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On "Overestimation-free Computational Version of Interval Analysis"

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The transformation of interval parameters into trigonometric functions, proposed in *Int. J. Comput. Meth. Eng. Sci. Mech.*, vol. 13, pp. 319–328 (2012), is not motivated in comparison to the infinitely many equivalent algebraic transformations. The conclusions about the efficacy of the methodology used are based on incorrect comparisons between solutions of different problems. We show theoretically, and in the examples considered in the commented article, that changing the number of the parameters in a system of linear algebraic equations may change the initial problem, respectively, its solution set. We also correct various misunderstandings and bugs that appear in the article noted above.

Keywords Interval analysis, reparameterization, stepped bar

1. INTRODUCTION

In their recent article in this journal [1] (see also [2]), Elishakoff and Miglis propose to transform the interval variables involved in a given problem into trigonometric functions depending on other interval parameters. The authors applied the following computational procedure when solving systems of linear algebraic equations involving interval parameters:

c1. Find the analytic solution of the new system obtained after trigonometric transformations of the original interval parameters.
c2. Find the global extrema of the expressions involved at each component of the above analytic solution in the domain of the new interval parameters.

The global minimum and maximum of a function in a given interval domain provides the exact range (in exact arithmetic) of the function in the considered domain. Thus the above procedure (c1.–c2.), if it is successful, provides the exact interval hull[1] (in exact arithmetic) of the united solution set[2] of a system of linear algebraic equations involving interval parameters.

The main conclusion in [1] is that the proposed trigonometric transformation provides overestimation-free results. It is not the trigonometric transformation but the computational procedure (c1.–c2.), in which the authors have no contribution, that provides overestimation-free results. In other words, the authors conclude that the trigonometric transformation of the interval parameters followed by the computational procedure (c1.–c2.) delivers sharper intervals than the intervals obtained by some naive interval methods applied to the initial non-transformed problem. In the following, we show that these false conclusions drawn for all examples considered in [1] are based on comparisons between the solutions of different problems, since reducing the number of interval parameters changes the original problem and its solution set.

2. TRANSFORMATIONS OF INTERVAL PARAMETERS

A real compact interval is $[a] = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. For $[a] = [\underline{a}, \overline{a}]$, define mid-point $\mu([a]) := (\underline{a} + \overline{a})/2$ and radius $\varrho([a]) := (\overline{a} - \underline{a})/2$. It is well known that any interval $[a]$ has the following equivalent representations

$$[a] = f_1([t]), \quad f_1(t) := \mu([a]) + t, \quad t \in [-\varrho([a]), \varrho([a])], \tag{1}$$

$$= f_2([t]), \quad f_2(t) := \mu([a]) + \varrho([a])t, \quad t \in [-1, 1]. \tag{2}$$

Elishakoff and Miglis propose in [1, Eq. (5)] (see also [2]),

$$[a] = \mu([a]) + \varrho([a])\sin(t), \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]. \tag{3}$$

In fact, any interval $[a]$ has infinitely many equivalent representations; for example,

$$[a] = f_3([t]), \quad f_3(t) := \mu([a]) + \frac{\varrho([a])}{\alpha}t, \quad t \in [-\alpha, \alpha], \alpha \in \mathbb{R}. \tag{4}$$

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[1]The smallest interval vector containing the solution set.

[2]The united solution set is defined in Section 2.

Above, $f_i([t])$ denotes the range of $f_i(t)$ on $[t]$.

Let us consider a system of linear algebraic equations

$$A(p)x = b(p), \qquad (5a)$$

where the parameters $p = (p_1, \ldots, p_K)^\top$ vary within given intervals $[p] = ([p_1], \ldots, [p_K])^\top$ and

$$A(p) = A_0 + \sum_{k=1}^{K} p_k A_k, \quad b(p) = b_0 + \sum_{k=1}^{K} p_k b_k \qquad (5b)$$

for some numerical matrices $A_k \in \mathbb{R}^{m\times n}$ and vectors $b_k \in \mathbb{R}^m$, $k = 0, 1, \ldots, K$. The system (5) is called an interval *parametric* system. The "solution" of a problem involving interval data can be defined in a variety of ways, depending on the particular goals. Most often, the so-called *united* parametric solution set

$$\begin{aligned}\Sigma^p_{uni} &= \Sigma_{uni}(A(p), b(p), [p]) \\ &:= \{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\}\end{aligned} \qquad (6)$$

is considered in the applications and its outer interval estimation $[y] \in \mathbb{IR}^n$, $\Sigma^p_{uni} \subseteq [y]$ is sought. In a *nonparametric* interval linear system, usually denoted by $[A]x = [b]$, the elements of the matrix $[A] := ([a_{ij}]) \in \mathbb{IR}^{m\times n}$ and of the right-hand side vector $[b] := ([b_i]) \in \mathbb{IR}^m$ are independent parameters varying within their intervals. Thus, the nonparametric interval system can be considered as a special case of a parametric system involving $mn + m$ interval parameters.

A parameter-dependent linear system (5) provides a more precise setting and more precise results for a real-life problem involving interval parameters than if it is approximated by the corresponding nonparametric system $A([p])x = b([p])$[3]. This is because the united parametric solution set $\Sigma_{uni}(A(p), b(p), [p])$ is much smaller and it is contained in the united solution set of the corresponding nonparametric system $\Sigma_{uni}(A([p]), b([p]))$. This fact has been known for a long time by the many examples given in the interval literature. The mathematical proof of the inclusion relations between the parametric and the corresponding nonparametric solution sets is given in [5]. Theorem 5.6 in [5] proves that reducing (by [5, Lemma 5.5]) the number of the interval parameters involved in a linear system, the united solution set of the new system becomes (in general) smaller and it is contained in the solution set of the original system. On the other hand, if some or all of the parameters in a linear system are changed by any of the transformations (1)–(4), without changing the number of the parameters involved, then the problem and its united solution set do not change. We call such a reparametrization equivalent. Hence, a fixed method applied to two or more linear systems, obtained by equivalent reparametrization, should produce the same result up to the round-off errors due to the reparametrization.

[3] $A([p])$, $b([p])$ denote the ranges of the corresponding functions in $A(p)$, $b(p)$ on $[p]$.

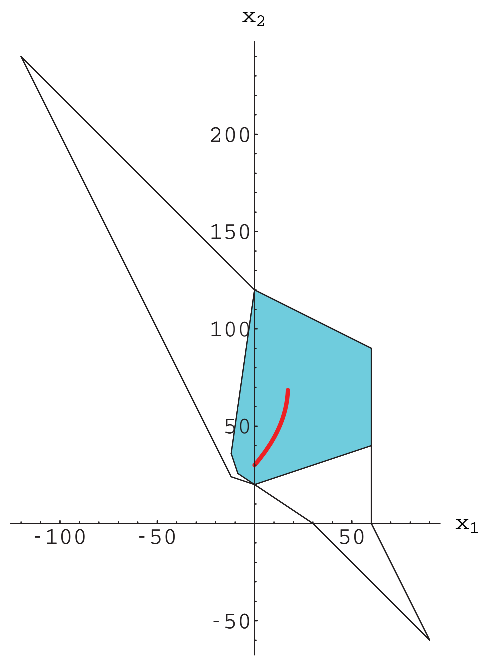

FIG. 1. Σ^p_{uni} to three different parametric linear systems related to the Hansen's example. (Color figure available online.)

3. COMPARING THE TRIGONOMETRIC TRANSFORMATION

In Section 3.3 of [1], the authors first apply the trigonometric transformation (3) equivalently to every one of the six interval parameters involved in a two-dimensional nonparametric interval system considered in [3] and called the Hansen's example. Since the transformation is equivalent, the interval enclosure of the solution set to the new linear system coincides[4] with the Hansen's enclosure of the solution set to the original nonparametric interval system. Next, by trigonometric transformations, Eq. (32) in [1], Elishakoff and Miglis transform the original system involving six independent parameters, T_1, T_2, V_i, $i = 1, \ldots, 4$, into another system involving three independent parameters, $\alpha_1, \alpha_2, \alpha_3$. Since the new system involves fewer parameters than the original one, the new system presents another problem, different from the original one. According to [5, Theorem 5.6], the solution set of the new system, presented in Fig. 1 by the polyhedron in gray, is contained in the solution set of the original system (the transparent polyhedron). Not accounting that they compare the solutions to different problems, Elishakoff and Miglis erroneously conclude that due to their trigonometric transformation the newly obtained interval solution is sharper than the enclosure of the original system provided by Hansen in [3]. One can easily check that applying (c1.–c2.) to a new system, obtained by any other of the equivalent transformations

[4] We remark that the last equalities in Eq. (28) and Eq. (32) in [1] should be read as $C_{T_2} = 150 + 90\sin(\alpha_6)$ instead of $C_{T_2} = 150 + 60\sin(\alpha_6)$ and x_1 in Eq. (31) of [1] should be read as $x_1 \in [-120, 90]$ instead of $x_1 \in [-120, 60]$.

(1)–(4), will yield the same solution as the one obtained by the trigonometric transformation. Further on, the authors treat all six interval parameters in the original Hansen's system as fully dependent on each other and reparametrize them using a single parameter α. The new parametric system, given by Eq. (36)[5] in [1], presents another parametric problem different from the original system considered by Hansen. The solution set of the new system involving one interval parameter is part of a curve, as presented in Fig. 1. A side remark: in [1] the nominators of the expressions in Eq. (37) are wrong as well as the solution given in Eq. (38) is also wrong—the correct interval hull of the solution set is $([0, \frac{120}{7}], [30, \frac{480}{7}])^\top$.

The linear system, specified by Eq. (44) in [1], involves six independent interval parameters. The united solution set to this system is unbounded in both coordinates. Elishakoff and Miglis apply an interval version of Cramer's rule in order to obtain an interval enclosure of the solution set to (44). Since the denominators in the expressions for the two solution components are intervals containing zero, both the resulting intervals should be $[-\infty, \infty]$. Instead, in Eq. (45) and Eq. (46), the authors obtain finite bounded intervals. To the six intervals involved in the system (44), Elishakoff and Miglis apply the trigonometric transformation assuming that all original independent parameters depend on a single parameter α. Solving symbolically the new parametric system, one obtains the algebraic solution $x_1 = x_2 = 1$, which is also the united parametric solution set of the new system. The authors of [1] again compare the solution of the new different system to the united solution set of the original system and erroneously conclude: "As we clearly see, the proposed parameterization procedure overcomes the large overestimation yielded by classical interval analysis." We stress that it is not the trigonometric reparametrization which reduces the new solution, since any other of the reparametrizations (1)–(4), applied the same way, will yield the same result. The above examples only show that when reducing the number of the parameters (that is, involving more dependencies), we generally change the problem and its solution set. As a consequence, the exact hull of the reduced solution set underestimates the solution set of the original problem.

In [6], Rao and Berke consider a two-step bar problem subjected to external load acting along the axis. The worst case variation of the axial displacements of the two sections are described by the united solution set to the following parametric linear system:

$$\begin{pmatrix} \dfrac{A_1E_1}{l_1} + \dfrac{A_2E_2}{l_2} & -\dfrac{A_2E_2}{l_2} \\ -\dfrac{A_2E_2}{l_2} & \dfrac{A_2E_2}{l_2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ p_2 \end{pmatrix}, \qquad (7)$$

where the parameters vary within given intervals. Solving the same problem in [1], Elishakoff and Miglis apply the trigonometric transformation equivalently to each of the seven interval parameters, then by (c1.–c2.) they find the exact interval hull of the united solution set of (7). The authors would find the same intervals if they had applied (c1.–c2.) to the naturally parametrized system (7), or to the latter equivalently transformed by any other of the transformations (1)–(4). On the other hand, the united parametric solution set to (7) is equivalent to the united parametric solution set to the following system involving only three interval parameters (instead of seven)

$$\begin{pmatrix} q_1 + q_2 & -q_2 \\ -q_2 & q_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ p_2 \end{pmatrix}, \qquad (8)$$

where $q_1 \in \frac{[A_1][E_1]}{[l_1]}$, $q_2 \in \frac{[A_2][E_2]}{[l_2]}$. The analytic solution to (8) is $(\frac{p_2}{q_2}, \frac{p_2(q_1-q_2)}{q_1+q_2})^\top$, which is much simpler than the analytic solution [1, Eq. (58)] of the system obtained from (7) by trigonometrically transformed parameters.

In the beginning of Section 3.1 of [1], the authors make an unfair comparison computing the range of f_1 by the natural interval extension and computing the range of their parametrized expression C_{f_1} by applying minimization/maximization procedures. Based on this comparison, four lines before the end of page 320, the authors make the false conclusion: "It is seen that the parameterization method overcomes the overestimation property in this case." In fact, it is not the parameterization method which overcomes the overestimation but the minimization/maximization procedures applied for computing the range of C_{f_1}. If the authors evaluate C_{f_1} in (13) in the same way as they evaluated f_1 in (10), they will obtain an even worse interval enclosure $[-3, 4]$, while the minimization/maximization of f_1 also yields the exact range.

In Section 3.2 of [1], the authors consider a model of a person rowing with and against the current and write down the following two equations (denoted Eq. (18) in [1])

$$\begin{aligned} [3, 4](x + y) &= 12 \\ [6, 8](x - y) &= 12. \end{aligned} \qquad (9)$$

First, Elishakoff and Miglis solve by the interval version of Cramer's rule an algebraic system

$$\begin{aligned} [3, 4]x + [3, 4]y &= 12 \\ [6, 8]x - [6, 8]y &= 12, \end{aligned} \qquad (10)$$

erroneously[6] considering it as equivalent to (9). Next, they solve (9) by (i) algebraic transformations and by (ii) trigonometric parametrization of the two intervals in (9), depending on only one parameter. Obtaining the same intervals in both cases (i) and (ii), Elishakoff and Miglis conclude, "It is remarkable that the parametrization method straightforwardly results in

[5]The coefficient of x_2 in the first equation should be $0.5 + 0.5\sin(\alpha)$ instead of $0.5\sin(\alpha)$. The variables in the second equation should be x_1, x_2 instead of x, y.

[6]Interval multiplication is not distributive. Therefore the second equations of (9) and (10) are not equivalent.

sharper bounds," compared to the result obtained by the interval Cramer's rule, which solves a completely different problem (10).

4. CONCLUSION

Reducing the number of interval parameters in a system of linear algebraic equations should be done carefully and only if there are either domain-specific or mathematical reasons for doing that; that is, if we are sure that the reparametrization does not change the solution set of the problem we solve. An equivalent trigonometric reparametrization, itself, does not provide overestimation-free results. The use of exactly this transformation is not motivated in [1]. The solution methodology (c1.–c2.), which follows the reparametrization and provides overestimation-free results, is not appropriate for large systems of symbolic equations, respectively, for finding global extrema of complicated trigonometric expressions. On the other hand, there is a highly efficient (in both computation time and sharpness of the interval enclosure) interval method [4] designed for large parametric linear systems that emerge from finite element analysis of truss structures.

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