

STRONG REGULARITY OF PARAMETRIC INTERVAL MATRICES

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We define strong regularity of a parametric interval matrix and give conditions that characterize it. The new conditions give a better estimation for regularity of a parametric matrix than the conditions used so far. Verifiable sufficient regularity conditions are also presented for parametric matrices. The new sufficient conditions motivate a generalization of Rump's parametric fixed-point iteration method.

1. Introduction. Regularity of interval matrices plays an important role in the theory of linear interval equations. In practical computations we usually use verifiable sufficient conditions for regularity of interval matrices. In order to cover a possibly wide class of interval matrices, it is recommendable to have more such conditions, since some conditions may be better suited for specific classes of interval matrices. In this paper we consider strong regularity of a parameter-dependent interval matrix as a sufficient condition for its regularity, which is important for the verification methods for solving parametric interval linear systems.

Consider a parametric linear system

$$A(p) \cdot x = b(p).$$

When p varies within a range $[p] \in \mathbb{IR}^k$, the parametric solution set, is

$$\Sigma(A(p), b(p), [p]) := \{x \in \mathbb{R}^n \mid A(p) \cdot x = b(p) \text{ for some } p \in [p]\}.$$

Denote by $A([p]) := \square\{A(p) \in \mathbb{R}^{n \times n} \mid p \in [p]\}$, $b([p]) := \square\{b(p) \in \mathbb{R}^n \mid p \in [p]\}$

the non-parametric interval matrix, resp. interval vector, that correspond and are obtained from the parametric ones. Hence, $A([p]) \cdot x = b([p])$ is the non-parametric interval linear system corresponding to the parametric one. The non-parametric solution set

$$\Sigma(A([p]), b([p])) := \{x \in \mathbb{R}^n \mid A \cdot x = b \text{ for some } A \in A([p]), b \in b([p])\}$$

corresponding to the parametric one usually has much bigger volume than the latter.

We shall use the following notations. For an interval matrix $[A] = [\underline{A}, \bar{A}] = \{A \mid \underline{A} \leq A \leq \bar{A}\}$, denote the mid-point matrix by $\check{A} = \frac{1}{2}(\underline{A} + \bar{A})$ and the radius matrix by $\text{rad}([A]) = \Delta = \frac{1}{2}(\bar{A} - \underline{A})$. Then an interval matrix can be written also as $[A] = [\check{A} - \Delta, \check{A} + \Delta]$. The absolute value of a matrix $A = (a_{ij})$ is denoted by $|A| = (|a_{ij}|)$.

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For two matrices of the same size matrix (vector) inequalities $A \leq B$ and the interval subset relations $[A] \subseteq [B]$ are understood componentwise. $A < B$ if $A \leq B$ and $A \neq B$, analogously $[A] \subset [B]$ if $[A] \subseteq [B]$ and $[A] \neq [B]$. The above matrix notations apply to vectors, considered as one-column matrices, as well. For $\Sigma \subseteq \mathbb{R}^n$, define $\square : \mathbb{R}^n \rightarrow \mathbb{IR}^n$

$$\square(\Sigma) := [\inf \Sigma, \sup \Sigma] = \cap \{[x] \in \mathbb{IR}^n \mid \Sigma \subseteq [x]\}.$$

$\varrho(A)$ is the spectral radius of a matrix A . I denotes the unit matrix. An interval matrix $[A]$ is called regular if each $A \in [A]$ is nonsingular. A parameter-dependent matrix $A(p)$ is called regular over a box $[p] \in \mathbb{IR}^k$ if $A(\tilde{p})$ is regular for all $\tilde{p} \in [p]$. Since $A(p) \in A([p])$ for all $p \in [p]$, regularity of $A([p])$ presents a sufficient condition for regularity of $A(p)$ over a given $[p] \in \mathbb{IR}^k$. Regularity of $A([p])$ is only a sufficient condition since a parametric matrix may be regular while the corresponding nonparametric matrix may not, as it is shown by the following example.

Example 1. Consider the parametric linear system defined by [3]

$$A(p) = \begin{pmatrix} 2p & 1 \\ -1 & 2p-1 \end{pmatrix}, \quad b(p) = \begin{pmatrix} 2p \\ 2 \end{pmatrix}, \quad p \in [0, 1].$$

The parametric solution set (dashed arch curve) is presented on Fig. 1. The corresponding non-parametric matrix contains a singular matrix $\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$ and the non-parametric solution set is unbounded and disconnected.

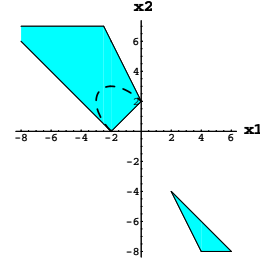


Fig. 1

A well-known sufficient condition for regularity of an interval matrix is its strong regularity, introduced by Neumaier [5]. An interval matrix $[A]$ is strongly regular if $\check{A}^{-1}[A]$ is regular. Hence, strong regularity of the corresponding non-parametric matrix is another sufficient condition for regularity of the parametric matrix. Moreover, all verification methods for solving interval linear systems require strong regularity. In particular, Rump's **parametric** iteration method [7] requires strong regularity of $A([p])$.

2. Strongly regular parametric matrices. Following the motivation of A. Neumaier [5] for introduction of the class of strongly regular non-parametric interval matrices, we define a class of strongly regular parametric matrices.

Definition 1. An $n \times n$ parametric matrix $A(p)$ with $p \in [p] \in \mathbb{IR}^k$ is called strongly regular if either of the following two matrices is regular

$$(1) \quad [B] := \square \{A^{-1}(\check{p})A(p) \mid p \in [p]\}, \quad [B'] := \square \{A(p)A^{-1}(\check{p}) \mid p \in [p]\}.$$

Theorem 1. Let $A(p)$ be an $n \times n$ parametric matrix with $p \in [p] \in \mathbb{IR}^k$. Suppose that $A(\check{p})$ is regular and $[B], [B']$ are defined by (1). Then the following conditions are equivalent:

- (i) $A(p)$ is strongly regular for $p \in [p]$;
- (ii) $[B]$ is regular or $[B']$ is regular;
- (iii) $\varrho(\text{rad}([B])) < 1$ or $\varrho(\text{rad}([B'])) < 1$;
- (iv) $\|I - [B]\|_u < 1$ for some $u > 0$ or $\|I - [B']\|_v < 1$ for some $v > 0$;
- (v) $[B]$ is a H -matrix or $[B']$ is a H -matrix.

Proof. The proof follows from [5], Proposition 4.1.1. with $[B]$ instead of $\check{A}^{-1}[A]$. Here we give the proof for $[B']$. The conditions for both $[B]$ and $[B']$ hold true if $A(p)$ is

symmetric, that is $A(p) = A^\top(p)$ for all $p \in [p]$.

(i) and (ii) are equivalent by Definition 1.

(ii) \Rightarrow (iii) Let us assume to the contrary that $\varrho(\text{rad}([B'])) \geq 1$. According to [5, Cor.3.2.3, (6)] if $0 \leq A \in \mathbb{R}^{n \times n}$ and $0 < \alpha \in \mathbb{R}$, then $\varrho(A) \geq \alpha \Leftrightarrow \exists u > 0 : Au \geq \alpha u \neq 0$. Hence, $\exists 0 \neq u \geq 0$ with $\text{rad}([B'])u \geq u \neq 0$. By [5, Cor.3.4.5, (iii)] $[A] = [\check{A} \pm \text{rad}([A])]$ is regular $\Leftrightarrow \tilde{x} \in \mathbb{R}^n$, $|\check{A}\tilde{x}| \leq \text{rad}([A])|\tilde{x}| \Rightarrow \tilde{x} = 0$. With $\tilde{x} = u$, due to $\check{B}' = I$, it follows that $Iu \leq \text{rad}([B'])u \Rightarrow u = 0$ which contradicts to $u \neq 0$. Hence, if $[B']$ is regular, we must have $\varrho(\text{rad}([B'])) < 1$.

(iii) \Rightarrow (iv) Since $\varrho(\text{rad}([B'])) < 1$, by [5, 3.2.3 (5)] (if $0 \leq A \in \mathbb{R}^{n \times n}$ and $0 < \alpha \in \mathbb{R}$, then $\varrho(A) < \alpha \Leftrightarrow \exists u > 0 : Au < \alpha u$) we have $\text{rad}([B'])u < u$ for some $u > 0$. However, $\text{rad}([B'])u = |\check{B}' - [B']|u = |I - [B']|u$, which gives (iv).

(iv) \Rightarrow (v) by [5, Proposition 3.7.2];

(v) \Rightarrow (ii) by [5, Proposition 3.7.5]. ■

Next Theorem follows from some well-known results for nonnegative matrices [1].

Theorem 2. Let $A(p)$ with $p \in [p]$ be a parametric matrix. Suppose that $[B]$ is defined by (1) and $\check{A} := A(\check{p})$ is regular. Then (i) $\varrho(\text{rad}([B])) \leq \varrho(\text{rad}(\check{A}^{-1}A([p])))$; (ii) if $[B] \subset \check{A}^{-1}A([p])$ and $\text{rad}([B]) + \text{rad}(\check{A}^{-1}A([p]))$ is irreducible then $\varrho(\text{rad}([B])) < \varrho(\text{rad}(\check{A}^{-1}A([p])))$.

The relations specified by Theorem 2 hold also for $[B']$ and $A([p])\check{A}^{-1}$. Theorem 2 says that the conditions for strong regularity of a parametric interval matrix give better estimations for the regularity of a parametric interval matrix than the conditions based on the corresponding non-parametric matrix.

3. Affine-linear dependencies. The results from the previous section are valid for matrices involving arbitrary affine-linear dependencies of the parameters. In what follows we give some details for parametric matrices whose elements are affine-linear functions of some parameters and present some examples. An $n \times n$ parametric matrix $A(p)$ involving affine-linear dependencies of a parameter vector $p \in \mathbb{R}^k$ can be represented as

$$(2) \quad A(p) = A^{(0)} + p_1 A^{(1)} + \dots + p_k A^{(k)}, \quad \text{where } A^{(i)} \in \mathbb{R}^{n \times n}, \quad i = 0, 1, \dots, k.$$

Proposition 1. Let $A(p)$ be an $n \times n$ parametric matrix involving affine-linear dependencies of the parameter vector $p \in [p] \in \mathbb{IR}^k$. Suppose that $A(\check{p})$ is regular, then for $[B]$

$$\text{defined by (1) we have: (i) } [B] = A^{-1}(\check{p})A^{(0)} + \sum_{\nu=1}^k [p_\nu] (A^{-1}(\check{p})A^{(\nu)})$$

$$(ii) \quad \text{mid}([B]) = I \text{ and } \text{rad}(B) = \sum_{\nu=1}^k \text{rad}([p_\nu]) |A^{-1}(\check{p})A^{(\nu)}|.$$

Proof. (i) Denote $\check{A} = A(\check{p})$.

$$\begin{aligned} [B] &:= \square\{\check{A}^{-1}A(p) \mid p \in [p]\} = \square\{\check{A}^{-1}A^{(0)} + \sum_{\nu=1}^k \check{A}^{-1}(p_\nu A^{(\nu)}) \mid p \in [p]\} \\ &= \square\{\check{A}^{-1}A^{(0)} + \sum_{\nu=1}^k p_\nu (\check{A}^{-1}A^{(\nu)}) \mid p \in [p]\} = \check{A}^{-1}A^{(0)} + \sum_{\nu=1}^k [p_\nu] (\check{A}^{-1}A^{(\nu)}). \end{aligned}$$

The last equality holds since every interval parameter p_ν occurs at most once (and to the first power) in the previous matrix expression [4, page 23].

(ii) For $\nu = 1, \dots, k$, there exist $0 < \delta_\nu \in \mathbb{R}$ such that $[p_\nu] = [\check{p} - \delta_\nu, \check{p} + \delta_\nu]$. Then, denoting $\check{A} = A(\check{p})$, we have

$$\begin{aligned} [B] &= \check{A}^{-1}A^{(0)} + \sum_{\nu=1}^k \check{p}_\nu \left(\check{A}^{-1}A^{(\nu)} \right) + \sum_{\nu=1}^k [-\delta_\nu, \delta_\nu] \left(\check{A}^{-1}A^{(\nu)} \right) \\ &= \check{A}^{-1} \left(A^{(0)} + \sum_{\nu=1}^k \check{p}_\nu A^{(\nu)} \right) + \sum_{\nu=1}^k [-\delta_\nu, \delta_\nu] \left(\check{A}^{-1}A^{(\nu)} \right). \end{aligned}$$

Hence $\text{mid}([B]) = I$ and $\text{rad}(B) = \sum_{\nu=1}^k \text{rad}([p_\nu]) |A^{-1}(\check{p})A^{(\nu)}|$. ■

Next Proposition establishes a relation between a preconditioned non-parametric interval matrix $\check{A}^{-1}A([p])$ and the interval hull of the preconditioned parametric matrix.

Proposition 2. *Let $A(p)$ be an $n \times n$ parametric matrix with $p \in [p] \in \mathbb{IR}^k$ and $A([p])$ be the corresponding non-parametric interval matrix. Suppose that $A(\check{p})$ is regular and $[B]$ is defined by (1). Then $[B] \subseteq A^{-1}(\check{p}) \cdot A([p])$.*

Proof. Denote $[A] := A([p])$ and $\check{A} = \text{mid}(A([p]))$. It is obvious that $\check{A} = A(\check{p})$.

$$\check{A}^{-1}[A] = \check{A}^{-1}A^{(0)} + \check{A}^{-1} \sum_{\nu=1}^k [p_\nu]A^{(\nu)} = \check{A}^{-1}A^{(0)} + \sum_{\nu=1}^k [p_\nu] \left(\check{A}^{-1}A^{(\nu)} \right).$$

For every $\nu = 1, \dots, k$, $\check{A}^{-1}([p_\nu]A^{(\nu)})$ is an inclusion monotonic interval extension of $g(p_\nu) := \check{A}^{-1}(p_\nu A^{(\nu)})$ and $\square\{\check{A}^{-1}(p_\nu A^{(\nu)}) \mid p_\nu \in [p_\nu]\} = [p_\nu] (\check{A}^{-1}A^{(\nu)})$. Then by Proposition 1, (i) and a Theorem by Moore [4, Th.3.1], we have $[B] \subseteq A^{-1}(\check{p})A([p])$. ■

From Proposition 2 it follows that $\text{rad}([B]) \leq \text{rad}(A^{-1}(\check{p})A([p]))$.

Premultiplying $A(p)$ by $\check{A}^{-1} \in \mathbb{R}^{n \times n}$ introduces linear transformations on the columns of $A(p)$. Next we define a class of parametric matrices for which the preconditioning is effective in the sense of Theorem 2.

Definition 2. *A parametric matrix $A(p) \in \mathbb{R}^{n \times n}$, defined by (2), with a parameter vector $p \in [p] \in \mathbb{IR}^k$, is called column-dependent parametric matrix if for some $m \in \{1, \dots, k\}$ and some $j \in \{1, \dots, n\}$ $\text{Card}(\mathcal{I}) \geq 2$, where $\mathcal{I} := \{i \mid 1 \leq i \leq n, a_{ij}^{(m)} \neq 0\}$.*

Example 2. Consider Milnes matrix $M(p)$, defined by [2]

$$m_{ij}(p) = \begin{cases} p_j & \text{if } i > j, \\ 1 & \text{otherwise,} \end{cases} \quad \text{with } p_i \in [i + 1 \pm 10\%], \quad i, j = 1, \dots, n.$$

Denote the corresponding non-parametric matrix by $[M]$ and by $[B] := \square\{\check{M}^{-1}M(p) \mid p \in [p]\}$. Although $\text{rad}([B]) < \text{rad}(\check{M}^{-1}[M])$, $\text{rad}([B]) + \text{rad}(\check{M}^{-1}[M])$ is not irreducible and $\varrho(\text{rad}([B])) = \varrho(\text{rad}(\check{M}^{-1}[M])) = 0.2$.

Example 3. In [5], Example 3.4.6, non-parametric interval matrices $A(t, p)$ defined by $a_{ii} = t$, $a_{ij} = p$, $1 \leq i, j \leq n$ are proven to be regular for $p = [0, 2]$ and $0 \leq t \in \mathbb{R}$ such that $t > n$ for n even, and $t > (n^2 - 1)^{1/2}$ for n odd. It can be easily verified that for $n = 3$ e.g., $A(3, [0, 2])$ is not strongly regular (see [4, Ex.4.1.8]) with $\varrho(\text{rad}(\check{A}^{-1}A(3, [0, 2]))) = 1.2$, while $A(3, p)$ with $p \in [0, 2]$ is strongly regular with $\text{rad}([B]) = 0.8$.

Example 4. Define $n \times n$ parametric matrices $Q(p)$ by

$$q_{ij}(p) = \begin{cases} p_j & \text{if } j \geq i, \\ 0 & \text{if } j = i - 2, \text{ with } p_j \in [j + 1 \pm 10\%], \quad 1 \leq i, j \leq n. \\ 1 & \text{otherwise.} \end{cases}$$

Since $Q(p)$ is column-dependent matrix, with $[B] = \square\{\check{Q}^{-1}Q(p) \mid p \in [p]\}$, $\text{rad}([B]) < \text{rad}(\check{Q}^{-1}Q([p]))$. It happens that $\text{rad}([B]) + \text{rad}(\check{Q}^{-1}Q([p]))$ is irreducible and for $3 < n \leq 50$, $1 < \varrho(\text{rad}(\check{Q}^{-1}Q([p])))$ increases with n , while $\varrho(\text{rad}([B]))$ remains equal to 0.2.

4. Verifiable sufficient conditions. Since using the inverse matrix computed in a finite precision arithmetic may affect validity of the above conditions, it is advantageous to formulate them in terms of an approximate inverse R instead of the exact inverse \check{A}^{-1} . The first such formulation for nonparametric linear systems (matrices) is due to S. M. Rump. Next Theorems can be used for the computational verification of the regularity of a parametric interval matrix.

Theorem 3. Let $A(p)$ be an $n \times n$ parametric matrix with $p \in [p] \in \mathbb{IR}^k$. Let $R \in \mathbb{R}^{n \times n}$ be given and let

$$[C] := \begin{cases} I - \square\{A(p) \cdot R \mid p \in [p]\} & \text{if } A^\top(p) \text{ is column-dependent,} \\ I - \square\{R \cdot A(p) \mid p \in [p]\} & \text{otherwise.} \end{cases}$$

For $[x^0] \in \mathbb{IR}^n$ define the iteration $[x^{l+1}] := [C] \diamond [x^l] \diamond [e^l]$ for $l \in \mathbb{N}$, where $[e^l] \in \mathbb{IR}^n$, $[e^l] \rightarrow [e] \in \mathbb{IR}^n$ with $^* 0 \in \text{int}([e])$ and all operations \diamond , \diamond are outwardly-rounded computer interval operations. If $[C] \diamond [x^l] \subseteq \text{int}([x^l])$ for some $l \in \mathbb{N}$, then R and every $A(p)$ with $p \in [p]$ are regular.

In practical applications, especially for large matrices, it may be superior to go from intervals to an absolute value iteration.

Theorem 4. Let $A(p)$ be an $n \times n$ parametric matrix with $p \in [p] \in \mathbb{IR}^k$. Let $R \in \mathbb{R}^{n \times n}$ be given and $0 < x \in \mathbb{R}^n$. Let $C(p) \in \mathbb{R}^{n \times n}$ with $C(p) := |I - \square\{R \cdot A(p) \mid p \in [p]\}|$ and define $x^{(l)}, y^{(l)} \in \mathbb{R}^n$ for $l \geq 0$ by $y_i^{(l)} := \{C(p) \cdot u\}_i$ with $u := (y_1^{(l)}, \dots, y_{i-1}^{(l)}, x_i^{(l)}, \dots, x_n^{(l)})^\top$ and $x^{(l+1)} := y^{(l)} + \varepsilon$ for $1 \leq i \leq n$ and some $0 < \varepsilon \in \mathbb{R}^n$. If $y^{(l)} < x^{(l)}$ for some $l \in \mathbb{N}$, then R and every $A(p)$ with $p \in [p]$ are regular.

Proof. Lemma 1.6 from [7] implies that $\varrho(C(p)) < 1$. Therefore for every $p \in [p]$, $\varrho(I - R \cdot A(p)) \leq \varrho(|I - R \cdot A(p)|) < 1$. Hence R and every $A(p)$, $p \in [p]$ are regular. ■

The verification of the assumptions of Theorem 4 needs only upward rounding. In fact, Theorems 3 and 4 verify strong regularity of $A(p)$ over a box $[p] \in \mathbb{IR}^k$.

5. Conclusion. To our knowledge by now all verification methods for solving parametric interval linear systems require strong regularity of the corresponding non-parametric matrix. In view of Theorem 2, illustrated by the Examples 2 and 3, the new strong regularity conditions give a better regularity estimation for parametric matrices.

We have used the above results to generalize Rump's parametric fixed-point iteration method [7], expanding thus its scope of applicability over the class of column-dependent parametric matrices. The implementation of the generalized method is described in [6].

* $\text{int}([x])$ denotes the topological interior.

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