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Outer enclosures to the parametric AE solution set

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Abstract We consider systems of linear equations, where the elements of the matrix and of the right-hand side vector are linear functions of interval parameters. We study parametric AE solution sets, which are defined by universally and existentially quantified parameters, and the former precede the latter. Based on a recently obtained explicit description of such solution sets, we present three approaches for obtaining outer estimations of parametric AE solution sets. The first approach intersects inclusions of parametric united solution sets for all combinations of the end-points of the universally quantified parameters. Polynomially computable outer bounds for parametric AE solution sets are obtained by parametric AE generalization of a single-step Bauer–Skeel method. In the special case of parametric tolerable solution sets, we derive an enclosure based on linear programming approach; this enclosure is optimal under some assumption. The application of these approaches to parametric tolerable and controllable solution sets is discussed. Numerical examples accompanied by graphic representations illustrate the solution sets and properties of the methods.

Keywords linear systems · dependent data · AE solution set · tolerable solution set · controllable solution set

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1 Introduction

Consider a system of linear algebraic equations

$$A(p)x = b(p) \tag{1}$$

which has a linear uncertainty structure

$$A(p) = A_0 + \sum_{k=1}^{K} A_k p_k, \quad b(p) = b_0 + \sum_{k=1}^{K} b_k p_k, \tag{2}$$

where $A_k \in \mathbb{R}^{n\times n}$, $b_k \in \mathbb{R}^n$, $k = 0, 1, \ldots, K$, and $p = (p_1, \ldots, p_K)$. The parameters p_k, $k \in \mathcal{K} = \{1, \ldots, K\}$ are considered as uncertain and varying within given intervals $\boldsymbol{p}_k = [\underline{p}_k, \overline{p}_k]$. Such systems are common in many engineering analysis or design problems (see Elishakoff and Ohsaki (2010) and the references therein), control engineering (Matcovschi and Pastravanu, 2007; Sokolova and Kuzmina, 2008; Busłowicz, 2010), robust Monte Carlo simulations (Lagoa and Barmish, 2002), etc. Usually, the set of solutions to (1)–(2) which is sought for is the so-called parametric *united* solution set

$$\begin{aligned}\Sigma^p_{uni} &= \Sigma_{uni}(A(p), b(p), \boldsymbol{p})\\ &:= \{x \in \mathbb{R}^n \mid (\exists p \in \boldsymbol{p})(A(p)x = b(p))\}.\end{aligned}$$

The united parametric solution set generalizes the united non-parametric solution set to an interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, which is defined

$$\begin{aligned}\Sigma_{uni} &= \Sigma_{uni}(\boldsymbol{A}, \boldsymbol{b})\\ &:= \{x \in \mathbb{R}^n \mid (\exists A \in \boldsymbol{A})(\exists b \in \boldsymbol{b})(Ax = b)\}.\end{aligned}$$

However, the solutions of many practical problems involving uncertain (interval) data have quantified formulation involving the universal quantifier (∀) besides the existential quantifier (∃). We consider quantified solution sets where all universally quantified parameters precede all existentially quantified ones. Such solution sets are called *AE* solution sets, after Shary (2002). Examples of several mathematical and engineering problems formulated in terms of quantified solution sets can be found, for example, in Shary (2002); Pivkina and Kreinovich (2006); Wang (2008). *AE* solution sets are of particular interest also for interval-valued fuzzy relational equations, see Wang, S. et al. (2003) where the concepts of so-called tolerable and controllable solution sets of interval-valued fuzzy relational equations are introduced, their structure and relations are discussed. The literature on control engineering contains many papers that explore problems related to linear dynamical systems with uncertainties bounded by interval matrices, see, e.g., Sokolova and Kuzmina (2008) and the references in Matcovschi and Pastravanu (2007); Busłowicz (2010). The tolerable solution set is utilized in Sokolova and Kuzmina (2008) for parameter identification problems and in controllability analysis. As in the other problem domains, the uncertain data involve more parameter dependencies than in an interval matrix with independent elements. So, the more realistic approaches consider linear dynamical systems with linear dependencies between state parameters as in Matcovschi and Pastravanu (2007), or structural perturbations of state matrices as in Busłowicz (2010).

Although the non-parametric *AE* solution sets are studied, e.g., in Shary (1995, 1997, 2002); Goldsztejn (2005); Goldsztejn and Chabert (2006); Pivkina and Kreinovich (2006), there are a few results on the more general case of linear parameter dependency. A special case of parametric tolerable solution sets is dealt with in Sharaya and Shary (2011). A characterization of the general parametric solution set is given in Popova and Krämer (2011), and a Fourier–Motzkin type elimination of parameters is applied in Popova (2011) to derive explicit descriptions of the parametric *AE* solution sets.

In this paper we are interested in obtaining outer bounds for the parametric *AE* solution sets. To our knowledge, this is the first systematic approach to outer estimations of parametric *AE* solution sets in their general form. In Section 3 we prove that (inner or outer) estimations of parametric *AE* solution sets can be obtained by using only some corresponding estimations of parametric united solution sets. In Section 4 we generalize a Bauer–Skeel method (see Rohn (2010) and the references therein), applied so far for bounding (parametric) united solution sets. The method is derived in a form which leads to the same conclusion, proven in Section 3, and requires intersecting the bounds of parametric united solution sets for all combinations of the end-points of the universally quantified parameters. Another, single-step single-application parametric Bauer–Skeel *AE* method is derived in Section 5 and both approaches are compared on several numerical examples. The derivation of both forms of Bauer–Skeel parametric *AE* method is self-contained and no knowledge of the original method is required. The special cases of parametric tolerable and controllable solution sets are discussed. In the tolerable case, an enclosure based on linear programming approach is derived in Section 6. Numerical examples accompanied by graphic representations illustrate the solution sets and properties of the methods.

2 Notations

Denote by $\mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real vectors with n components and the set of real $n\times m$ matrices, respectively. A real compact interval is

$$\boldsymbol{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}.$$

As a generalization of real compact intervals, *an interval matrix* $\boldsymbol{A}$ with independent components is defined as a family

$$\boldsymbol{A} = [\underline{A}, \overline{A}] := \{A \in \mathbb{R}^{n\times m} \mid \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{n\times m}$, $\underline{A} \le \overline{A}$, are given matrices. Similarly we define interval vectors. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n\times m$ matrices, respectively. For $\boldsymbol{a} \in \mathbb{IR}$, define *mid-point* $a^c := (\underline{a} + \overline{a})/2$ and *radius* $a^\Delta := (\overline{a} - \underline{a})/2$. These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (2), we assume that $p_k^\Delta > 0$ for all $k \in \mathcal{K}$. The spectral radius of a matrix M is denoted by $\rho(M)$. The identity matrix of dimension n is denoted by I. For a given index set $I = \{i_1, \ldots, i_k\}$ denote $p_I := (p_{i_1}, \ldots, p_{i_k})$. Next, $Card(S)$ denotes the cardinality of a set S. The following definitions are recalled from Popova and Krämer (2011).

Definition 1 A parameter p_k, $1 \le k \le K$, is of *1st class* if it occurs in only one equation of the system (1).

Definition 2 A parameter p_k, $k \in \mathcal{K}$, is of *2nd class* if it is involved in more than one equation of the system (1).

Let $\mathcal{E}$ and $\mathcal{A}$ be two disjoint sets such that $\mathcal{E} \cup \mathcal{A} = \mathcal{K}$. The parametric AE solution set is defined as

$$\Sigma^p_{AE} = \Sigma_{AE}(A(p), b(p), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}})(A(p)x = b(p))\}.$$

Beside the parametric *united* solution set, there are several other special cases of AE solutions:

- A parametric *tolerable* solution set is such that universal quantifiers concern only the constraint matrix and existential quantifiers only the right-hand side. That is, $A_k = 0$ for every $k \in \mathcal{E}$ and $b_k = 0$ for every $k \in \mathcal{A}$. The parametric tolerable solution set is

$$\Sigma^p_{tol} = \Sigma_{AE}(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}}), (\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}) (A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}.$$

- In contrast to the tolerable solutions, a parametric *controllable* solution set is such that existential quantifiers concern only the constraint matrix and universal quantifiers only the right-hand side. Thus, $A_k = 0$ for every $k \in \mathcal{A}$ and $b_k = 0$ for every $k \in \mathcal{E}$. The parametric controllable solution set is denoted shortly by Σ^p_{cont}.

$$\Sigma^p_{cont} = \Sigma_{AE}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}}), (\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}) (A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

For a given parametric system and index sets $\mathcal{A}$, $\mathcal{E}$, there is a unique non-parametric system, resp. non-parametric AE solution set $\Sigma(A(\boldsymbol{p}_{\mathcal{A}}, \boldsymbol{p}_{\mathcal{E}}), b(\boldsymbol{p}_{\mathcal{A}}, \boldsymbol{p}_{\mathcal{E}}))$, where

$$A(\boldsymbol{p}_{\mathcal{A}}, \boldsymbol{p}_{\mathcal{E}}) := A_0 + \sum_{k\in\mathcal{A}} A_k \boldsymbol{p}_k + \sum_{k\in\mathcal{E}} A_k \boldsymbol{p}_k,$$
$$b(\boldsymbol{p}_{\mathcal{A}}, \boldsymbol{p}_{\mathcal{E}}) := b_0 + \sum_{k\in\mathcal{A}} b_k \boldsymbol{p}_k + \sum_{k\in\mathcal{E}} b_k \boldsymbol{p}_k.$$

On the other hand, every non-parametric system, resp. non-parametric AE solution set, can be considered as a parametric system, resp. parametric AE solution set, involving n^2+n quantified parameters. Thus, every non-parametric AE solution set presents a special case of parametric AE solution set involving $n^2 + n$ quantified parameters.

For a nonempty and bounded set $\mathcal{S} \subset \mathbb{R}^n$, define its *interval hull* by $\square\mathcal{S} := \bigcap\{\boldsymbol{y} \in \mathbb{IR}^n \mid \mathcal{S} \subseteq \boldsymbol{y}\}$. For two intervals $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{IR}$, $\boldsymbol{u} \subseteq \boldsymbol{v}$, the percentage by which $\boldsymbol{v}$ overestimates $\boldsymbol{u}$ is defined by

$$\mathcal{O}(\boldsymbol{u}, \boldsymbol{v}) := 100(1 - u^{\Delta}/v^{\Delta}).$$

In Popova and Krämer (2011), it was shown that every $x \in \Sigma^p_{AE}$ satisfies the following inequality

$$|A(p^c)x - b(p^c)| \le \sum_{k\in\mathcal{E}} |A_k x - b_k| p^{\Delta}_k - \sum_{k\in\mathcal{A}} |A_k x - b_k| p^{\Delta}_k. \quad (3)$$

Moreover, for parametric systems involving only 1st class existentially quantified parameters, this system of nonlinear inequalities describes exactly the set Σ^p_{AE}.

3 End-Point Bounds for Σ^p_{AE}

It follows from the explicit representation of the parametric AE solution sets (Popova, 2011) that the interval hull of Σ^p_{AE} is attained at particular end-points of the intervals for the 1st class existentially quantified parameters and the universally quantified parameters. Here we exploit this property to develop a methodology for obtaining outer bounds of the parametric AE solution set using only solvers for bounding parametric united solution sets.

For a given index set $I = \{i_1, \ldots, i_k\}$, define

$$\mathcal{B}_I := \{(p^c_{i_1} + \delta_{i_1} p^{\Delta}_{i_1}, \ldots, p^c_{i_k} + \delta_{i_k} p^{\Delta}_{i_k}) \mid \delta_1, \ldots, \delta_k \in \{\pm 1\}\}.$$

Theorem 1 *We have*

$$\Sigma^p_{AE} = \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}} \Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}}). \quad (4)$$

Proof It follows from the set-theoretic representation of Σ^p_{AE} (see (Popova and Krämer, 2011, Theorem 3.1)) that

$$\Sigma^p_{AE} = \bigcap_{\tilde{p}_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}}} \Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}}).$$

Then, the assertion of the theorem follows from the relation

$$(\forall p \in [p] : b_1 \le f(p) \le b_2) \Leftrightarrow \left(b_1 \le \min_{p\in[p]} f(p)\right) \wedge \left(\max_{p\in[p]} f(p) \le b_2\right) \quad (5)$$

and because the polynomials involved in the explicit description of $\Sigma(A(p_{\mathcal{A}}, p_{\mathcal{E}}), b(p_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}})$ are linear with respect to all ∀-parameters. □

The next theorem gives a sufficient condition for a non-empty parametric AE solution set to be bounded.

Theorem 2 *Let Σ^p_{AE} be non-empty and for some $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$ the matrix $A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}})$ be regular for all $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$. Then Σ^p_{AE} is bounded.*

Proof Σ^p_{AE} is not empty iff the intersection in (4) is not empty. If for some $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$ the matrix $A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}})$ is regular for all $p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}}$, then $\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}})$ is bounded and its intersection (which is not empty) with bounded or unbounded solution sets for the remaining $p_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$ will be bounded. □

By Theorem 1, one can obtain (inner or outer) estimations of a bounded parametric AE solution set by intersecting at most $\mathrm{Card}(\mathcal{B}_{\mathcal{A}})$ corresponding estimations of the united parametric solution sets

$$\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}})), \quad \tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}.$$

In particular, we have

Corollary 1 *For a bounded parametric AE solution set* $\Sigma^p_{AE} \neq \emptyset$ *and a set* $\mathcal{B}'_{\mathcal{A}}$*, such that* $\mathcal{B}'_{\mathcal{A}} \subseteq \mathcal{B}_{\mathcal{A}}$ *and* $\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}})$ *is bounded for all* $\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}$*, we have*

$$\Box\Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \Box\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}}).$$

If the parametric system involves some 1st class $\exists$-parameters p_k, $k \in \mathcal{E}$, we can further sharpen the estimation of the parametric AE solution set. Denote by $\mathcal{E}_1$, $\mathcal{E}_1 \subseteq \mathcal{E}$, the set of indices of all $\exists$-parameters which occur in only one equation of the system. Since inf/sup of $\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \boldsymbol{p}_{\mathcal{E}})$ is attained at particular end-points of $\boldsymbol{p}_{\mathcal{E}_1}$, we have

$$\Sigma^p_{AE} = \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \bigcup_{\tilde{p}_{\mathcal{E}_1} \in \mathcal{B}_{\mathcal{E}_1}} \Sigma_{\mathcal{A},\mathcal{E},\mathcal{E}_1},$$

and

$$\Box\Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \Box\left(\bigcup_{\tilde{p}_{\mathcal{E}_1} \in \mathcal{B}_{\mathcal{E}_1}} \Box\, \Sigma_{\mathcal{A},\mathcal{E},\mathcal{E}_1}\right), \tag{6}$$

where

$$\Sigma_{\mathcal{A},\mathcal{E},\mathcal{E}_1} := \Sigma(A(\tilde{p}_{\mathcal{A}}, \tilde{p}_{\mathcal{E}_1}, p_{\mathcal{E}\setminus\mathcal{E}_1}), b(\tilde{p}_{\mathcal{A}}, \tilde{p}_{\mathcal{E}_1}, p_{\mathcal{E}\setminus\mathcal{E}_1}), \boldsymbol{p}_{\mathcal{E}\setminus\mathcal{E}_1}).$$

By a methodology based on solving derivative systems with respect to every parameter (Popova, 2006) one can prove that the interval hull of a united parametric solution set can be attained at particular end-points of the parameters, which are not only of 1st class. The parameters, for which we can prove this property, can be joined to the set $\mathcal{E}_1$ in relation (6).

4 Bauer–Skeel Method for Parametric *AE* Solution Sets

Bauer–Skeel bounds were used to enclose bounded and connected non-parametric united solution sets (Stewart, 1998; Rohn, 2010) and later bounded and connected parametric united solution sets (Skalna, 2006; Hladík, 2012). In this section, we extend the Bauer–Skeel method to the case of non-empty bounded and connected parametric AE solution sets. Since the following is a generalization of the Bauer–Skeel theorem, we do not state the original one explicitly.

Theorem 3 *For a fixed* $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$ *in the form of* $\tilde{p}_k = p^c_k + \tilde{\delta}_k p^\Delta_k$*,* $\tilde{\delta}_k \in \{\pm 1\}$*,* $k \in \mathcal{A}$*, suppose that* $A(\tilde{p}_{\mathcal{A}}, p^c_{\mathcal{E}})$ *be regular and define*

$$C := \left(A(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k A_k p^\Delta_k\right)^{-1} = A^{-1}(\tilde{p}_{\mathcal{A}}, p^c_{\mathcal{E}}),$$

$$x^* := C\left(b(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k b_k p^\Delta_k\right) = Cb(\tilde{p}_{\mathcal{A}}, p^c_{\mathcal{E}}),$$

$$M := \sum_{k\in\mathcal{E}} |CA_k| p^\Delta_k.$$

If $\rho(M) < 1$*, then every* $x \in \Sigma^p_{AE}$ *satisfies*

$$|x - x^*| \le (I - M)^{-1} \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)| p^\Delta_k.$$

Proof We precondition (1) by C, so (3) reads

$$|CA(p^c)x - Cb(p^c)| \le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)| p^\Delta_k - \sum_{k\in\mathcal{A}} |C(A_k x - b_k)| p^\Delta_k, \tag{7}$$

that is

$$|CA(p^c)x - Cb(p^c)| + \sum_{k\in\mathcal{A}} |C(A_k x - b_k)| p^\Delta_k \le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)| p^\Delta_k. \tag{8}$$

Since $|u| + |v| \ge |u \pm v|$, we have

$$\left|CA(p^c)x - Cb(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k C(A_k x - b_k) p^\Delta_k\right| \le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)| p^\Delta_k.$$

Rearranging we get

$$\left|C\left(A(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k A_k p^\Delta_k\right) x - C\left(b(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k b_k p^\Delta_k\right)\right| \le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)| p^\Delta_k,$$

or,

$$|x - x^*| \le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)|p_k^\Delta.$$

Now, we approximate the right-hand side from above

$$\begin{aligned}|x - x^*| &\le \sum_{k\in\mathcal{E}} |C(A_k x - b_k)|p_k^\Delta \\ &\le \sum_{k\in\mathcal{E}} |CA_k(x - x^*)|p_k^\Delta + \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta \\ &\le \sum_{k\in\mathcal{E}} |CA_k||x - x^*|p_k^\Delta + \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta \\ &= M|x - x^*| + \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta.\end{aligned}$$

Hence

$$(I - M)|x - x^*| \le \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta.$$

Since $M \ge 0$ and $\rho(M) < 1$, we have $(I - M)^{-1} \ge 0$ and

$$|x - x^*| \le (I - M)^{-1} \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta.$$

□

The application of Theorem 3 to the special case of parametric united solution set has the following form, which is identical with the Bauer–Skeel method generalized to parametric united solution sets in Skalna (2006); Hladík (2012).

Corollary 2 *Let $A(p^c)$ be regular and define*

$$\begin{aligned}C &:= A^{-1}(p^c), \\ x^* &:= Cb(p^c), \\ M &:= \sum_{k\in\mathcal{E}} |CA_k|p_k^\Delta.\end{aligned}$$

If $\rho(M) < 1$, then every $x \in \Sigma_{uni}^p$ satisfies

$$|x - x^*| \le (I - M)^{-1} \sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)|p_k^\Delta.$$

In the special case of parametric tolerable solution set we have

Corollary 3 *For a fixed $\tilde{p}_\mathcal{A} \in \mathcal{B}_\mathcal{A}$ let $A(\tilde{p}_\mathcal{A})$ be regular and define*

$$C := \left(A(p^c) + \sum_{k\in\mathcal{A}} \tilde{\delta}_k A_k p_k^\Delta\right)^{-1} = A^{-1}(\tilde{p}_\mathcal{A}),$$

$$x^* := Cb(p^c) = Cb(p_\mathcal{E}^c).$$

Then every $x \in \Sigma_{tol}^p$ satisfies

$$|x - x^*| \le \sum_{k\in\mathcal{E}} |Cb_k|p_k^\Delta.$$

The special case of parametric controllable solution set is discussed thoroughly in Section 5.

Corollary 4 *The intersection of the solution enclosures obtained by Theorem 3 (respectively Corollary 3) for all $\tilde{p}_\mathcal{A} \in \mathcal{B}_\mathcal{A}$ is equal to the intersection of the solution enclosures obtained by Corollary 2 for all $\tilde{p}_\mathcal{A} \in \mathcal{B}_\mathcal{A}$.*

Proof The proof follows immediately from the equivalent representation of C and x^* presented in the formulation of Theorem 3. □

Thus, the derivation of the parametric AE version of Bauer–Skeel method confirms Corollary 1.

Corollary 1 with using Bauer–Skeel enclosures for the particular united solution sets gives the same result as the intersection of all enclosures by Theorem 3. However, Corollary 1 with some other methods for enclosing parametric united solution sets may give better bounds.

Example 1 Let us consider the example from Popova and Krämer (2011)

$$\begin{pmatrix} p_1 & p_1 + 1 \\ p_2 + 1 & -2p_4 \end{pmatrix} x = \begin{pmatrix} p_3 \\ -3p_2 + 1 \end{pmatrix},$$

where $p_1, p_2 \in [0, 1]$ and $p_3, p_4 \in [-1, 1]$. For the sake of simplicity, $\Sigma_{\forall p_4 \exists p_{123}}$ denotes the parametric AE solution set where the universal quantifier is applied to p_4 and the existential one elsewhere. Similar notation is used for other combinations of quantifiers.

In case of $\Sigma_{\forall p_1 \exists p_{234}}$, see Fig. 1,

$$\square\left(\Sigma_{\exists p_{234}}(A(\underline{p}_1)) \bigcap \Sigma_{\exists p_{234}}(A(\overline{p}_1))\right) = \square\Sigma_{\exists p_{234}}(A(\underline{p}_1)).$$

That is why, enclosing sharply

$$\square\Sigma_{\exists p_{234}}(A(\underline{p}_1)) = ([-2, 3], [-1, 1])^\top,$$

we enclose $\square\Sigma_{\forall p_1 \exists p_{234}}$ in a best way, although the set $\Sigma_{\exists p_{234}}(A(\overline{p}_1))$ is unbounded. The parametric Bauer–Skeel method for $\Sigma_{\exists p_{234}}(A(\underline{p}_1), \boldsymbol{p}_{234})$ gives the enclosure $([-11/3, 3], [-1, 1])^\top$ and the 25% overestimation of x_1 is because the method cannot account well the row-dependencies in p_2. Therefore, applying (6), we compute

$$\square \bigcup_{p_2 \in \{0,1\}} \Sigma_{\exists p_{234}}(A(\underline{p}_1)) = ([-2, 3], [-1, 1])^\top.$$

For $\Sigma_{\forall p_2 \exists p_{134}}$ we cannot obtain an enclosure since the assumption $\rho(M) < 1$ is not fulfilled.

In case of $\Sigma_{\forall p_4 \exists p_{123}}$, see Fig. 2,

$$\Sigma_{\exists p_{123}}(A(\underline{p}_4)) \bigcap \Sigma_{\exists p_{123}}(A(\overline{p}_4)) \subset \square\Sigma_{\exists p_{123}}(A(\overline{p}_4)).$$

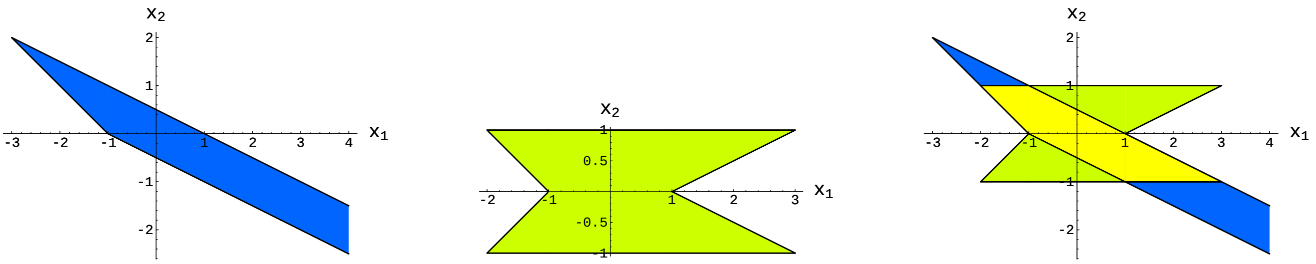

Fig. 1 Solution sets $\Sigma_{\forall p_1 \exists p_{234}}$ of the linear system from Example 1 for $p_1 \in \{1, 0\}$ (blue, green) and their intersection (yellow).

Since $\Sigma_{\exists p_{123}}(A(\underline{p}_4))$ is unbounded, we cannot find $\square\Sigma_{\forall p_4 \exists p_{123}}$ and approximate the latter outwardly by $\square\Sigma_{\exists p_{123}}(A(\overline{p}_4))$.

Applying the Bauer–Skeel method for parametric united solution sets, we obtain

$$\square\Sigma_{\exists p_{123}}(A(\tilde{p}_4), \boldsymbol{p}_{123}) = ([-4.9161, 4.4546], [-2.7203, 2.8742])^\top.$$

The overestimation is due to the row-dependencies in p_1 and p_2. Therefore, applying (6), we compute

$$\square \bigcup_{\tilde{p}_1, \tilde{p}_2 \in \{0,1\}} \Sigma_{\exists p_3}(A(\tilde{p}_1, \tilde{p}_2, p_3, \overline{p}_4)) = ([-2, 3], [-1, 1])^\top,$$

which is the interval hull of $\Sigma_{\exists p_{123}}(A(\overline{p}_4))$.

Remark 1 The formulation of Bauer–Skeel method is in real arithmetic, therefore its implementation in floating-point arithmetic will not provide a guaranteed enclosure, especially for intervals with very small radii or ill-conditioned problems. All computations below based on Bauer–Skeel method were done in rational arithmetic to avoid uncontrolled round-off errors. Instead of Bauer–Skeel method for bounding a parametric united solution set one can use the parametric fixed-point iteration, see Popova and Krämer (2007), which provides guaranteed enclosures of comparable quality under the same requirement for strong regularity of the parametric matrix. In fact, most of the general-purpose methods for bounding a parametric united solution set require strong regularity of the parametric matrix.

5 Another Form of the Bauer–Skeel Method

Below, we derive another form of the parametric Bauer–Skeel method under stronger assumptions.

Theorem 4 *Let $A(p^c)$ be regular and define*

$$C := A^{-1}(p^c),$$
$$x^* := Cb(p^c),$$
$$M := \sum_{k \in \mathcal{K}} |CA_k| p_k^\Delta.$$

If $\rho(M) < 1$, then every $x \in \Sigma_{AE}^p$ satisfies

$$|x - x^*| \le (I - M)^{-1} \left(\sum_{k \in \mathcal{E}} |C(A_k x^* - b_k)| p_k^\Delta - \sum_{k \in \mathcal{A}} |C(A_k x^* - b_k)| p_k^\Delta \right).$$

Proof Consider the preconditioned parametric system $C \cdot A(p)x = C \cdot b(p)$. The characterization (3) for the preconditioned system reads

$$|x - x^*| = |CA(p^c)x - Cb(p^c)| \le \sum_{k \in \mathcal{E}} |C(A_k x - b_k)| p_k^\Delta - \sum_{k \in \mathcal{A}} |C(A_k x - b_k)| p_k^\Delta.$$

For the right-hand side of the above inequality, due to

$$|u| - |v| \le |u + v| \le |u| + |v|$$

we have

$$\sum_{k \in \mathcal{E}} |C(A_k x - A_k x^* + A_k x^* - b_k)| p_k^\Delta - \sum_{k \in \mathcal{A}} |C(A_k x - A_k x^* + A_k x^* - b_k)| p_k^\Delta \le$$
$$|x - x^*| \sum_{k \in \mathcal{E}} |CA_k| p_k^\Delta + \sum_{k \in \mathcal{E}} |C(A_k x^* - b_k)| p_k^\Delta + |x - x^*| \sum_{k \in \mathcal{A}} |CA_k| p_k^\Delta - \sum_{k \in \mathcal{A}} |C(A_k x^* - b_k)| p_k^\Delta,$$

which implies

$$\left(I - \sum_{k \in \mathcal{K}} |CA_k| p_k^\Delta \right) |x - x^*| \le \sum_{k \in \mathcal{E}} |C(A_k x^* - b_k)| p_k^\Delta - \sum_{k \in \mathcal{A}} |C(A_k x^* - b_k)| p_k^\Delta.$$

Since $M \ge 0$ and $\rho(M) < 1$, we have $(I - M)^{-1} \ge 0$ and thus, the statement of the theorem. $\square$

In the special case of a united parametric solution set, Theorem 4 has the same form as Corollary 2. In the special case of a parametric tolerable solution set, Theorem 4 is the following.

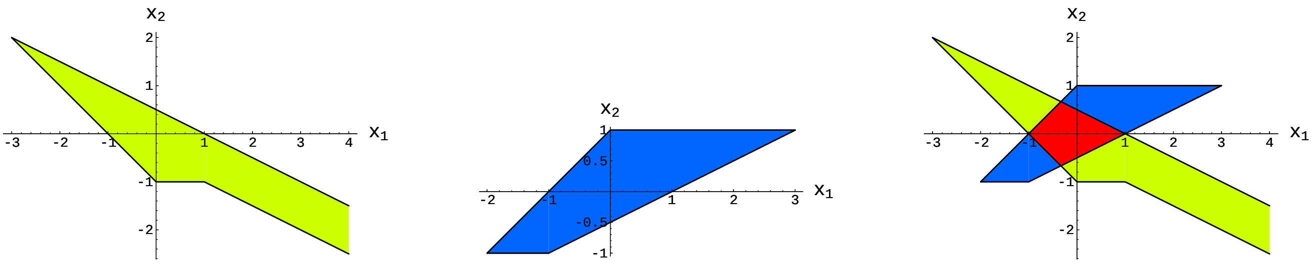

Fig. 2 Solution sets $\Sigma_{\forall p_4 \exists p_{123}}$ of the linear system from Example 1 for $p_4 \in \{-1, 1\}$ (green, blue) and their intersection (red).

Corollary 5 *Let $A(p^c) = A(p^c_{\mathcal{A}})$ be regular and*

$$C := A^{-1}(p^c_{\mathcal{A}}),$$
$$x^* := Cb(p^c) = Cb(p^c_{\mathcal{E}}),$$
$$M := \sum_{k\in\mathcal{A}} |CA_k| p^{\Delta}_k.$$

If $\rho(M) < 1$, then every $x \in \Sigma^p_{tol}$ satisfies

$$|x - x^*| \leq (I - M)^{-1} \left(\sum_{k\in\mathcal{E}} |Cb_k| p^{\Delta}_k - \sum_{k\in\mathcal{A}} |CA_k x^*| p^{\Delta}_k \right).$$

In the special case of parametric controllable solution sets, Theorem 4 is the following.

Corollary 6 *Let $A(p^c) = A(p^c_{\mathcal{E}})$ be regular and*

$$C := A^{-1}(p^c_{\mathcal{E}}),$$
$$x^* := Cb(p^c) = Cb(p^c_{\mathcal{A}}),$$
$$M := \sum_{k\in\mathcal{E}} |CA_k| p^{\Delta}_k.$$

If $\rho(M) < 1$, then every $x \in \Sigma^p_{cont}$ satisfies

$$|x - x^*| \leq (I - M)^{-1} \left(\sum_{k\in\mathcal{E}} |CA_k x^*| p^{\Delta}_k - \sum_{k\in\mathcal{A}} |Cb_k| p^{\Delta}_k \right).$$

The application of Corollary 4 requires strong regularity of the parametric matrix $A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}})$ on the domain $\boldsymbol{p}_{\mathcal{E}}$ for some $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$. Theorem 4 has a stronger requirement: strong regularity of $A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}})$ on $\boldsymbol{p}_{\mathcal{E}}$ for all $\tilde{p}_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}}$, resp. for all $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$. Therefore Corollary 1, resp. Corollary 4, have a larger scope of applicability (and a bigger computational complexity) than Theorem 4. Let us compare the two approaches for bounding parametric tolerable and controllable solution sets.

Example 2 Obtain outer enclosures of the parametric tolerable solution set for

$$A(p) = \begin{pmatrix} p_1 & p_2 + \frac{1}{2} \\ -2p_2 & p_1 + 1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} q_1 \\ q_1 - q_2 \end{pmatrix}$$

and $p_1 \in [0, 1]$, $p_2 \in [\frac{1}{3}, 1]$, $q_1, q_2 \in [-1, 2]$. The exact interval hull of the parametric tolerable solution set is $([-\frac{2}{5}, \frac{4}{5}], [-\frac{2}{3}, \frac{4}{3}])^\top$. Applying Corollary 5 we obtain the enclosure

$$([-36.904, 37.555], [-24.80, 25.38])^\top$$

which overestimates the hull by more than 95%. The application of Corollary 4 yields the interval hull. The conservative enclosure of the tolerable solution set produced by Corollary 5 is natural. Since every parametric tolerable solution set is a convex polyhedron (Popova, 2011), its interval hull is attained at particular endpoints of the parameters, which is the approach exploited by Corollary 4. Indeed, shrinking the interval for p_2 to $[\frac{999}{1000}, 1]$ the overestimation produced by Theorem 4 is reduced to 45%, resp. 35%. On the contrary, when we enlarge the interval for p_2 the parametric matrix is no more strongly regular.

While the application of Theorem 4 is not suitable for bounding parametric tolerable solution sets, this theorem gives a better enclosure for a parametric controllable solution set than the enclosure obtained by Corollary 4 (the intersection of the solution enclosures obtained by Theorem 3 for all $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$).

Proposition 1 *Under the same assumptions, the enclosure of the parametric controllable solution set computed by Corollary 6 is a subset of the enclosure computed by Corollary 4.*

Proof For a fixed endpoint of a fixed solution component, the intersection of the solution enclosures obtained by Theorem 3 for all $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$ is attained at a particular $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$. Let us consider an upper bound attained at a particular $\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}$. With the notations from Corollary 6, that particular right endpoint of the Bauer–Skeel enclosure by Theorem 3 is

$$x^* + C \sum_{j\in\mathcal{A}} \tilde{\delta}_j b_j p^{\Delta}_j + (I - M)^{-1} \sum_{k\in\mathcal{E}} \left| CA_k \left(x^* + C \sum_{j\in\mathcal{A}} \tilde{\delta}_j b_j p^{\Delta}_j \right) \right| p^{\Delta}_k.$$

We estimate the right endpoint from below as

$$
\begin{aligned}
&x^* - \sum_{j\in\mathcal{A}} |Cb_j| p_j^\Delta + (I-M)^{-1} \sum_{k\in\mathcal{E}} |CA_k x^*| p_k^\Delta \\
&- (I-M)^{-1} \sum_{k\in\mathcal{E}} |CA_k| p_k^\Delta \sum_{j\in\mathcal{A}} |Cb_j| p_j^\Delta \\
&= x^* + (I-M)^{-1} \sum_{k\in\mathcal{E}} |CA_k x^*| p_k^\Delta \\
&\qquad - \left(I + (I-M)^{-1}M\right) \sum_{j\in\mathcal{A}} |Cb_j| p_j^\Delta .
\end{aligned}
$$

Using $(I-M)^{-1} = I + (I-M)^{-1}M$, we obtain

$$
x^* + (I-M)^{-1} \left(\sum_{k\in\mathcal{E}} |CA_k x^*| p_k^\Delta - \sum_{j\in\mathcal{A}} |Cb_j| p_j^\Delta \right),
$$

which is the right endpoint of the enclosure by Corollary 6. Similarly we prove a corresponding relation between the left endpoints of the enclosures. □

Example 3 Consider a parametric linear system where

$$
A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}
$$

and $p_1 \in [0, \frac{1}{2}]$, $p_2 \in [1, \frac{3}{2}]$, $q \in [1, \frac{3}{2}]$. The exact interval hull of the parametric controllable solution set is $([2, 12/5], [-2, -6/5])^\top$, see Fig. 3. Applying Corollary 4 we obtain the enclosure

$$\Sigma^p_{cont} \subseteq ([1.186, 2.902], [-2.286, -0.263])^\top,$$

overestimating the components of the interval hull by more than 76%, resp. 60%. However, by Theorem 4 (Corollary 6), we obtain the enclosure

$$
\Sigma^p_{cont} \subseteq ([1.7802, 2.8352], [-2.2198, -0.8571])^\top,
$$

and the overestimation is 62%, resp. 41%.

Bauer–Skeel method, in any of its forms, requires strong regularity of the parametric matrix. Strong regularity (in the present formulation $\rho(M) < 1$ or $(I - M)^{-1} \geq 0$) must be checked when implementing the method. Since it is a sufficient condition for a parametric matrix to be regular, Bauer–Skeel method may fail for some regular matrices which are not strongly regular, see the next example.

Example 4 Consider the parametric system from Example 3 with other domains for the parameters: $p_1 \in [\frac{1}{2}, \frac{3}{2}]$, $p_2 \in [0, 1]$ and $q \in [1, 2]$. The parametric matrix is regular but not strongly regular. Therefore, by Theorem 4 (resp. Corollary 6), we cannot find outer bounds for the parametric controllable solution set which is connected and bounded, see Fig. 4, and has interval hull $([8/3, 2(1+\sqrt{2})], [0, 4])^\top$.

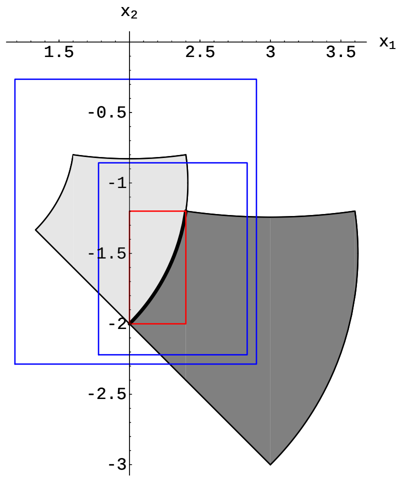

Fig. 3 The controllable solution set for the linear system from Example 3 represented as intersection of the united solution sets for $q = 1$ (light gray) and $q = 3/2$ (dark gray) together with its interval hull and its enclosures obtained by Corollary 4 and Corollary 6.

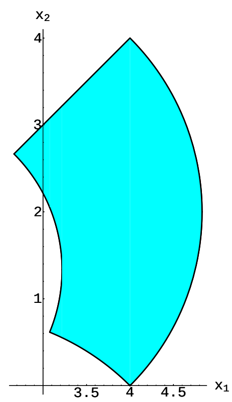

Fig. 4 The parametric controllable solution set for the linear system from Example 4.

Example 5 We look for the controllable solution set of the parametric system from Example 3, enlarging the domain for q to $q \in [1, \frac{5}{2}]$. Although the parametric matrix is strongly regular on the domain for p_1, p_2, the inequality

$$
\sum_{k\in\mathcal{E}} |C(A_k x^* - b_k)| p_k^\Delta \;\geq\; \sum_{k\in\mathcal{A}} |C(A_k x^* - b_k)| p_k^\Delta
$$

does not hold, which means that $\Sigma_{cont} = \emptyset$. Thus, by Theorem 4 we not only compute enclosures of the controllable solution set, but also can sometimes detect emptiness.

6 LP Enclosure for the Parametric Tolerable Solution Set

Besides the united solution set, tolerable solutions are the most studied AE solutions to interval linear systems. In the non-parametric case, there are plenty of results, see Shary (1995); Beaumont and Philippe (2001); Shary (2002); Pivkina and Kreinovich (2006); Rohn (2006); Wang (2008), among others. The only generalization to a special class of parametric tolerable solution sets is found in Sharaya and Shary (2011).

Corollary 4 provides an enclosure to the tolerable solution set Σ^p_{tol} which is much sharper than the enclosure provided by Theorem 4. By a careful inspection of the characterization (3) we can derive a polyhedral approximation of Σ^p_{tol}.

Proposition 2 *For every $x \in \Sigma^p_{tol}$ there are $y^k \in \mathbb{R}^n$, $k \in \mathcal{A}$, such that*

$$A(p^c)x + \sum_{k\in\mathcal{A}} p_k^{\Delta} y^k \le \sum_{k\in\mathcal{E}} |b_k| p_k^{\Delta} + b(p^c), \tag{9a}$$

$$-A(p^c)x + \sum_{k\in\mathcal{A}} p_k^{\Delta} y^k \le \sum_{k\in\mathcal{E}} |b_k| p_k^{\Delta} - b(p^c), \tag{9b}$$

$$A_k x \le y^k, \; -A_k x \le y^k, \quad \forall k \in \mathcal{A}. \tag{9c}$$

Moreover, for parametric systems involving only 1st class existentially quantified parameters, the x solutions to (9) form Σ^p_{tol}.

Proof By (3), each $x \in \Sigma^p_{tol}$ satisfies

$$|A(p^c)x - b(p^c)| + \sum_{k\in\mathcal{A}} |A_k x| p_k^{\Delta} \le \sum_{k\in\mathcal{E}} |b_k| p_k^{\Delta},$$

or,

$$A(p^c)x + \sum_{k\in\mathcal{A}} |A_k x| p_k^{\Delta} \le \sum_{k\in\mathcal{E}} |b_k| p_k^{\Delta} + b(p^c),$$

$$-A(p^c)x + \sum_{k\in\mathcal{A}} |A_k x| p_k^{\Delta} \le \sum_{k\in\mathcal{E}} |b_k| p_k^{\Delta} - b(p^c).$$

Substituting $y^k := |A_k x|$ we get (9). □

The system (9) consists of linear inequalities with respect to x and y^ks, so we can employ linear programming techniques to obtain lower and upper bounds for the components of x.

Proposition 2 also shows that the parametric tolerable solution set is a convex polyhedron for parametric linear systems involving only 1st class parameters. This is in accordance with the results from Sharaya and Shary (2011); Popova (2011).

Linear programming (LP) techniques are well studied for bounding non-parametric AE solution sets, see Beaumont and Philippe (2001). Proposition 2 generalizes the LP approach for *parametric* tolerable solution sets and provides exact bounds when the involved ∃-parameters are only of 1st class. Recall that a parametric matrix $A(p)$ is row-independent if for every $k = 1,\dots,K$ and every $i = 1,\dots,n$ the following set has cardinality at most one:

$$\{j \in \{1,\dots,n\} \mid (A_k)_{ij} \ne 0\}.$$

Due to the equality relation in (Popova, 2011, eq. (5.3)), inner and outer inclusions of a tolerable solution set, where the matrix involves only row-independent parameters and the right-hand side vector involves only 1st class parameters, can be computed by methods for the non-parametric case. Therefore, Proposition 2 is particularly useful for linear systems involving row-dependent parameters in the matrix and right-hand side vector with independent components.

By using a standard linear programming technique to calculate lower and upper bounds on x solutions of (9), we have to solve $2n$ linear programs, each of them with $n(1 + Card(\mathcal{A}))$ variables and $2n(1 + Card(\mathcal{A}))$ constraints. For a non-parametric tolerable system $\boldsymbol{A}x = \boldsymbol{b}$, this number is too conservative. The system (9) may be furhter reduced (Fiedler et al, 2006; Rohn, 1986) and the interval hull of the tolerable solution set is determined by solving $2n$ linear programs, each of them with only $2n$ variables and $4n$ constraints. If we call Corollary 1 to compute an enclosure and linear programming to calculate the subordinate interval hulls, then we have to solve $2n \cdot 2^{Card(\mathcal{A})}$ linear programs, each with n variables and $2n$ constraints.

Example 6 Motivated by Example 5.2, Popova (2011), let

$$A^{(1)}(p) = \begin{pmatrix} p_1 & p_2 \\ p_3 & p_1 + 1 \end{pmatrix}, \quad A^{(2)}(r) = \begin{pmatrix} r & r + \frac{1}{2} \\ -2r & r + 1 \end{pmatrix},$$

$$A^{(3)}(s) = \begin{pmatrix} s_1 & s_1 + \frac{1}{2} \\ -2s_2 & s_2 + 1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} q_1 \\ q_1 - q_2 \end{pmatrix},$$

where $p_1, r, s_1, s_2 \in [0,1]$, $p_2 \in [\frac{1}{2}, \frac{3}{2}]$, $p_3 \in [-2,0]$ and $q_1, q_2 \in [-1,2]$. Relaxing the parametric dependencies in the interval systems $A^{(1)}(p)x = b(q)$, $A^{(2)}(r)x = b(q)$, and $A^{(3)}(s)x = b(q)$ we get a standard interval system $\boldsymbol{A}x = \boldsymbol{b}$ drawing

$$\begin{pmatrix} [0,1] & [\frac{1}{2}, \frac{3}{2}] \\ [-2,0] & [1,2] \end{pmatrix} x = \begin{pmatrix} [-1,2] \\ [-3,3] \end{pmatrix}.$$

Consider first the interval systems $A^{(1)}(p)x = b(q)$ and $A^{(1)}(p)x = \boldsymbol{b}$. Applying Corollary 4 we obtain

$$\Sigma_{tol}(A^{(1)}(p), b(q), \boldsymbol{p}, \boldsymbol{q}) \subseteq ([-\tfrac{2}{5}, \tfrac{4}{5}], [-\tfrac{2}{3}, \tfrac{4}{3}])^{\top},$$

$$\Sigma_{tol}(A^{(1)}(p), \boldsymbol{b}, \boldsymbol{p}) \subseteq ([-1.167, 1.7], [-0.667, 1.334])^{\top}.$$

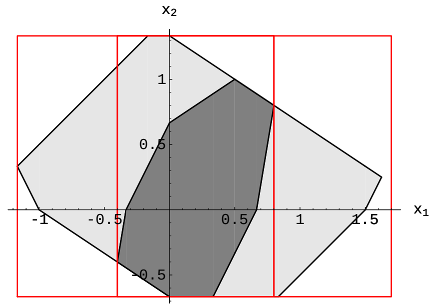

Fig. 5 The tolerable solution sets for the linear systems $A^{(1)}(p)x = b(q)$ (dark gray) and $A^{(1)}(p)x = \boldsymbol{b}$ (light gray) from Example 6 together with the enclosing boxes obtained by Corollary 4.

The two parametric AE solution sets and the corresponding enclosing boxes are presented on Fig. 5. Theorem 4 cannot be applied since the parametric matrix is not strongly regular. A linear programming approach based on Proposition 2 gives

$$\begin{aligned}
&\Sigma_{tol}(A^{(1)}(p), b(q), \boldsymbol{p}, \boldsymbol{q}) \subseteq ([-0.4, 0.8], [-1, 1.286])^\top,\\
&\Sigma_{tol}(A^{(1)}(p), \boldsymbol{b}, \boldsymbol{p}) \subseteq ([-1.3, 1.7], [-1, 1.4])^\top,\\
&\Sigma_{tol}(\boldsymbol{A}, \boldsymbol{b}) \subseteq ([-1.167, 1.625], [-0.667, 1.334])^\top,
\end{aligned}$$

The LP enclosures to $\Sigma_{tol}(A^{(1)}(p), b(q), \boldsymbol{p}, \boldsymbol{q})$ and $\Sigma_{tol}(A^{(1)}(p), \boldsymbol{b}, \boldsymbol{p})$ are not optimal since the system involves a 2nd class existentially quantified parameter p_1. Since the matrix $A^{(1)}(p)$ involves only row-independent parameters,

$$\Sigma_{tol}(A^{(1)}(p), \boldsymbol{b}, \boldsymbol{p}) = \Sigma_{tol}(\boldsymbol{A}, \boldsymbol{b}) \subseteq ([-1.167, 1.625], [-0.667, 1.334])^\top,$$

which is the interval hull.

Now, we consider the systems $A^{(2)}(r)x = b(q)$ and $A^{(2)}(r)x = \boldsymbol{b}$. For these systems Corollary 4 gives the exact interval hulls

$$\begin{aligned}
&\Sigma_{tol}(A^{(2)}(r), \boldsymbol{b}, \boldsymbol{r}) \subseteq ([-1.3, 1.7], [-1, 1.4])^\top,\\
&\Sigma_{tol}(A^{(2)}(r), b(q), \boldsymbol{r}, \boldsymbol{q}) \subseteq ([-0.4, 0.8], [-1, 1.4])^\top.
\end{aligned}$$

Proposition 2 gives the same enclosures.

For the parametric interval system $A^{(3)}(s)x = \boldsymbol{b}$, Corollary 4 yields the exact hull

$$\Sigma_{tol}(A^{(3)}(s), \boldsymbol{b}, \boldsymbol{s}) \subseteq ([-1.3, 1.7], [-1, 1.4])^\top.$$

Since all parameters are of 1st class only, Proposition 2 gives the same result.

7 Conclusion

This paper presents a first attempt to propose and investigate methods providing outer bounds for parametric AE solution sets. The methods are general ones — they are applicable to linear systems involving arbitrary linear dependencies between interval parameters; the parametric AE solution sets may be defined so that $\mathcal{A}$- and $\mathcal{E}$-parameters are mixed in both sides of the equations. Being the most general, these methods are applicable to the special cases of non-parametric AE solution sets, in particular non-parametric tolerable or controllable solution sets.

From a methodological point of view, the methods we consider are based on a simple (though not always complete) Oettli-Prager-type description (3) of the parametric AE solution sets. This allows us to obtain bounds for the parametric AE solution sets either by bounding only parametric united solution sets or by using only real arithmetic and the properties of classical interval arithmetic. This makes the main methodological and computational difference between the methodology employed in this paper and the methodology that is used so far for estimating non-parametric AE solution sets (Shary, 1995, 1997, 2002; Goldsztejn, 2005; Goldsztejn and Chabert, 2006), based on the arithmetic of proper and improper intervals (called Kaucher interval arithmetic).

The methods we present here provide outer bounds for non-empty, connected and bounded parametric AE solution sets. The first approach intersects inclusions of parametric united solution sets for all combinations of the end-points of $\mathcal{A}$-parameters. This approach has exponential computational complexity, however provides very sharp estimations of the AE solution sets, especially for tolerable solution sets and for general parametric AE solution sets when combined with sharp bounds for the linear $\mathcal{E}$-parameters. The second method we discuss is a parametric AE generalization of the single-step Bauer–Skeel method used so far for bounding parametric united solution sets. In the special cases of non-parametric (tolerable, controllable) AE solution sets, this new method expands the range of available methods for outer enclosures. However, while most of the known methods for enclosing non-parametric AE solution sets are based on Kaucher interval arithmetic, the present method is based on the classical interval arithmetic. Also, it is a direct method and therefore a fast one. Finally, for parametric tolerable solution sets, we proposed a linear programming based method, which utilizes a polyhedral approximation of the set. When each existentially quantified parameter is involved in only one equation of the system, this method yields the interval hull of the parametric AE solution set.

We demonstrated that the approach intersecting enclosures of parametric united solution sets for all combinations of the end-points of $\mathcal{A}$-parameters is applicable to a larger class of parametric AE solution sets compared to the parametric Bauer–Skeel AE-method. Despite its computational complexity, the first approach may be more suitable for bounding tolerable solution set of large-scale parametric systems if one exploits distributed computations and modern methods for solving large-scale point systems which do not invert the matrix. On the other hand, the parametric Bauer–Skeel AE method provides better bounds for the parametric controllable solution sets. This method implies a simple necessary (sometimes and sufficient) condition for any parametric AE solution set to be non-empty.

The present formulation of the parametric Bauer–Skeel AE method is in real arithmetic, therefore its implementation in floating-point arithmetic will not provide a guaranteed enclosure, unless combined with suitably chosen directed rounding. A self-verified method, which corresponds to the present parametric Bauer–Skeel AE method, and provides guaranteed outer bounds for nonempty connected and bounded parametric AE solution sets will be presented in a separate paper.

The parametric Bauer–Skeel AE method and the intersection of enclosures obtained by a self-verified method can be used for bounding only connected and bounded solution sets. However, the interval Gauss-Seidel method, where the interval division is extended to allow division by interval containing zero (Goldsztejn and Chabert, 2006), can be used to enclose bounded disconnected solution sets. So, a parametric generalization of the Gauss-Seidel method, may be helpful sometimes.

The parametric Bauer–Skeel AE method and most of the general-purpose parametric self-verified methods do not provide sharp enclosures of the parametric united solution set when the system involves row-dependent parameters. A parametric generalization of the right preconditioning process, considered in Goldsztejn (2005) for non-parametric AE systems, may be also helpful.

Searching best estimations of parametric AE solution sets one has to take into account the inclusion relations between such solution sets (Popova, 2011), and the properties of the methods.

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