C)} c^{-\nu(A)\sigma(C)\sigma(S)} &=& a^{-\sigma(C)} c^{-\sigma(S)} + b^{-\sigma(C)} c^{-\sigma(S)} \\ a^{\sigma(C)} c^{\nu(A)\sigma(C)\sigma(S)} + b^{\sigma(C)} c^{-\nu(A)\sigma(C)\sigma(S)} &=& a^{\sigma(C)} c^{\sigma(S)} + b^{\sigma(C)} c^{\sigma(S)} \end{array} \right.$$

For $\nu(A) = \sigma(C)$ or $\nu(A) \neq \sigma(C)$, these equalities hold true iff either $C = c \in \mathbb{R}$, or $a^{-\nu(A)} = b^{-\nu(B)} = 0$. That is why, for $S \in \mathbb{D}\backslash\mathbb{T}$ and $\nu(A) \neq \nu(B)$ the equality (20) is more general than the equality (21).

For $S \in \mathbb{T}$, the equality (21) is equivalent (due to (8.3), (7), (5)) to the equality

$$S_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = (A+B)_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)},$$

which obviously holds true.

For $S \in \mathbb{T}$ equality (20) is equivalent to the equality

$$S_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)} \times_h c^{\nu(A)\sigma(C)\nu(S)\tau(S)} + B_{\sigma(C)} \times_h c^{\nu(B)\sigma(C)\nu(S)\tau(S)}.$$

For $\nu(A) = \nu(B) = \nu(S)$, the equality (20) is equivalent to the equality (21) and holds true.

Note 3. *Although $\mathcal{N}(S) = \{+, -\}$ for $\chi(S) = -1$, in the situation we consider now, there is only one distributive relation*

$$(A+B) \times C \;=\; A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)}$$

due to the additional requirements $\nu(S) = \nu(A) = \nu(B)$ which fix the value of $\nu(S)$.

For $\nu(A) = \nu(B) \neq \nu(S)$, the equality (20) is equivalent to the equality

$$S_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = (A+B)_{\sigma(C)} \times_h c^{-\sigma(C)\tau(S)},$$

which holds true iff $S = 0$.

Note 4. *For $S = 0$, due to $\nu(A) = \nu(B) \neq \nu(S)$, the value of $\nu(S)$ is fixed, but $\mathcal{T}(S) = \{+, -\}$ and we have two distributive representations:*

$$(A+B) \times C \;=\; A \times C_{-\tau(A)} + B \times C_{-\tau(B)} \;=\; A \times C_{\tau(A)} + B \times C_{\tau(B)}.$$

For $\nu(S) = \nu(A) \neq \nu(B)$, equality (20) is equivalent to the equality

$$B_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = B_{\sigma(C)} \times_h c^{-\sigma(C)\tau(S)},$$

which does not hold true for $B \neq 0$ and $C \notin \mathbb{R}$.
Analogously, for $\nu(A) \neq \nu(B) = \nu(S)$ and $C \notin \mathbb{R}$, the equality (20) does not hold true.

Thus, for $S \in \mathbb{T}$ the equality (21) is more general than the equality (20).

Applying Propositions 3.5 and 3.6, we generalize the whole case:

$$\text{For } A, B \in \mathbb{T}\setminus\{0\}, \quad C \in \mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R}) \text{ and } \lambda\in\Lambda \tag{26}$$

$$(A+B)\times C = \begin{cases} A\times C_{\nu(A)\tau(A)\sigma(S)} + B\times C_{\nu(B)\tau(B)\sigma(S)}, & \text{iff } S\in\mathbb{D}\setminus\mathbb{T},\ \chi(A)=\chi(B)=0; \\ A\times C_{\tau(A)\tau(S)} + B\times C_{\tau(B)\tau(S)}, & \text{iff } S\in\mathbb{T}\setminus\{0\}; \\ A\times C_{\lambda} + B\times C_{-\lambda}, & \text{iff } S=0. \end{cases}$$

• Consider $A, B \in \mathbb{T}\setminus\{0\}$, $A+B=S\in\mathbb{D}\setminus\mathbb{T}$ and $C\in\mathbb{T}\setminus\{0\}$.

Equality (20) is equivalent to the equality

$$s^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} = \begin{cases} 0, & \text{if } \nu(A)=\nu(B)\neq\tau(C)\sigma(S); \\ a^{\nu(A)}\times_h C_{\sigma(S)} + b^{\nu(B)}\times_h C_{\sigma(S)}, & \text{if } \nu(A)=\nu(B)=\tau(C)\sigma(S), \\ & \quad \chi(C)\le\min\{\chi(A),\chi(B)\}; \\ A_{\nu(C)\tau(C)}\times_h c^{\nu(C)} + b^{\nu(B)}\times_h C_{\sigma(S)}, & \text{if } \nu(A)=\nu(B)=\tau(C)\sigma(S), \\ & \quad \chi(A)\le\chi(C)\le\chi(B); \\ a^{\nu(A)}\times_h C_{\sigma(S)} + B_{\nu(C)\tau(C)}\times_h c^{\nu(C)}, & \text{if } \nu(A)=\nu(B)=\tau(C)\sigma(S), \\ & \quad \chi(B)\le\chi(C)\le\chi(A); \\ A_{\nu(C)\tau(C)}\times_h c^{\nu(C)} + B_{\nu(C)\tau(C)}\times_h c^{\nu(C)}, & \text{if } \nu(A)=\nu(B)=\tau(C)\sigma(S), \\ & \quad \max\{\chi(A),\chi(B)\}\le\chi(C); \\ b^{\nu(B)}\times_h C_{\sigma(S)}, & \text{if } \nu(A)\neq\nu(B)=\tau(C)\sigma(S), \chi(C)\le\chi(B); \\ B_{\nu(C)\tau(C)}\times_h c^{\nu(C)}, & \text{if } \nu(A)\neq\nu(B)=\tau(C)\sigma(S), \chi(C)\ge\chi(B); \\ a^{\nu(A)}\times_h C_{\sigma(S)}, & \text{if } \tau(C)\sigma(S)=\nu(A)\neq\nu(B),\ \chi(C)\le\chi(A); \\ A_{\nu(C)\tau(C)}\times_h c^{\nu(C)}, & \text{if } \tau(C)\sigma(S)=\nu(A)\neq\nu(B),\ \chi(C)\ge\chi(A). \end{cases}$$

The first of the above 9 relations does not hold true for $S\in\mathbb{D}\setminus\mathbb{T}$.
The second of the above 9 relations holds true.
The third of the above 9 relations holds true iff $\tau(C)=\tau(A)$ and $\chi(C)=\chi(A)\le\chi(B)$.
The fourth of the above 9 relations holds true iff $\tau(C)=\tau(B)$ and $\chi(C)=\chi(B)\le\chi(A)$.
(The third and the fourth relations are special cases of the second relation.)
The fifth of the above 9 relations does not hold true.
The sixth of the above 9 relations holds true iff $\chi(A)=0$.
The seventh of the above 9 relations holds true iff $\chi(A)=0$ and $\tau(C)=\tau(B)$, $\chi(C)=\chi(B)$.
(The seventh relation is a special case of the sixth relation.)
The eight of the above 9 relations holds true iff $\chi(B)=0$.
The ninth of the above 9 relations holds true iff $\chi(B)=0$ and $\tau(C)=\tau(A)$, $\chi(C)=\chi(A)$.
(The ninth relation is a special case of the eight relation.)

Equality (21) is equivalent to the equality

$$s^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} =$$
$$\begin{cases} 0, & \text{if } \tau(C) \neq \sigma(S); \\ a^{\nu(A)} \times_h C_{\nu(A)\sigma(S)} + b^{\nu(B)} \times_h C_{\nu(B)\sigma(S)}, & \text{if } \tau(C) = \sigma(S),\ \chi(C) \leq \min\{\chi(A), \chi(B)\}; \\ A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + b^{\nu(B)} \times_h C_{\nu(B)\sigma(S)}, & \text{if } \tau(C) = \sigma(S),\ \chi(A) \leq \chi(C) \leq \chi(B); \\ a^{\nu(A)} \times_h C_{\nu(A)\sigma(S)} + B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \tau(C) = \sigma(S),\ \chi(B) \leq \chi(C) \leq \chi(A); \\ A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \tau(C) = \sigma(S),\ \max\{\chi(A), \chi(B)\} \leq \chi(C). \end{cases}$$

First of the above 5 relations does not hold true for $S \in \mathbb{D} \backslash \mathbb{T}$.
The second relation holds true iff either $\nu(A) = \nu(B) = +,\ \chi(C) \leq \min\{\chi(A), \chi(B)\}$ (this case is a special case of the relation (20) – the second of the above nine); or $\nu(A) \neq \nu(B) = +$, $\chi(C) = \chi(A) = -1$ (this case is also a special case of the relation (20) – the second of the above nine, because the two representations (20) and (21) are identical and due to $\chi(A) = -1$ we can take $\nu(A) = \nu(B) = +$ instead of $\nu(A) \neq \nu(B) = +$); or $+ = \nu(A) \neq \nu(B),\ \chi(C) = \chi(B) = -1$ (due to commutativity with the previous one, this case is also a special case of the relation (20)
The third and the fourth of the above 5 relations hold true iff $\nu(A) = \nu(B) = +$ and $\chi(C) = \min\{\chi(A), \chi(B)\}$ (which is a special case of the relation (20)).
The fifth of the above 5 relations does not hold true for $S \in \mathbb{D} \backslash \mathbb{T}$.
Thus, relation (20) is more general than the relation (21).

Summarizing the whole case:

For $A, B \in \mathbb{T} \setminus \{0\},\ A + B = S \in \mathbb{D} \backslash \mathbb{T}$ and $C \in \mathbb{T} \setminus \{0\}$ (27)

$$(A + B) \times C =$$
$$\begin{array}{rl} A \times C_{\nu(A)\tau(A)\sigma(S)} + B \times C_{\nu(B)\tau(B)\sigma(S)}, & \text{iff } \nu(A) = \nu(B) = \tau(C)\sigma(S), \\ & \quad \chi(C) \leq \min\{\chi(A), \chi(B)\}; \\ B \times C_{\nu(B)\tau(B)\sigma(S)}, & \text{iff } \nu(A) \neq \nu(B) = \tau(C)\sigma(S),\ \chi(A) = 0, \\ & \quad \chi(C) \leq \chi(B); \\ A \times C_{\nu(A)\tau(A)\sigma(S)} & \text{iff } \tau(C)\sigma(S) = \nu(A) \neq \nu(B),\ \chi(B) = 0, \\ & \quad \chi(C) \leq \chi(A). \end{array}$$

Note 5. *Although A or B may be such that $\chi(A) = -1$ or $\chi(B) = -1$, the above conditionally distributive relations are unique because their conditions fix the ν-values in an unique way.*

- Consider $A, B, C \in \mathbb{T} \setminus \{0\}$ and $A + B = S \in \mathbb{T}$, such that $\tau(C) \neq \tau(S)$.

Because

$$\tau(C_{\tau(A)\tau(S)}) \overset{Pr.3.1.2.}{=} \tau(C)\tau(A)\tau(S) = -\tau(A) \quad \text{and} \quad \tau(C_{\tau(B)\tau(S)}) = -\tau(B),$$

according to (8.6), relation (21) is equivalent to the equality $0 = 0$ and holds true.

For $\tau(C) \neq \tau(S)$ the left-hand side of the relation (20) is 0 and due to

$$\tau(C_{\nu(A)\tau(A)\nu(S)\tau(S)}) = \tau(C)\nu(A)\tau(A)\nu(S)\tau(S) = -\nu(A)\nu(S)\tau(A)$$
$$= \begin{cases} -\tau(A), & \text{if } \nu(A) = \nu(S); \\ \tau(A), & \text{if } \nu(A) \neq \nu(S), \end{cases}$$

the right-hand side of this relation is
$A \times C_{\nu(A)\tau(A)\nu(S)\tau(S)} \ + \ B \times C_{\nu(B)\tau(B)\nu(S)\tau(S)} \ =$

$$0 + 0 \qquad \text{iff} \quad \nu(A) = \nu(B) = \nu(S); \tag{28.1}$$

$$0 + B \times C_{\tau(B)\tau(S)} \qquad \text{iff} \quad \nu(S) = \nu(A) \neq \nu(B); \tag{28.2}$$

$$A \times C_{\tau(A)\tau(S)} + 0 \qquad \text{iff} \quad \nu(A) \neq \nu(B) = \nu(S); \tag{28.3}$$

$$A \times C_{-\tau(A)\tau(S)} + B \times C_{-\tau(B)\tau(S)} \qquad \text{iff} \quad \nu(A) = \nu(B) \neq \nu(S). \tag{28.4}$$

Equality (20), in case (28.1), is equivalent to the equality (21) and holds true.
In case (28.2) equality (20) is equivalent to the equalities

$$0 = \begin{cases} b^{\nu(B)} \times_h C_{\nu(B)\tau(S)}, & \text{if } \chi(C) \leq \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \chi(C) \geq \chi(B). \end{cases}$$

These equalities hold true iff $B = 0$ or $C = 0$ which contradict to the initial assumptions. Thus the equality (20) does not hold true in case (28.2) and due to the commutativity of the addition operation the equality does not hold true in case (28.3), too.

Let $\nu(A) = \nu(B) \neq \nu(S)$ (case (28.4)).
For $\chi(C) \leq \min\{\chi(A), \chi(B)\}$ equality (20) is equivalent to the equalities

$$0 = a^{\nu(A)} \times_h C_{-\nu(A)\tau(S)} + b^{\nu(B)} \times_h C_{-\nu(B)\tau(S)}$$
$$0 = s^{-\nu(S)} \times_h C_{\nu(S)\tau(S)},$$

which hold true iff $s^{-\nu(S)} = 0$, that is $\chi(S) = 0$. In case (28.4) $s^{-\nu(S)} = 0$ means also that $a^{\nu(A)} = -b^{\nu(B)}$.

For $\chi(A) \leq \chi(C) \leq \chi(B)$, applying (8.6), (8.4), (8.5) and Proposition 3.1, we get that equality (20) is equivalent to the equality

$$0 = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + b^{\nu(B)} \times_h C_{-\nu(B)\tau(S)}.$$

The latter is equivalent to

$$\left| \begin{array}{lcl} a^{-\nu(C)\tau(C)}c^{\nu(C)} + b^{\nu(B)}c^{\nu(B)\tau(S)} & = & 0 \\ a^{\nu(C)\tau(C)}c^{\nu(C)} + b^{\nu(B)}c^{-\nu(B)\tau(S)} & = & 0 \end{array} \right.$$

For $\nu(B)\tau(S) = \nu(C)$, or $\nu(B)\tau(S) \neq \nu(C)$, the last equalities hold true iff $a^{\nu(B)} = -b^{\nu(B)}$ and $\chi(C) = \chi(A)$. This way, we obtained that the equality (20) holds true iff $\chi(C) = \chi(A) \leq \chi(B)$ and $a^{\nu(B)} = -b^{\nu(B)}$.
Due to the commutativity of the addition operation, we obtained that the equality (20) also

holds true iff $\chi(C) = \chi(B) \leq \chi(A)$ and $a^{\nu(A)} = -b^{\nu(A)}$. Summarizing the last 2 cases, we get that the equality (20) holds true iff $\chi(C) = \min\{\chi(A), \chi(B)\}$ and $a^{\nu(A)} = -b^{\nu(B)}$. These conditions are a special case of the conditions $\chi(C) \leq \min\{\chi(A), \chi(B)\}$ and $\chi(S) = 0$ under which the relation (20) holds true.

For $\chi(C) \geq \max\{\chi(A), \chi(B)\}$, equality (20) is equivalent to the equalities

$$\begin{aligned} 0 &= A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + B_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \\ 0 &= S_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, \end{aligned}$$

which hold true iff $S = 0$.

Note 6. *Whenever $S = 0$ there are two equal distributive relations:*

$$(A+B) \times C \;=\; 0 \;=\; A \times C_{\tau(A)} + B \times C_{\tau(B)} \;=\; A \times C_{-\tau(A)} + B \times C_{-\tau(B)}$$

Although some conditions impose restrictions on $\nu(S)$, $\mathcal{T}(S) = \{+,-\}$.

Note 7. *For $A, B, C \in \mathbb{T}\setminus\{0\}$ such that $\nu(A) = \nu(B) \neq \nu(S)$, $\chi(S) = 0$ and $\chi(C) \leq \min\{\chi(A), \chi(B)\}$, both relations (20) and (21) hold true, implying two equal distributive representations in this case, too.*

Example 3. *For the intervals $A = [2, -5]$, $B = [-3, 5]$, $C = [4, -5]$, satisfying the conditions of Note 7, there are two distributive representations:*

$$(A+B) \times C = \begin{cases} A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)} = A \times C_{-} + B \times C = 0 + 0 \\ A \times C_{-\tau(A)\tau(S)} + B \times C_{-\tau(B)\tau(S)} = A \times C + B \times C_{-} = [25, -20] + [-25, 20] \end{cases}$$

Summarizing, we get:

$$\text{For } A, B, C \in \mathbb{T}\setminus\{0\}, \quad A + B = S \in \mathbb{T} \text{ and } \lambda \in \Lambda \tag{29}$$

$$\begin{aligned} 0 = (A+B) \times C & \\ = \; A \times C_{\lambda} + B \times C_{-\lambda} & \quad \text{iff } S = 0; \\ A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)} & \quad \text{iff } S \neq 0,\ \tau(C) \neq \tau(S); \\ A \times C_{-\tau(A)\tau(S)} + B \times C_{-\tau(B)\tau(S)} & \quad \text{iff } \nu(A) = \nu(B) \neq \nu(S),\ \chi(S) = 0, \\ & \quad \tau(C) \neq \tau(S),\ \chi(C) \leq \min\{\chi(A), \chi(B)\}. \end{aligned}$$

- Consider $A, B, C \in \mathbb{T}\setminus\{0\}$ and $A + B = S \in \mathbb{T}$, such that $\tau(C) = \tau(S)$, $\chi(C) \geq \chi(S)$.

Lemma 3.10. *For $A, B, C \in \mathbb{T}\setminus\{0\}$, such that $A + B = S \in \mathbb{T}$ and $\tau(C) = \tau(S)$,*

$$A \times C_{\nu(A)\tau(A)\nu(S)\tau(S)} \ + \ B \times C_{\nu(B)\tau(B)\nu(S)\tau(S)} \ =$$

$$A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)} \qquad \textit{iff} \quad \nu(A) = \nu(B) = \nu(S); \tag{30.1}$$

$$A \times C_{\tau(A)\tau(S)} + 0 \qquad \textit{iff} \quad \nu(S) = \nu(A) \neq \nu(B); \tag{30.2}$$

$$0 + B \times C_{\tau(B)\tau(S)} \qquad \textit{iff} \quad \nu(A) \neq \nu(B) = \nu(S); \tag{30.3}$$

$$0 + 0 \qquad \textit{iff} \quad \nu(A) = \nu(B) \neq \nu(S). \tag{30.4}$$

Proof. Follows from

$$\tau(C_{\nu(A)\tau(A)\nu(S)\tau(S)}) = \tau(C)\nu(A)\tau(A)\nu(S)\tau(S) = \nu(A)\nu(S)\tau(A)$$
$$= \begin{cases} \tau(A), & \text{if } \ \nu(A) = \nu(S); \\ -\tau(A), & \text{if } \ \nu(A) \neq \nu(S) \end{cases}$$

and (8.6). □

Let $\chi(C) \geq \max\{\chi(A), \chi(B)\}$.

Relation (21) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}$$

which obviously holds true.

In cases (30.1), equality (20) is equivalent (due to (8.4) and Proposition 3.1) to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + B_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which obviously holds true.

In case (30.2), equality (20) is equivalent to the equality $B_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = 0$, which holds true iff either $B = 0$ or $C = 0$, both contradicting to the initial assumptions.

Analogously, in case (30.3) equality (20) does not hold true.

In case (30.4) equality (20) holds true iff $S = 0$.

Thus, for $\chi(C) \geq \max\{\chi(A), \chi(B)\}$, relation (21) is more general than the relation (20).

Let $\chi(A) \leq \chi(C) \leq \chi(B)$.

Equality (21) is equivalent to the equality

$$B_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = b^{\nu(B)} \times_h C_{\nu(B)\tau(S)},$$

which, according to Lemma 3.4, holds true iff $\tau(C) = \tau(B)$ and $\chi(C) = \chi(B)$. This is a special case of the relation (21) in the previous case.

In case (30.1), equality (20) is equivalent to the equality

$$B_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = b^{\nu(B)} \times_h C_{\nu(B)\tau(S)},$$

which, according to Lemma 3.4, holds true iff $\tau(C) = \tau(B)$ and $\chi(C) = \chi(B)$. This is also a special case of the relation (21) for $\chi(C) \geq \max\{\chi(A), \chi(B)\}$.

In case (30.2) equality (20) is equivalent to the equality $B_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = 0$, which does not hold true for $B, C \in \mathbb{T}\backslash\{0\}$.

Analogously, in case (30.3) equality (20) does not hold true.
In case (30.4), equality (20) holds true iff $S = 0$.
Thus, the case $\chi(A) \le \chi(C) \le \chi(B)$ is a special case of the case $\chi(C) \ge \max\{\chi(A), \chi(B)\}$.

Let $\chi(B) \le \chi(C) \le \chi(A)$. This case is commutative to the previous one.

Let $\chi(C) \le \min\{\chi(A), \chi(B)\}$.

Equality (21) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = a^{\nu(A)} \times_h C_{\nu(A)\tau(C)} + b^{\nu(B)} \times_h C_{\nu(B)\tau(C)}. \tag{31}$$

For $\nu(A) = \nu(B) = \nu(S)$ equality (31) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = s^{\nu(S)} \times_h C_{\nu(S)\tau(C)},$$

which according to Lemma 3.4 holds true iff $\chi(C) = \chi(S)$.
For $\nu(A) = \nu(B) \ne \nu(S)$ equality (31) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = s^{-\nu(S)} \times_h C_{-\nu(S)\tau(C)}$$

which according to Lemma 3.7 holds true iff $\chi(C) = \chi(S) = -1$ (in particular $\chi(S = 0) = -1$). But, for $\chi(S) = -1$, $\mathcal{N}(S) = \{+, -\}$. That is why, this case can be considered as a special case of the previous one for $\nu(A) = \nu(B) = \nu(S)$.

For $\nu(C)\tau(C) = \nu(A) = -\nu(B)$ equality (31) is equivalent to

$$\left| \begin{array}{lcll} a^{-\nu(A)}c^{-\nu(C)} & = & a^{-\nu(A)}c^{\nu(C)} & \Longrightarrow \chi(C) = \chi(A) \\ b^{-\nu(A)}c^{-\nu(C)} & = & b^{\nu(A)}c^{\nu(C)} & \Longrightarrow \chi(C) = \chi(B). \end{array} \right.$$

However, for $\nu(A) \ne \nu(B)$, $\chi(S) \le \chi(C) = \chi(A) = \chi(B)$, iff $\tau(A) = \tau(B)$. Analogously, for $-\nu(A) = \nu(B) = \nu(C)\tau(C)$ equality (31) holds true iff $\chi(C) = \chi(A) = \chi(B)$ and $\tau(A) = \tau(B)$.

In case (30.1) equality (20) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = (a + b)^{\nu(S)} \times_h C_{\nu(S)\tau(S)}$$

which, according to Lemma 3.4, holds true iff $\chi(C) = \chi(S)$. Thus, in case (30.1) relation (20) is equivalent to the relation (21).
In case (30.2) equality (20) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} = a^{\nu(A)} \times_h C_{\nu(A)\tau(S)}$$

which does not hold true. (This equality implies $\chi(C) = \chi(S)$, $b^{-\nu(B)} = 0$, $\tau(A) \ne \tau(B)$ and $0 < |b^{\nu(B)}| \le |a^{-\nu(A)}|$, but under these conditions the initial assumption $\chi(C) = \chi(S) \le \chi(A)$ is not fulfilled.)
Analogously in case (30.3) equality (20) does not hold true.
In case (30.4) equality (20) holds true iff $S = 0$.

Summarizing the whole case, which is described by the equality (21), we obtain:

For $A, B, C \in \mathbb{T}\backslash\{0\}$, such that $A+B=S \in \mathbb{T}$, $\tau(C)=\tau(S)$ and $\lambda \in \Lambda$ (32)

$$(A+B)\times C = \begin{cases} A\times C_{\lambda}+B\times C_{-\lambda}, & \text{iff } S=0; \\ A\times C_{\tau(A)\tau(S)}+B\times C_{\tau(B)\tau(S)}, & \text{iff either } \chi(C)\geq \max\{\chi(A),\chi(B),\chi(S)\}; \\ & \text{or } \chi(C)=\chi(S)\leq \min\{\chi(A),\chi(B)\}, \\ & \qquad \nu(A)=\nu(B)=\nu(S). \end{cases}$$

• Consider $A, B, C \in \mathbb{T}\backslash\{0\}$ and $A+B=S \in \mathbb{T}$, such that $\tau(C)=\tau(S)$, $\chi(C)\leq\chi(S)$.

Let $\chi(C)\leq \min\{\chi(A),\chi(B)\}$.

In case (30.1) equality (20) is equivalent to the equality

$$s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=(a+b)^{\nu(S)}\times_h C_{\nu(S)\tau(S)}$$

which obviously holds true.
In case (30.2) equality (20) is equivalent to the equality

$$s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=a^{\nu(S)}\times_h C_{\nu(S)\tau(S)}$$

which holds true iff $b^{-\nu(B)}=0$.
Case (30.3) is commutative to case (30.2) and equality (20) holds true iff $a^{-\nu(A)}=0$.
In case (30.4) equality (20) is equivalent to the equality $s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=0$, which holds true iff $s^{\nu(S)}=0$, that is $S=0$, that is $a^{-\nu(A)}=-b^{-\nu(B)}$ for $\nu(A)=\nu(B)\neq\nu(S)$.

Equality (21) is equivalent to the equality

$$s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=a^{\nu(A)}\times_h C_{\nu(A)\tau(S)}+b^{\nu(B)}\times_h C_{\nu(B)\tau(S)}. \tag{33}$$

For $\nu(A)=\nu(B)=\nu(S)$ equality (33) holds true.
For $\nu(A)=\nu(B)\neq\nu(S)$ equality (33) is equivalent to the equality

$$s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=(a+b)^{-\nu(S)}\times_h C_{-\nu(S)\tau(S)},$$

which according to Proposition 3.2 holds true iff $\chi(C)=\chi(S)=-1$.
For $\nu(A)\neq\nu(B)=\nu(S)$ equality (33) is equivalent to the equality

$$a^{-\nu(A)}\times_h C_{-\nu(A)\tau(S)}=a^{\nu(A)}\times_h C_{\nu(A)\tau(S)},$$

which according to Proposition 3.2 holds true iff $\tau(C)=\tau(A)$, $\chi(C)=\chi(A)=-1$.
Analogously, for $\nu(S)=\nu(A)\neq\nu(B)$ equality (33) holds true iff $\tau(C)=\tau(B)$, $\chi(C)=\chi(B)=-1$.

Let $\chi(A)\leq\chi(C)\leq\chi(B)$.

In case (30.1) equality (20) is equivalent to the equality

$$s^{\nu(S)}\times_h C_{\nu(S)\tau(S)}=A_{\nu(C)\tau(C)}\times_h c^{\nu(C)}+b^{\nu(B)}\times_h C_{\nu(S)\tau(S)}$$

which is equivalent to the equality

$$a^{\nu(A)} \times_h C_{\nu(A)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)}$$

which according to Lemma 3.4 holds true iff $\tau(C) = \tau(A), \ \chi(C) = \chi(A)$.
In case (30.2) equality (20) is equivalent to the equality

$$a^{\nu(A)} \times_h C_{\nu(A)\tau(S)} + b^{-\nu(B)} \times_h C_{-\nu(B)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which holds true iff $b^{-\nu(B)} = 0$ and $\tau(C) = \tau(A), \ \chi(C) = \chi(A)$.
Case (30.3) is commutative to case (30.2) and we have $a^{-\nu(A)} = 0$ and $\tau(C) = \tau(A), \ \chi(C) = \chi(B)$. But $a^{-\nu(A)} = 0$, that is $\chi(A) = 0$ contradicts to the initial assumptions, that is why, in case (30.3) equality (20) does not hold true.
In case (30.4) equality (20) holds true iff $S = 0$.

Equality (21) is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + b^{\nu(B)} \times_h C_{\nu(B)\tau(S)}. \tag{34}$$

For $\nu(A) = \nu(B) = \nu(S)$ equality (34) is equivalent to the equality

$$a^{\nu(A)} \times_h C_{\nu(A)\tau(A)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)}$$

which according to Lemma 3.4 holds true iff $\tau(C) = \tau(A), \ \chi(C) = \chi(A)$.
For $\nu(A) = \nu(B) \neq \nu(S)$ equality (34) is equivalent to the equality

$$a^{\nu(S)} \times_h C_{\nu(S)\tau(S)} + b^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + b^{-\nu(S)} \times_h C_{-\nu(S)\tau(S)},$$

which holds true iff $\chi(C) = \chi(A) = \chi(B) = -1$.
For $\nu(A) \neq \nu(B) = \nu(S)$ equality (34) is equivalent to the equality

$$a^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which according to Lemma 3.4 holds true iff $\tau(C) = \tau(A), \ \chi(C) = \chi(A)$.
For $\nu(S) = \nu(A) \neq \nu(B)$ equality (34) is equivalent to the equality

$$a^{\nu(A)} \times_h C_{\nu(A)\tau(S)} + b^{-\nu(B)} \times_h C_{-\nu(B)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} + b^{\nu(B)} \times_h C_{\nu(B)\tau(S)},$$

which holds true iff $\chi(C) = \chi(A) = \chi(B) = -1$.

Let $\chi(B) \leq \chi(C) \leq \chi(A)$. This case is commutative to the previous one.

Let $\chi(C) \geq \max\{\chi(A), \chi(B)\}$.

In case (30.1) equality (20) is equivalent (due to (8.4) and Proposition 3.1) to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = (A + B)_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which according to Lemma 3.4 holds true iff $\chi(C) = \chi(S)$.
In case (30.2) equality (20) is equivalent to the equality

$$a^{\nu(A)} \times_h C_{\nu(A)\tau(S)} + b^{-\nu(B)} \times_h C_{\nu(S)\tau(S)} = A_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which holds true iff $b^{-\nu(B)} = 0$ and $\chi(C) = \chi(A)$. But these conditions contradict the initial assumptions. Thus in case (30.2) equality (20) does not hold true.

Case (30.3) is commutative to case (30.2) and equality (20) does not hold true.
In case (30.4) equality (20) holds true iff $S = 0$.

Equality (21) is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} = (A+B)_{\nu(C)\tau(C)} \times_h c^{\nu(C)}.$$

which according to Lemma 3.4 holds true iff $\chi(C) = \chi(S)$.

Summarizing the whole case, we obtain:

For $A, B, C \in \mathbb{T}\backslash\{0\}$, such that $A + B = S \in \mathbb{T}$, $\tau(C) = \tau(S)$ and $\lambda \in \Lambda$ (35)

$$(A+B) \times C = \begin{cases} A \times C_{\lambda} + B \times C_{-\lambda}, & \text{iff } S = 0,\ \chi(C) = -1; \\ A \times C_{\tau(A)\tau(S)} + B \times C_{\tau(B)\tau(S)}, & \text{iff either } \chi(C) \le \min\{\chi(A), \chi(B), \chi(S)\} \\ & \qquad \nu(A) = \nu(B) = \nu(S), \\ & \quad \text{or } \chi(C) = \chi(A) \le \min\{\chi(B), \chi(S)\} \\ & \qquad \tau(C) = \tau(A), \quad \nu(A) \ne \nu(B) = \nu(S), \\ & \quad \text{or } \chi(C) = \chi(S) \ge \max\{\chi(A), \chi(B)\}; \\ A \times C_{\tau(A)\tau(S)}, & \text{iff } \nu(S) = \nu(A) \ne \nu(B),\ \chi(B) = 0, \\ & \qquad \chi(C) \le \min\{\chi(A), \chi(S)\}; \\ B \times C_{\tau(B)\tau(S)}, & \text{iff } \nu(A) \ne \nu(B) = \nu(S),\ \chi(A) = 0, \\ & \qquad \chi(C) \le \min\{\chi(B), \chi(S)\}. \end{cases}$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T} \setminus \{0\}$, $A + B = S \in \mathbb{D}\backslash\mathbb{T}$ and $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$.

Equality (21) is equivalent to the equality

$$S_{\sigma(C)} \times_h C_{\sigma(S)} = A_{\sigma(C)} \times_h C_{\sigma(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)},$$

which is equivalent to the equality

$$B_{\sigma(C)} \times_h C_{\sigma(S)} = B_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)},$$

which, according to Lemma 3.3, holds true iff $\nu(B) = +$ and $b^{-\nu(B)} = 0$.

Equality (20) is equivalent to the equality

$$S_{\sigma(C)} \times_h C_{\sigma(S)} = A_{\sigma(C)} \times_h C_{\sigma(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\nu(B)\sigma(S)},$$

which, according to Lemma 3.3, holds true iff $b^{-\nu(B)} = 0$.

For the case, we consider, relation (20) is more general than relation (21).

Note, that

$$\sigma(S) = \sigma(A) \qquad \text{for } A \in \mathbb{D}\backslash\mathbb{T},\ B \in \mathbb{T}\backslash\{0\} \text{ and } A + B = S \in \mathbb{D}\backslash\mathbb{T}. \qquad (36)$$

Applying this property, we get:

$$\text{For } A \in \mathbb{D}\backslash\mathbb{T},\ B \in \mathbb{T}\backslash\{0\},\ A+B=S \in \mathbb{D}\backslash\mathbb{T}, \text{ and } C \in \mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R}) \tag{37}$$
$$(A+B)\times C \;=\; A\times C \;+\; B\times C_{\nu(B)\tau(B)\sigma(S)}, \qquad \text{iff } \chi(B)=0.$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T}\backslash\{0\}$, and $A+B=S \in \mathbb{D}\backslash\mathbb{T}$ and $C \in \mathbb{T}\backslash\{0\}$.

Because $\tau(C_{\tau(B)\sigma(S)}) = \tau(B)\sigma(S)\tau(C) = \begin{cases} \tau(B), & \text{if } \tau(C)=\sigma(S); \\ -\tau(B), & \text{if } \tau(C)\neq\sigma(S), \end{cases}$ equality (21) is equivalent to the equality

$$s^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} \;=\; a^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} \;+$$
$$\begin{cases} b^{\nu(B)} \times_h C_{\nu(B)\sigma(S)}, & \text{if } \tau(C)=\sigma(S),\ \chi(C)\le\chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \tau(C)=\sigma(S),\ \chi(C)\ge\chi(B); \\ 0, & \text{iff } \tau(C)\neq\sigma(S). \end{cases}$$

Applying Property (36), Proposition 3.2 and Lemma 3.4, we obtain that relation (21) holds true iff
either $\nu(B)=+$, $\tau(C)=\sigma(S)$, $\chi(C)\le\chi(B)$, or $\nu(B)=-$, $\tau(C)=\sigma(S)$, $\chi(C)=\chi(B)=-1$, or $b^-=0$, $\tau(C)\neq\sigma(S)$.

Equality (20) is equivalent to the equality

$$s^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} \;=\; a^{\tau(C)\sigma(S)} \times_h C_{\sigma(S)} \;+$$
$$\begin{cases} b^{\nu(B)} \times_h C_{\sigma(S)}, & \text{if } \tau(C)=\nu(B)\sigma(S),\ \chi(C)\le\chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \tau(C)=\nu(B)\sigma(S),\ \chi(C)\ge\chi(B); \\ 0, & \text{if } \tau(C)\neq\nu(B)\sigma(S). \end{cases}$$

Thus, relation (20) holds true iff either $\tau(C)=\nu(B)\sigma(S)$, $\chi(C)\le\chi(B)$, or $b^{-\nu(B)}=0$, $\tau(C)\neq\nu(B)\sigma(S)$. Relation (20) is more general than the relation (21).

Summarizing we get:

$$\text{For } A \in \mathbb{D}\backslash\mathbb{T},\ B \in \mathbb{T}\backslash\{0\}, \text{ and } A+B=S \in \mathbb{D}\backslash\mathbb{T},\ C \in \mathbb{T}\backslash\{0\} \tag{38}$$
$$(A+B)\times C = \begin{cases} A\times C + B\times C_{\nu(B)\tau(B)\sigma(S)}, & \text{iff } \nu(B)=\tau(C)\sigma(S),\ \chi(C)\le\chi(B); \\ A\times C, & \text{iff } \nu(B)\neq\tau(C)\sigma(S),\ \chi(B)=0. \end{cases}$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T}\setminus\{0\}$, and $A+B=S \in \mathbb{T}$, $C \in \mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R})$.

Equality (21) is equivalent to the equality

$$S_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)} \times_h C_{\tau(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)},$$

which holds true iff $C = c \in \mathbb{R}$.

Equality (20) is equivalent to the equality

$$S_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)} \times_h C_{\nu(S)\tau(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\nu(B)\nu(S)\tau(S)}. \tag{39}$$

For $\nu(S) = \nu(B)$, equality (39) is equivalent to the equality

$$A_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)} \times_h C_{\nu(S)\tau(S)},$$

which does not hold true for $C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R})$.
For $\nu(S) \neq \nu(B)$, equality (39) is equivalent to the equality

$$A_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} + B_{\sigma(C)} \times_h c^{\sigma(C)\tau(S)} = A_{\sigma(C)} \times_h C_{\nu(S)\tau(S)} + B_{\sigma(C)} \times_h c^{-\sigma(C)\tau(S)},$$

which is equivalent to

$$\left| \begin{array}{lcl} a^{-\sigma(C)}c^{\sigma(C)\tau(S)} + b^{-\sigma(C)}c^{\sigma(C)\tau(S)} & = & a^{-\sigma(C)}c^{-\nu(S)\tau(S)} + b^{-\sigma(C)}c^{-\sigma(C)\tau(S)} \\ a^{\sigma(C)}c^{\sigma(C)\tau(S)} + b^{\sigma(C)}c^{\sigma(C)\tau(S)} & = & a^{\sigma(C)}c^{\nu(S)\tau(S)} + b^{\sigma(C)}c^{-\sigma(C)\tau(S)} \end{array} \right.$$

For $\sigma(C) = \nu(S)$ we have

$$\left| \begin{array}{rclcl} (a+b)^{-\sigma(C)}c^{\sigma(C)\tau(S)} & = & (a+b)^{-\sigma(C)}c^{-\sigma(C)\tau(S)} & \Longrightarrow & s^{-\nu(S)} = 0 \\ a^{\sigma(C)}c^{\sigma(C)\tau(S)} + b^{\sigma(C)}c^{\sigma(C)\tau(S)} & = & a^{\sigma(C)}c^{\sigma(C)\tau(S)} + b^{\sigma(C)}c^{-\sigma(C)\tau(S)} & \Longrightarrow & b^{\sigma(C)} = 0 \end{array} \right.$$

For $\sigma(C) \neq \nu(S)$ we have

$$\left| \begin{array}{rclcl} a^{-\sigma(C)}c^{\sigma(C)\tau(S)} + b^{-\sigma(C)}c^{-\sigma(C)\tau(S)} & = & a^{-\sigma(C)}c^{\sigma(C)\tau(S)} + b^{-\sigma(C)}c^{\sigma(C)\tau(S)} & \Longrightarrow & b^{-\sigma(C)} = 0 \\ (a+b)^{\sigma(C)}c^{\sigma(C)\tau(S)} & = & (a+b)^{\sigma(C)}c^{-\sigma(C)\tau(S)} & \Longrightarrow & s^{-\nu(S)} = 0 \end{array} \right.$$

Thus, relation (20) holds true iff $\nu(S) \neq \nu(B)$, $\chi(S) = \chi(B) = 0$, and relation (20) is more general than the relation (21).

Note, that

$$\tau(S) = \tau(B) \qquad \text{for} \;\; A \in \mathbb{D}\backslash\mathbb{T},\; B \in \mathbb{T} \setminus \{0\} \;\; \text{and} \;\; A + B = S \in \mathbb{T}. \tag{40}$$

Applying the above property and Propositions 3.8 and 3.9, we get:

$$\boxed{\begin{array}{c} \text{For } A \in \mathbb{D}\backslash\mathbb{T},\; B \in \mathbb{T} \setminus \{0\},\; A + B = S \in \mathbb{T} \text{ and } C \in \mathbb{D} \setminus (\mathbb{T} \cup \mathbb{R}) \qquad (41) \\ (A+B) \times C \;=\; A \times C + B \times C_{-}, \qquad \text{iff} \;\; \chi(B) = 0 \text{ and } \chi(S) = 0. \end{array}}$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}, B \in \mathbb{T} \setminus \{0\}$, and $A + B = S \in \mathbb{T}$, $C \in \mathbb{T}\backslash\{0\}$, such that $\tau(C) \neq \tau(S)$.

Equality (21) is equivalent to the equality

$$0 \;=\; a^{\tau(C)\tau(S)} \times_h C_{\tau(S)} \;+\; 0,$$

which does not hold true for $A \in \mathbb{D}\backslash\mathbb{T}$ and $C \in \mathbb{T} \setminus \{0\}$.

Equality (20) is equivalent to the equality

$$0 \;=\; a^{-\nu(S)} \times_h C_{\nu(S)\tau(S)} \;+\; \begin{cases} b^{\nu(B)} \times_h C_{\nu(S)\tau(S)}, & \text{if } \nu(B) \neq \nu(S),\ \chi(C) \leq \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \nu(B) \neq \nu(S),\ \chi(C) \geq \chi(B); \\ 0, & \text{if } \nu(B) = \nu(S). \end{cases} \tag{42}$$

First of the equalities (42) holds true iff $s^{-\nu(S)} = 0,\ \nu(B) \neq \nu(S),\ \chi(C) \leq \chi(B)$.
For $\nu(B) \neq \nu(S),\ \chi(C) \geq \chi(B)$ the second of the equalities (42) is equivalent to the equality

$$-a^{-\nu(S)} \times_h C_{\nu(S)\tau(S)} \;=\; B_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which is equivalent to

$$\left|\begin{array}{rcl} -a^{-\nu(S)} c^{-\nu(S)\tau(S)} & = & b^{-\nu(C)\tau(C)} c^{\nu(C)} \\ -a^{-\nu(S)} c^{\nu(S)\tau(S)} & = & b^{\nu(C)\tau(C)} c^{\nu(C)} \end{array}\right.$$

For $\nu(S)\tau(S) = \nu(C)$ last equalities imply

$$\left|\begin{array}{rclcl} -a^{-\nu(S)} c^{-\nu(C)} & = & b^{\nu(S)} c^{\nu(C)} & \Longrightarrow & \chi(C) = \chi(B) \\ -a^{-\nu(S)} c^{\nu(C)} & = & b^{-\nu(S)} c^{\nu(C)} & \Longrightarrow & -a^{-\nu(S)} = b^{-\nu(S)} \end{array}\right.$$

For $\nu(S)\tau(S) \neq \nu(C)$, we obtain the same conditions.
This way for the second of the equalities (42), we obtained the conditions $s^{-\nu(S)} = 0,\ \nu(B) \neq \nu(S),\ \chi(C) = \chi(B)$, which are a special case of the conditions for the first of the three equalities (42).
The third of the three equalities (42) does not hold true for $A \in \mathbb{D}\backslash\mathbb{T}$ and $C \in \mathbb{T} \setminus \{0\}$.

Summarizing the whole case:

$$\begin{array}{c} \text{For } A \in \mathbb{D}\backslash\mathbb{T},\ B \in \mathbb{T} \setminus \{0\},\ A + B = S \in \mathbb{T},\ C \in \mathbb{T} \setminus \{0\}, \text{ such that } \tau(C) \neq \tau(S) \\ (A + B) \times C = 0 = A \times C + B \times C_-, \quad \text{iff } \ \chi(S) = 0,\ \nu(B) \neq \nu(S),\ \chi(C) \leq \chi(B). \end{array} \tag{43}$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T}\setminus\{0\}$, and $A+B = S \in \mathbb{T}$, $C \in \mathbb{T}\setminus\{0\}$, such that $\tau(C) = \tau(S)$, $\chi(C) \geq \chi(S)$.

Equality (21) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \;=\; a^+ \times_h C_{\tau(S)} \;+\; \begin{cases} b^{\nu(B)} \times_h C_{\nu(B)\tau(S)}, & \text{if } \chi(C) \leq \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \chi(C) \geq \chi(B). \end{cases}$$

If $\chi(C) \leq \chi(B)$ and $\nu(B) = + = \nu(S)$, according to Lemma 3.4, the equality holds true iff $\chi(C) = \chi(S)$.

If $\chi(C) \leq \chi(B)$ and $\nu(B) = + \neq \nu(S)$, according to Lemma 3.7, the equality holds true iff $\chi(C) = \chi(S) = -1$. Last condition implies $\chi(B) = \chi(S) = -1$, which is not possible for $A \in \mathbb{D}\backslash\mathbb{T}$.

If $\chi(C) \leq \chi(B)$ and $\nu(B) = -$, the equality is equivalent to

$$\left|\begin{array}{rcl} a^{-\nu(C)\tau(C)} c^{\nu(C)} + b^{-\nu(C)\tau(C)} c^{\nu(C)} & = & a^+ c^{-\tau(S)} + b^- c^{\tau(S)} \\ a^{\nu(C)\tau(C)} c^{\nu(C)} + b^{\nu(C)\tau(C)} c^{\nu(C)} & = & a^+ c^{\tau(S)} + b^- c^{-\tau(S)} \end{array}\right.$$

which imply $\chi(C) = a^-/a^+$ — not possible for $C \in \mathbb{T}$ and $A \in \mathbb{D}\backslash\mathbb{T}$.

If $\chi(C) \geq \chi(B)$, the equality is equivalent to the equality

$$A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ a^+ \times_h C_{\tau(C)}$$

which implies $\chi(C) = a^-/a^+$ — not possible for $C \in \mathbb{T}$ and $A \in \mathbb{D}\backslash\mathbb{T}$.

Equality (20) is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ a^{\nu(S)} \times_h C_{\nu(S)\tau(S)} + \begin{cases} b^{\nu(B)} \times_h C_{\nu(S)\tau(S)}, & \text{if } \nu(B) = \nu(S),\ \chi(C) \leq \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \nu(B) = \nu(S),\ \chi(C) \geq \chi(B); \\ 0, & \text{if } \nu(B) \neq \nu(S). \end{cases}$$

First of these equalities is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ s^{\nu(S)} \times_h C_{\nu(S)\tau(S)}$$

which, according to Lemma 3.4, holds true iff $\chi(C) = \chi(S)$.
The second of the above three equalities is equivalent to the equality

$$A_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ a^{\nu(S)} \times_h C_{\nu(S)\tau(S)}$$

which implies $\chi(C) = a^{-\nu(S)}/a^{\nu(S)}$ — not possible for $C \in \mathbb{T}$ and $A \in \mathbb{D}\backslash\mathbb{T}$.
The third of the above three equalities is equivalent to the equality

$$S_{\nu(C)\tau(C)} \times_h c^{\nu(C)} \ = \ a^{\nu(S)} \times_h C_{\nu(S)\tau(S)}$$

which is equivalent to

$$\left| \begin{array}{rcl} s^{-\nu(C)\tau(C)} c^{\nu(C)} & = & a^{\nu(S)} c^{-\nu(S)\tau(S)} \\ s^{\nu(C)\tau(C)} c^{\nu(C)} & = & a^{\nu(S)} c^{\nu(S)\tau(S)} \end{array} \right.$$

For $\nu(S)\tau(S) = \nu(C)$ and $\nu(S)\tau(S) \neq \nu(C)$ this system implies $\chi(C) = s^{-\nu(S)}/a^{\nu(S)}$. Since $\chi(C) \leq 0$, $s^{-\nu(S)}/a^{\nu(S)}$ should be negative or zero. But $s^{-\nu(S)}/a^{\nu(S)} \leq 0$, iff $\sigma(b^{-\nu(S)}) \neq \sigma(A) = \sigma(b^{\nu(S)})$ and $|b^{-\nu(S)}| \geq |a^{-\nu(S)}|$.
Let $|b^{-\nu(S)}| > |a^{-\nu(S)}|$. However, according to the initial assumptions $\chi(C) \geq \chi(S)$, that is

$$\chi(C) = s^{-\nu(S)}/a^{\nu(S)} \geq \chi(S) = s^{-\nu(S)}/s^{\nu(S)},$$

that is $a^{\nu(S)} \leq a^{\nu(S)} + b^{\nu(S)}$. The last relation is not possible because $a^{\nu(S)}$ and $b^{\nu(S)}$ have equal signs and $B \neq 0$.
If $|b^{-\nu(S)}| = |a^{-\nu(S)}|$, then $\chi(C) = \chi(S) = 0 \geq \chi(B)$ and for $\nu(S) \neq \nu(B)$ relation (20) holds true.

Summarizing the whole case:

For $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T} \setminus \{0\}$, $A + B = S \in \mathbb{T}$, $C \in \mathbb{T} \setminus \{0\}$, such that $\tau(C) = \tau(S)$ (44)

$$(A+B) \times C = \begin{cases} A \times C_{\sigma(A)\nu(S)\tau(S)} + B \times C, & \text{iff } \ \nu(B) = \nu(S),\ \chi(C) = \chi(S) \leq \chi(B); \\ A \times C, & \text{iff } \ \nu(B) \neq \nu(S),\ \chi(C) = \chi(S) = 0. \end{cases}$$

• Consider $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T}\setminus\{0\}$, and $A+B = S \in \mathbb{T}$, $C \in \mathbb{T}\setminus\{0\}$, such that $\tau(C) = \tau(S)$, $\chi(C) \le \chi(S)$.

Equality (21) is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} \;=\; a^+ \times_h C_{\tau(S)} \;+\; \begin{cases} b^{\nu(B)} \times_h C_{\nu(B)\tau(S)}, & \text{if } \chi(C) \le \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \chi(C) \ge \chi(B). \end{cases}$$

If $\chi(C) \le \chi(B)$ and $\nu(B) = \nu(S)$, the equality is equivalent to the equality

$$a^{\nu(S)} \times_h C_{\nu(S)\tau(S)} \;=\; a^+ \times_h C_{\tau(S)}$$

which holds true iff $\nu(S) = +$.

If $\chi(C) \le \chi(B)$ and $\nu(B) \ne \nu(S)$, the equality is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} \;=\; a^+ \times_h C_{\tau(S)} \;+\; b^{-\nu(S)} \times_h C_{-\nu(S)\tau(S)},$$

which holds true iff either $\nu(S) = +$, $\chi(C) = \chi(B) = -1$, or $\nu(S) = -$, $\chi(C) = \chi(S) = -1$.

For $\chi(C) \ge \chi(B)$ the equality holds true iff either $\nu(S) = \nu(B) = +$, $\chi(C) = \chi(B)$, or $+ = \nu(S) \ne \nu(B)$, $\chi(C) = \chi(B) = -1$.

Equality (20) is equivalent to the equality

$$s^{\nu(S)} \times_h C_{\nu(S)\tau(S)} \;=\; a^{\nu(S)} \times_h C_{\nu(S)\tau(S)} + \begin{cases} b^{\nu(B)} \times_h C_{\nu(S)\tau(S)}, & \text{if } \nu(B) = \nu(S),\ \chi(C) \le \chi(B); \\ B_{\nu(C)\tau(C)} \times_h c^{\nu(C)}, & \text{if } \nu(B) = \nu(S),\ \chi(C) \ge \chi(B); \\ 0, & \text{if } \nu(B) \ne \nu(S). \end{cases}$$

First of these equalities holds true.
Second of the above three equalities is equivalent to the equality

$$b^{\nu(B)} \times_h C_{\nu(B)\tau(B)} \;=\; B_{\nu(C)\tau(C)} \times_h c^{\nu(C)},$$

which, according to Lemma 3.4, holds true iff $\chi(C) = \chi(B)$ and this is a special case of $\nu(B) = \nu(S)$, $\chi(C) \le \chi(B)$.
The third of the above three equalities is equivalent to the equality

$$b^{-\nu(B)} \times_h C_{\nu(S)\tau(S)} \;=\; 0,$$

which holds true iff $b^{-\nu(B)} = 0$.
Applying Proposition 3.8, the third equality becomes

$$(A+B) \times C = A \times C, \qquad \text{iff } \ \chi(B) = 0,\ \nu(B) \ne \nu(S).$$

Summarizing the whole case:

For $A \in \mathbb{D}\backslash\mathbb{T}$, $B \in \mathbb{T}\setminus\{0\}$, $A+B = S \in \mathbb{T}$, $C \in \mathbb{T}\setminus\{0\}$, such that $\tau(C) = \tau(S)$ (45)

$$(A+B) \times C = \begin{cases} A \times C_{\sigma(A)\nu(S)\tau(S)} + B \times C, & \text{iff } \ \nu(B) = \nu(S), \\ & \qquad \chi(C) \le \min\{\chi(B), \chi(S)\}; \\ A \times C, & \text{iff } \ \nu(B) \ne \nu(S), \quad \chi(B) = 0, \\ & \qquad \chi(C) \le \chi(S). \end{cases}$$

Thus, we have proven the following Theorem which summarizes relations (23), (24)–(27), (29), (32), (35), (37), (38), (41), (43)–(45).

Theorem 3.11. *For two additive terms $A_1, A_2 \in \mathbb{D}\backslash\{0\}$, the following conditionally distributive relations, wherein $S = A_1 + A_2$, hold true:*

1. $\underline{S \in \mathbb{D}\backslash\mathbb{T}, \quad C \in \mathbb{D}\setminus(\mathbb{T}\cup\mathbb{R})}$

$$(A_1 + A_2)\times C \;=\; A_1 \times C_{\widehat{\mu}(A_1)\sigma(S)} + A_2 \times C_{\widehat{\mu}(A_2)\sigma(S)} \qquad \textit{iff} \quad \chi(A_i) = 0 \;\; \textit{for} \;\; A_i \in \mathbb{T}\setminus\{0\};$$

2. $\underline{S \in \mathbb{T}, \quad C \in \mathbb{D}\backslash(\mathbb{T}\cup\mathbb{R})}$

$$\begin{aligned}
&(A_1 + A_2)\times C \;=\\
&A_1 \times C_{\sigma(A_1)\nu(S)\tau(S)} + A_2 \times C_{\sigma(A_2)\nu(S)\tau(S)} && \textit{iff } A_1, A_2 \in \mathbb{D}\backslash\mathbb{T},\; (S = 0 \;\vee\; \chi(S) = 0)\,;\\
&A_1 \times C_{\tau(A_1)\tau(S)} + A_2 \times C_{\tau(A_2)\tau(S)} && \textit{iff } A_1, A_2 \in \mathbb{T}\setminus\{0\};\\
&A_{i_1} \times C + A_{i_2} \times C_{-} && \textit{iff } A_{i_1} \in \mathbb{D}\backslash\mathbb{T},\; A_{i_2} \in \mathbb{T}\backslash\{0\},\; \chi(A_{i_2}) = \chi(S) = 0;
\end{aligned}$$

3. $\underline{S \in \mathbb{D}\backslash\mathbb{T}, \quad C \in \mathbb{T}\backslash\{0\}}$

$$\begin{aligned}
&(A_1 + A_2)\times C \;=\\
&A_1 \times C_{\sigma(A_1)\sigma(S)} + A_2 \times C_{\sigma(A_2)\sigma(S)} && \textit{iff} \;\; A_i \in \mathbb{D}\backslash\mathbb{T},\; i = 1, 2;\\
&A_1 \times C_{\nu(A_1)\tau(A_1)\sigma(S)} + A_2 \times C_{\nu(A_2)\tau(A_2)\sigma(S)}, && \textit{iff} \;\; \textit{for } i = 1, 2,\; A_i \in \mathbb{T}\backslash\{0\},\\
& && ((\nu(A_i) = \tau(C)\sigma(S) \;\wedge\; \chi(C) \le \chi(A_i)) \quad \vee\\
& && \qquad (\nu(A_i) \ne \tau(C)\sigma(S) \;\wedge\; \chi(A_i) = 0))\,;\\
&A_{i_1} \times C + A_{i_2} \times C_{\nu(A_{i_2})\tau(A_{i_2})\sigma(S)} && \textit{iff} \;\; A_{i_1} \in \mathbb{D}\backslash\mathbb{T},\; A_{i_2} \in \mathbb{T}\backslash\{0\},\\
& && \qquad \nu(A_{i_2}) = \tau(C)\sigma(S), \quad \chi(C) \le \chi(A_{i_2});\\
&A_{i_1} \times C && \textit{iff} \;\; A_{i_1} \in \mathbb{D}\backslash\mathbb{T},\; A_{i_2} \in \mathbb{T}\backslash\{0\},\\
& && \qquad \nu(A_{i_2}) \ne \tau(C)\sigma(S), \quad \chi(A_{i_2}) = 0;
\end{aligned}$$

4. $\underline{S \in \mathbb{T}, \quad C \in \mathbb{T}\backslash\{0\} \;\; \textit{such that} \;\; \tau(C) \ne \tau(S)}$

$$\begin{aligned}
&(A_1 + A_2)\times C \;=\; 0 \;=\\
&A_1 \times C_{\sigma(A_1)\nu(S)\tau(S)} + A_2 \times C_{\sigma(A_2)\nu(S)\tau(S)}, && \textit{iff} \;\; A_1, A_2 \in \mathbb{D}\backslash\mathbb{T}, \quad (S = 0 \;\vee\; \chi(S) = 0);\\
&A_1 \times C_{\tau(A_1)\tau(S)} + A_2 \times C_{\tau(A_2)\tau(S)}, && \textit{iff} \;\; A_1, A_2 \in \mathbb{T}\backslash\{0\},\\
&A_2 \times C_{-\tau(A_2)\tau(S)} + A_2 \times C_{-\tau(A_2)\tau(S)}, && \textit{iff} \;\; \textit{for } i = 1, 2,\; A_i \in \mathbb{T}\backslash\{0\}, \quad \nu(A_i) \ne \nu(S),\\
& && \qquad \chi(C) \le \chi(A_i), \quad \chi(S) = 0;\\
&A_{i_1} \times C + A_{i_2} \times C_{-}, && \textit{iff} \;\; A_{i_1} \in \mathbb{D}\backslash\mathbb{T},\; A_{i_2} \in \mathbb{T}\backslash\{0\}, \quad \nu(S) \ne \nu(A_{i_2}),\\
& && \qquad \chi(C) \le \chi(A_{i_2}), \quad \chi(S) = 0;
\end{aligned}$$

5. $\underline{S \in \mathbb{T}, \quad C \in \mathbb{T}\backslash\{0\}, \quad such\ that \quad \tau(C) = \tau(S)}$

$(A_1 + A_2) \times C \ =$

$A_1 \times C_{\sigma(A_1)\nu(S)\tau(S)} + A_2 \times C_{\sigma(A_2)\nu(S)\tau(S)}$,	*iff* $A_1, A_2 \in \mathbb{D}\backslash\mathbb{T}, \quad \chi(C) \leq \chi(S)$;
$A_1 \times C_\lambda + A_2 \times C_{-\lambda} = 0$,	*iff* $S = 0, \quad \lambda \in \Lambda$;
$A_1 \times C_{\tau(A_1)\tau(S)} + A_2 \times C_{\tau(A_2)\tau(S)}$,	*iff for* $i = 1, 2, \quad A_i \in \mathbb{T}\backslash\{0\}$, *either* $\chi(C) \geq \max\{\chi(A_i), \chi(S)\}$; *or* $\nu(A_i) = \nu(S),\ \chi(C) \leq \min\{\chi(A_i), \chi(S)\}$; *or* $\chi(C) = \chi(A_{i_1}) \leq \min\{\chi(A_{i_2}), \chi(S)\}$, $\tau(C) = \tau(A_{i_1}),\ \nu(A_{i_1}) \neq \nu(A_{i_2}) = \nu(S)$;
$A_{i_1} \times C_{\tau(A_{i_1})\tau(S)}$,	*iff* $A_1, A_2 \in \mathbb{T}\backslash\{0\}, \quad \nu(S) = \nu(A_{i_1}) \neq \nu(A_{i_2})$, $\chi(A_{i_2}) = 0,\ \chi(C) \leq \min\{\chi(A_{i_1}), \chi(S)\}$;
$A_{i_1} \times C_{\sigma(A_{i_1})\nu(S)\tau(S)} + A_{i_2} \times C$,	*iff* $A_{i_1} \in \mathbb{D}\backslash\mathbb{T}, A_{i_2} \in \mathbb{T}\backslash\{0\}, \quad \nu(A_{i_2}) = \nu(S)$, $\chi(C) \leq \min\{\chi(A_{i_2}), \chi(S)\}$;
$A_1 \times C$	*iff* $A_1 \in \mathbb{D}\backslash\mathbb{T},\ A_2 \in \mathbb{T}\backslash\{0\}, \quad \nu(A_2) \neq \nu(S)$, $((\chi(C) \leq \chi(S),\ \chi(A_2) = 0) \ \vee\ \chi(C) = \chi(S) = 0)$.

Note 8. *Now, we can definitely say in which of the cases of Theorem 3.11 the relation (21) is more general than the relation (20) and vice verse. Relation (20) holds true in the cases 1., 2.1, 2.3, 3., 4.1, 4.3, 4.4, 5.1 and 5.4–5.6 of Theorem 3.11. Relation (21) is more general than the relation (20) in the cases 2.2, 4.2 and 5.3 of Theorem 3.11. There are also some situations in which both relations (20) and (21) hold true determining thus two equal distributive representations. Under the conditions of the case 4.3 of Theorem 3.11 both relations (4.2 and 4.3) hold true.*

Note 9. *For* $A \in \mathbb{T}\backslash\{0\}$, *the following three conditions are equivalent:*

$$1. \quad \chi(A) = 0, \qquad 2. \quad a^- a^+ = 0, \qquad 3. \quad a^{-\nu(A)} = 0.$$

4 Distributivity in Multiplication of a Finite Interval Sum

Define $I = \{1, ..., p\}, p \geq 0$, $I = \emptyset$ for $p = 0$; $J = \{1, ..., q\}, q \geq 0$, $J = \emptyset$ for $q = 0$; and $A_i \in \mathbb{D}\backslash\mathbb{T}$ for $i \in I$, $B_j \in \mathbb{T}\backslash\{0\}$ for $j \in J$. Denote $S = \sum_{i\in I} A_i + \sum_{j\in J} B_j$. With these notations, next Theorems of this section define some sufficient conditions for disclosing brackets in multiplying interval sum, without referring to the end points of the participating intervals.

Theorem 4.1. *If $S \in \mathbb{D}\backslash\mathbb{T}$, $C \in \mathbb{D}\backslash(\mathbb{T} \cup \mathbb{R})$ and there exist index sets I, J_j, $j = 1, ..., 4$, some of them may be empty sets, such that $\bigcup_{j=1}^4 B_j = J$, $\bigcup_{j=1}^4 B_j = \emptyset$ and*

- *$\chi(B_j) = 0$ for all $j \in J_1$;* • *$\Sigma_2 = \sum_{j\in J_2} B_j$ and $\chi(\Sigma_2) = 0$;*
- *$\nu(B_j) = +$ for all $j \in J_3$, $\Sigma_3 = \sum_{j\in J_3} B_j$ and $(\Sigma_3 = 0 \vee \chi(\Sigma_3) = 0)$;*
- *$\nu(B_j) = -$ for all $j \in J_4$, $\Sigma_4 = \sum_{j\in J_4} B_j$ and $(\Sigma_4 = 0 \vee \chi(\Sigma_4) = 0)$;*

then the following relation holds true

$$\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C = \sum_{i\in I} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{j\in J_1} \left(B_j \times C_{\nu(B_j)\tau(B_j)\sigma(S)}\right) +$$
$$\sum_{j\in J_2} \left(B_j \times C_{\nu(\Sigma_2)\tau(B_j)\sigma(S)}\right) + \sum_{j\in J_3} \left(B_j \times C_{\tau(B_j)\sigma(S)}\right) + \sum_{j\in J_4} \left(B_j \times C_{-\tau(B_j)\sigma(S)}\right).$$

Proof. Most of the conditions follow straightforward from Theorem 3.11.1. As an illustration, we shall prove the conditions for $p = 0 \neq q$. Let $\exists\ J_1 \subset J$, $J_2 \subset J$, such that $J_1 \neq \emptyset \neq J_2$, $J_1 \cup J_2 = J$, $J_1 \cap J_2 = \emptyset$ and $\Sigma_1 = \sum_{j\in J_1} B_j$, $\Sigma_2 = \sum_{j\in J_2} B_j$ are such that $\nu(B_j) \stackrel{j\in J_1}{=} +$, $\nu(B_j) \stackrel{j\in J_2}{=} -$ and $\chi(\Sigma_1) = \chi(\Sigma_2) = 0$.

$$\begin{aligned}
\sum_{j\in J} \left(B_j \times C_{\nu(B_j)\tau(B_j)\sigma(S)}\right) &= \sum_{j\in J_1} \left(B_j \times C_{\tau(B_j)\sigma(S)}\right) + \sum_{j\in J_2} \left(B_j \times C_{-\tau(B_j)\sigma(S)}\right) \\
&= \sum_{j\in J_1} \left((B_j)_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)}\right) + \sum_{j\in J_2} \left((B_j)_{\sigma(C)} \times_h c^{-\sigma(C)\sigma(S)}\right) \\
&= \left(\sum_{j\in J_1} B_j\right)_{\sigma(C)} \times_h c^{\sigma(C)\sigma(S)} + \left(\sum_{j\in J_2} B_j\right)_{\sigma(C)} \times_h c^{-\sigma(C)\sigma(S)} \\
&\stackrel{\text{Lemma 3.3}}{=} (\Sigma_1)_{\sigma(C)} \times_h C_{\sigma(S)} + (\Sigma_2)_{\sigma(C)} \times_h C_{\sigma(S)} \\
&= S_{\sigma(C)} \times_h C_{\sigma(S)} = S \times C.
\end{aligned}$$

□

Theorem 4.2. *If $S \in \mathbb{T}$, $C \in \mathbb{D}\backslash(\mathbb{T} \cup \mathbb{R})$ and there exist index subsets $I_i \subseteq I$, $i = 1, 2, 3$ and $J_j \subseteq J$, $j = 1, 2$, some of them possibly empty sets, $\bigcup_{i=1}^{3} I_i = I$, $\bigcap_{i=1}^{3} I_i = \emptyset$, $\bigcup_{j=1}^{2} J_i = J$, $\bigcap_{j=1}^{2} J_i = \emptyset$, such that* • $\sum_{i\in I_1} A_i = \Sigma_1 = 0$, • $\Sigma_2 = \sum_{i\in I_2} A_i$, $\chi(\Sigma_2) = 0$,

- $\Sigma_4=\sum_{j\in J_1} B_j$, $\chi(\Sigma_4)=0$, $\Sigma_5=\sum_{i\in I_3} A_i+\sum_{j\in J_1} B_j$, $\chi(\Sigma_5)=0$,

then the next distributive relation holds true.

$$\left(\sum_{i\in I} A_i+\sum_{j\in J} B_j\right)\times C \;=\; \sum_{i\in I_1}\left(A_i\times C_{\sigma(A_i)}\right) \;+\; \sum_{i\in I_2}\left(A_i\times C_{\sigma(A_i)\nu(\Sigma_2)\tau(S)}\right) \;+$$
$$\left(\sum_{i\in I_3}\left(A_i\times C_{\sigma(A_i)\nu(\Sigma_5)\tau(S)}\right) \;+\sum_{j\in J_1}\left(B_j\times C_{\nu(B_j)\tau(B_j)\nu(\Sigma_5)\tau(S)}\right)\right) \;+\sum_{j\in J_2}\left(B_j\times C_{\tau(B_j)\tau(S)}\right).$$

Proof. Let the assumptions of the theorem hold true, then

$$\begin{aligned}
&\sum_{i\in I_1}\left(A_i\times C_{\sigma(A_i)}\right) \;+\; \sum_{i\in I_2}\left(A_i\times C_{\sigma(A_i)\nu(\Sigma_2)\tau(S)}\right) \;+\; \sum_{j\in J_2}\left(B_j\times C_{\tau(B_j)\tau(S)}\right)\\
&= \sum_{i\in I_1}\left((A_i)_{\sigma(C)}\times_h C\right) \;+\; \sum_{i\in I_2}\left((A_2)_{\sigma(C)}\times_h C_{\nu(\Sigma_2)\tau(S)}\right) \;+\; \sum_{j\in J_2}\left((B_j)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)}\right)\\
&= \left(\sum_{i\in I_1} A_i\right)_{\sigma(C)}\times_h C \;+\; \left(\sum_{i\in I_2} A_i\right)_{\sigma(C)}\times_h C_{\nu(\Sigma_2)\tau(S)} \;+\; \left(\sum_{j\in J_2} B_j\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)}\\
&\overset{\text{Lemma 3.3}}{=} 0 \;+\; \left(\sum_{i\in I_2} A_2\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)} \;+\; \left(\sum_{j\in J_2} B_j\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)}\\
&= \left(\sum_{i\in I_1\cup I_2} A_i\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)} \;+\; \left(\sum_{j\in J_2} B_j\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)}.
\end{aligned}\tag{46}$$

Because $\chi(\Sigma_4)=0=\chi(\Sigma_5)$, then $\nu(\Sigma_4)\neq\nu(\Sigma_5)$ and

$$\begin{aligned}
&\left(\sum_{i\in I_3}\left(A_i\times C_{\sigma(A_i)\nu(\Sigma_5)\tau(S)}\right) \;+\; \sum_{j\in J_1}\left(B_j\times C_{\nu(B_j)\tau(B_j)\nu(\Sigma_5)\tau(S)}\right)\right)\\
&= \left(\sum_{i\in I_3} A_i\right)_{\sigma(C)}\times_h C_{\nu(\Sigma_5)\tau(S)} \;+\; \left(\sum_{j\in J_1} B_j\right)_{\sigma(C)}\times_h c^{-\sigma(C)\tau(S)}\\
&\overset{\text{Lemma 3.3}}{=} \left(\sum_{i\in I_3} A_i\right)_{\sigma(C)}\times_h C_{\nu(\Sigma_5)\tau(S)} \;+\; \left(\sum_{j\in J_1} B_j\right)_{\sigma(C)}\times_h C_{\nu(\Sigma_5)\tau(S)}\\
&= \left(\sum_{i\in I_3} A_i+\sum_{j\in J_1} B_j\right)_{\sigma(C)}\times_h C_{\nu(\Sigma_5)\tau(S)}\\
&\overset{\text{Lemma 3.3}}{=} \left(\sum_{i\in I_3} A_i+\sum_{j\in J_1} B_j\right)_{\sigma(C)}\times_h c^{\sigma(C)\tau(S)}.
\end{aligned}\tag{47}$$

From (46) and (47) we get the proof of the theorem. □

Theorem 4.3. *If $S \in \mathbb{D}\backslash\mathbb{T}$, $C \in \mathbb{T}\backslash\{0\}$ and there exist index sets $I, J, J_i \subseteq J$, $i = 1,2,3$, some of them but not all possibly empty sets, such that $\bigcup_{i=1}^{3} J_i = J$, $\bigcap_{i=1}^{3} J_i = \emptyset$ and*

- $\sum_{j\in J_1} b_j^{\tau(C)\sigma(S)} = 0$ *(that is $\chi(\sum_{j\in J_1} B_j) = 0$ and $\nu(\sum_{j\in J_1} B_j) \neq \tau(C)\sigma(S)$),*
- *for all $j \in J_2$ $\chi(C) \leq \chi(B_j)$ and either $\nu(B_j) = \tau(C)\sigma(S)$, or $\nu(B_j) \neq \tau(C)\sigma(S)$, $\chi(\sum_{j\in J_2} B_j) = 0$,*
- *for all $j \in J_3$ $\tau(C) = \tau(B_j)$, $\tau(C)\sigma(S) = \nu(\sum_{j\in J_3} B_j)$, $\chi(B_j) \leq \chi(\sum_{j\in J_3} B_j) = \chi(C)$,*

then the following distributive relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= \sum_{i\in I}\left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{j\in J_1}\left(B_j \times C_{-\tau(C)\tau(B_j)}\right)\\ &\quad + \sum_{j\in J_2\cup J_3}\left(B_j \times C_{\tau(C)\tau(B_j)}\right)\\ &= \sum_{i\in I}\left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{j\in J_2\cup J_3}\left(B_j \times C_{\tau(C)\tau(B_j)}\right).\end{aligned}$$

Proof. The proof goes analogously to the proofs of the previous Theorems. □

Theorem 4.4. *If $S \in \mathbb{T}$, $C \in \mathbb{T}\backslash\{0\}$, $\tau(C) \neq \tau(S)$ and there exist index sets $I, J, I_k \subseteq I$, $J_k \subseteq J$, $k = 1,2$, some of them but not all possibly empty sets, such that $J_1 \cup J_2 = J$, $J_1 \cap J_2 = \emptyset$, $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \emptyset$, and*

- $\Sigma_1 = \sum_{i\in I_1} A_i$ *is such that $(\Sigma_1 = 0 \ \vee\ \chi(\Sigma_1) = 0)$;*
- $\Sigma_2 = \sum_{i\in I_2} A_i + \sum_{j\in J_2} B_j$ *is such that $(\Sigma_2 = 0 \ \vee\ \chi(\Sigma_2) = 0)$ and for all $j \in J_2$ $\nu(B_j) \neq \nu(\Sigma_2)$, $\chi(C) \leq \chi(B_j)$,*

then the following distributive relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= 0\\ &= \sum_{i\in I_1}\left(A_i \times C_{\sigma(A_i)\nu(\Sigma_1)\tau(S)}\right) + \sum_{j\in J_1}\left(B_j \times C_{\tau(B_j)\tau(S)}\right) +\\ &\quad \sum_{i\in I_2}\left(A_i \times C_{\sigma(A_i)\nu(\Sigma_2)\tau(S)}\right) + \sum_{j\in J_2}\left(B_j \times C_{-\tau(B_j)\tau(S)}\right).\end{aligned}$$

Theorem 4.5. *If $S \in \mathbb{T}$, $C \in \mathbb{T}\backslash\{0\}$, $\tau(C) \in \mathcal{T}(S)$, $\chi(C) \leq \chi(S)$ and there exist index sets I, J, $J_k \subseteq J$, $k = 1,2,3$, $\bigcup_{k=1}^{3} J_k = J$, $\bigcap_{k=1}^{3} J_k = \emptyset$, some of them but not all possibly empty sets, such that*

- *for all $j \in J_1$, $\chi(C) \leq \chi(B_j)$, $\left(\sum_{j\in J_1} B_j = 0 \ \vee\ \nu(B_j) = \nu(S)\right)$;*
- *for all $j \in J_2$, $\chi(B_j) \leq \chi(\Sigma_2) = \chi(C)$, wherein $\sum_{j\in J_2} B_j = \Sigma_2 \in \mathbb{T}$ and either $\Sigma_2 = 0$ or $\tau(\Sigma_2) = \tau(S)$, $\nu(\Sigma_2) = \nu(S)$;*

- $\sum_{j\in J_3} B_j = \Sigma_3 \in \mathbb{T}$ *and for all* $j \in J_3$, $\nu(B_j) \neq \nu(S)$, $(\Sigma_3 = 0 \ \vee \ \chi(\Sigma_3) = 0)$,

then the following distributive relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= \sum_{i\in I}\left(A_i \times C_{\sigma(A_i)\nu(S)\tau(S)}\right) + \sum_{j\in J_1\cup J_2}\left(B_j \times C_{\nu(B_j)\tau(B_j)\nu(S)\tau(S)}\right) \\ &\quad + \sum_{j\in J_3}\left(B_j \times C_{-\tau(B_j)\tau(S)}\right) \\ &= \sum_{i\in I}\left(A_i \times C_{\sigma(A_i)\nu(S)\tau(S)}\right) + \sum_{j\in J_1\cup J_2}\left(B_j \times C_{\nu(B_j)\tau(B_j)\nu(S)\tau(S)}\right).\end{aligned}$$

Theorem 4.6. *If* $S \in \mathbb{T}$, $C \in \mathbb{T}\backslash\{0\}$, $\tau(C) \in \mathcal{T}(S)$, $\chi(S) < \chi(C)$ *and there exist index sets* I, J_1, J_2, *some of them but not all possibly empty sets, such that* $J_1 \cup J_2 = J \neq \emptyset$, $J_1 \cap J_2 = \emptyset$, *and*

- $\sum_{i\in I} A_i = \Sigma_1 \in \mathbb{T}$ *is such that* $\Sigma_1 = 0$ *or* $\chi(C) = \chi(\Sigma_1)$, $\tau(C) = \tau(\Sigma_1)$;
- *for all* $j \in J_1$, $\chi(B_j) \leq \chi(C)$;
- $\sum_{j\in J_2} B_j = \Sigma_2 \in \mathbb{T}$ *is such that* $\Sigma_2 = 0$ *or for all* $j \in J_2$, $\chi(C) = \chi(\Sigma_2) \leq \chi(B_j)$, $\tau(C) = \tau(\Sigma_2)$, $\nu(B_j) = \lambda \in \Lambda$,

then the following distributive relation holds true

$$\begin{aligned}\left(\sum_{i\in I} A_i + \sum_{j\in J} B_j\right) \times C &= \sum_{i\in I}\left(A_i \times C_{\sigma(A_i)\nu(\Sigma_1)\tau(S)}\right) + \sum_{j\in J_1}\left(B_j \times C_{\tau(B_j)\tau(S)}\right) \\ &\quad + \sum_{j\in J_2}\left(B_j \times C_{\tau(B_j)\tau(S)}\right).\end{aligned}$$

Note 10. *For fixed nonempty sets* I, $I_k \subseteq I$ *and/or* J, $J_m \subseteq J$, *the corresponding conditions, defined by Theorems 4.1–4.6, are necessary and sufficient for the validity of the corresponding distributive relation.*

Lemma 4.7. *For any* $A = [a^-, a^+] \in \mathbb{T} \setminus \{0\}$, *such that* $\chi(A) \neq 0$,

(1) there exist $A', A'' \in \mathbb{T}\backslash\{0\}$, *such that* $\chi(A') = \chi(A'') = 0$ *and* $A = A' + A'' = [a^-, 0] + [0, a^+]$;

(2) if $A' = [a^-, 0]$, $A'' = [0, a^+]$ *and* $A = A' + A''$, *then* $\nu(A')\tau(A') = \nu(A'')\tau(A'') = \tau(A)$.

By Lemma 4.7, Theorem 3.11 and Theorems 4.1–4.6, the next Corollaries 4.8–4.10 give more precise representation of the corresponding cases of Theorem 2.1.

Corollary 4.8. *For* $S, C \in \mathbb{D}\backslash\mathbb{T}$ *and the notations of Theorem 2.1*

$$\left(\sum_{i=1}^{n} A_i\right) \times C = \sum_{A_i\in\mathbb{D}\backslash\mathbb{T}}\left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{A_i\in\mathbb{T}\backslash\{0\}}\left([a_i^-, 0] \times C_{\tau(A_i)\sigma(S)} + [0, a_i^+] \times C_{\tau(A_i)\sigma(S)}\right).$$

By Lemma 4.7, as a consequence of Theorem 4.3, we obtain the following more precise representation of the corresponding case of Theorem 2.1.

Corollary 4.9. *For* $\sum_{i=1}^{n} A_i = S \in \mathbb{D}\backslash\mathbb{T}$ *and* $C \in \mathbb{T}\backslash\{0\}$

$$\left(\sum_{i=1}^{n} A_i\right) \times C = \sum_{A_i \in \mathbb{D}\backslash\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\sigma(S)}\right) + \sum_{A_i \in I_1} \left(A_i \times C_{-\tau(C)\tau(A_i)}\right) +$$

$$\sum_{A_i \in I_2 \cup I_3} \left(A_i \times C_{\tau(C)\tau(A_i)}\right) + \sum_{\substack{A_i \in \mathbb{T}\backslash\{0\} \\ A_i \notin I_1 \cup I_2 \cup I_3}} \left([a_i^{\tau(C)\sigma(S)}, 0]_{-\tau(C)\sigma(S)} \times C_{\tau(A_i)\sigma(S)}\right),$$

wherein $I_k,\ k = 1,2,3$ *are such that for all* $i \in I_k,\ \ A_i \in \mathbb{T}\backslash\{0\}$ *and* A_i *for* $i \in I_k$ *satisfy the conditions defined for the corresponding* $J_k,\ k = 1,2,3$ *in Theorem 4.3.*

By Lemma 4.7, as a consequence of Theorem 4.5, we obtain the following more precise representation of the corresponding case of Theorem 2.1.

Corollary 4.10. *For* $\sum_{i=1}^{n} A_i = S \in \mathbb{T}$ *and* $C \in \mathbb{T}\setminus\{0\}$*, such that* $\tau(C) = \tau(S)$ *and* $\chi(C) \le \chi(S)$

$$\left(\sum_{i=1}^{n} A_i\right) \times C = \sum_{A_i \in \mathbb{D}\backslash\mathbb{T}} \left(A_i \times C_{\sigma(A_i)\nu(S)\tau(S)}\right) + \sum_{i \in I_1 \cup I_2} \left(A_i \times C_{\nu(A_i)\tau(A_i)\nu(S)\tau(S)}\right)$$

$$+ \sum_{i \in I_3} \left(A_i \times C_{-\tau(A_i)\tau(S)}\right) + \sum_{\substack{\text{all else} \\ A_i \in \mathbb{T}\backslash\{0\}}} \left([0, a_i^{\nu(S)}]_{\nu(S)} \times C_{\tau(A_i)\nu(S)\tau(S)}\right).$$

wherein $I_k,\ k = 1,2,3$ *are such that for all* $i \in I_k,\ \ A_i \in \mathbb{T}\backslash\{0\}$ *and* A_i *for* $i \in I_k$ *satisfy the conditions defined for the corresponding* $J_k,\ k = 1,2,3$ *in Theorem 4.5.*

The application of Theorem 2.1 can be combined with Theorem 4.2 or with Theorem 4.6 analogously.

5 Equivalent Conditionally Distributive Relations

Various different forms of the conditionally distributive rela can be derived from Theorem 3.11 and the next Theorem.

Theorem 5.1. *Let* $A_1, A_2, C \in \mathbb{D}\backslash\{0\}$, $A_1 + A_2 = S \in \mathbb{D}$ *and either* $\mu(\cdot) = \hat{\mu}(\cdot)$*, or* $\mu(\cdot) = \tilde{\mu}(\cdot)$*, as defined in the beginning of Section 3.2. Denote by (C1) the conditions under which the relations, defined by Theorem 3.11, hold true. The relation*

$$(A_1 + A_2) \times C = A_1 \times C_{\mu(A_1)\mu(S)} + A_2 \times C_{\mu(A_2)\mu(S)}, \qquad (R1)$$

that is the corresponding relation, defined by Theorem 3.11, ***is equivalent*** *to the relation*

$$(A_1 + A_2) \times C_{\mu(S)} = A_1 \times C_{\mu(A_1)} + A_2 \times C_{\mu(A_2)}, \qquad (R2)$$

which is valid iff the conditions (C1), with the substitution $\tau(C) \to \mu(S)\tau(C)$*, are satisfied.*

Proof.
Let the relation (R1) holds true iff the conditions (C1) are satisfied.
We shall prove that the relation (R2) holds true iff the conditions (C1), wherein $\tau(C)$ is replaced by $\mu(S)\tau(C)$, are satisfied.
Denote $Y = C_{\mu(S)}$ and consider the relation

$$(A_1 + A_2) \times Y = A_1 \times Y_{\mu(A_1)\mu(S)} + A_2 \times Y_{\mu(A_2)\mu(S)}, \tag{48}$$

which is equivalent to the relation (R2). Now, we shall apply Theorem 3.11 to the relation (48). By Proposition 3.1 we have

$$\begin{aligned} \sigma(Y) &= \sigma(C); \qquad & \tau(Y) &= \mu(S)\tau(C); \\ \chi(Y) &= \chi(C); \qquad & \nu(Y) &= \mu(S)\nu(C). \end{aligned} \tag{49}$$

Then, according to Theorem 3.11, the relation (R2) holds true iff the conditions (C1), wherein $\tau(C)$ is replaced by $\mu(S)\tau(C)$, are satisfied.

Let the relation (R2) holds true iff the conditions (C1), wherein $\tau(C)$ is replaced by $\mu(S)\tau(C)$, are satisfied.
We shall prove that the relation (R1) holds true iff the conditions (C1) are satisfied.
In (R2) denote $Y = C_{\mu(S)}$ and obtain that the relation

$$(A_1 + A_2) \times Y = A_1 \times Y_{\mu(A_1)\mu(S)} + A_2 \times Y_{\mu(A_2)\mu(S)} \tag{50}$$

holds true iff the conditions (C1), wherein $\tau(C)$ is replaced by $\mu(S)\tau(C)$, are satisfied. Now, applying back the substitutions (49) to these conditions, we obtain the conditions (C1) for Y, A_1, A_2, S and the relation (50). □

Note 11. *Obtaining the equivalent relation (R2) from Theorem 3.11 and Theorem 5.1, one has to take into account Note 8.*

The substitutions, under which we get the necessary and sufficient conditions for the validity of the conditionally distributive relation (R2), affect the conditions 3.2–3.4, 5.3.3 as well as the condition 4. ($\tau(C) \neq \tau(S) \to \tau(C) = -$) and the condition 5. ($\tau(C) = \tau(S) \to \tau(C) = +$) from Theorem 3.11.

6 Concluding Remarks

A fundamental role of the conditionally distributive law is to connect the additive and the multiplicative groups of generalized intervals. The generalized conditionally distributive law can be also of particular theoretical interest for a more complete characterization of the distributive relations in the extended interval space involving inner (nonstandard) operations [3, 9].

The application of the generalized conditionally distributive law concerns development of some numerical methods involving proper and improper intervals, explicit solution of classes of interval algebraic equations, as well as development of a methodology for true symbolic-algebraic interval computations. Even not generally valid, the distributive law for generalized intervals turned out to be an indispensable tool for the reduction of interval algebraic equations, with multi-incidence on the unknown variable, to simpler ones. The latter would be helpful for the explicit solution of the corresponding equation and/or for the reduction of the round-off errors due to the reduced number of arithmetic operations in the simplified equation. An application of interval distributive relations for finding general normal form of pseudo-linear interval expressions and equations is discussed in [16].

The existence of inverse additive and multiplicative elements in the space of generalized intervals makes it possible to find algebraic solution to certain types interval equations just by applying elementary algebraic transformations on these equations. Such equations usually come from real-life practical problems, where modelling equations involve multiple occurrences of the interval parameters. Applying a theorem for eliminating the dependency problem [4] often leads to interval equations in generalized interval arithmetic. The validity of a conditionally distributive law in this space considerably extends the class of interval algebraic equations that can be solved explicitly. Some examples, illustrating the combined application of properties (4) and the distributive relations for solving interval algebraic equations in one variable, can be found in [1, 17].

The algebraic and distributive-like properties of generalized interval arithmetic can be easy and effectively exploited in the environment of a computer algebra system [1]. The computer algebra implementation of the generalized interval distributive relations provides automatic simplification of symbolic-numerical interval expressions [17]. An advanced methodology for symbolic algebraic interval computations involves explicit algebraic transformations on interval formulae, automatic simplification of interval expressions and algebraic solutions to interval equations. This way, the computer algebra tools for generalized interval arithmetic and symbolic-algebraic manipulations [1, 17] greatly facilitate the application and the computations with generalized intervals, which otherwise may seem too complicated.

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