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Maximal inner boxes in parametric AE-solution sets with linear shape

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Abstract

We consider linear systems of equations $A(p)x = b(p)$, where the parameters p are linearly dependent and come from prescribed boxes, and the sets of solutions (defined in various ways) which have linear boundary. One fundamental problem is to compute a box being inside a parametric solution set. We first consider parametric tolerable solution sets (being convex polyhedrons). For such solution sets we prove that finding a maximal inner box is an NP-hard problem. This justifies our exponential linear programming methods for computing maximal inner boxes. We also propose a polynomial heuristic that yields a large, but not necessarily the maximal, inner box. Next, we discuss how to apply the presented linear programming methods for finding large inner estimations of general parametric AE-solution sets with linear shape. Numerical examples illustrate the properties of the methods and their application.

Keywords: linear equations, dependent interval parameters, tolerable solution set, AE-solution set, inner estimation.
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1. Introduction

Denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real vectors with n components and the set of real $m \times n$ matrices, respectively. A real compact interval is

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$\boldsymbol{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \leq a \leq \overline{a}\}$, *an interval matrix* $\boldsymbol{A}$ is defined as

$$\boldsymbol{A} := [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n} \mid \underline{A} \leq A \leq \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, $\underline{A} \leq \overline{A}$, are given. Inequalities are understood componentwise throughout this paper. Interval vectors are considered as one-column matrices. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. We consider systems of linear algebraic equations having linear uncertainty structure

$$\begin{aligned} A(p)x &= b(p), \\ A(p) &:= A_0 + \sum_{k=1}^{K} p_k A_k, \qquad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k, \end{aligned} \tag{1}$$

where $A_k \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^m$, $k = 0, \ldots, K$ and the parameters $p = (p_1, \ldots, p_K)^\top$ are considered to be uncertain and varying within given intervals $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_K)^\top$. For $\boldsymbol{a} = [\underline{a}, \overline{a}]$, define the mid-point $a^c := (\underline{a}+\overline{a})/2$, the radius $a^\Delta := (\overline{a}-\underline{a})/2$ and the absolute value (magnitude) $|\boldsymbol{a}| := \max\{|\underline{a}|, |\overline{a}|\}$. For $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{IR}$,

$$\boldsymbol{a} \subseteq \boldsymbol{b} \quad \Longleftrightarrow \quad \underline{b} \leq \underline{a} \text{ and } \overline{a} \leq \overline{b}.$$

The above functions and the inclusion relation are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (1), we assume that $p_k^\Delta > 0$ for all $1 \leq k \leq K$.

The parameter dependent linear system (1) presents a generalization of the non-parametric interval linear system $\boldsymbol{A}x = \boldsymbol{b}$, $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$, $\boldsymbol{b} \in \mathbb{IR}^m$, which was mainly studied until recently. Thus, a non-parametric interval system $\boldsymbol{A}x = \boldsymbol{b}$ can be considered as a special case of a parametric system, namely the parametric system involving $mn+m$ interval parameters $a_{ij} \in \boldsymbol{a}_{ij}$, $b_i \in \boldsymbol{b}_i$, $1 \leq i \leq m$, $1 \leq j \leq n$. However, the practical problems, which require solving linear algebraic systems, are usually described by complicated dependencies between the uncertain model parameters. Therefore, the parameter dependent linear systems (1) usually provide more precise settings and more precise results for the real-life problems involving uncertainties, cf. [1].

We consider parametric AE-solution sets of the system (1), which are defined by

$$\begin{aligned} \Sigma^p_{AE} &= \Sigma_{AE}(A(p), b(p), \boldsymbol{p}) \\ &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}})(A(p)x = b(p))\}, \end{aligned} \tag{2}$$

where $\mathcal{A}$ and $\mathcal{E}$ are sets of indexes such that $\mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}$, $\mathcal{A} \cap \mathcal{E} = \emptyset$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$. Among the AE-solution sets most studied and of particular practical interest are: the (parametric) united solution set

$$\Sigma_{\text{uni}}(A(p), b(p), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\exists p \in \boldsymbol{p})(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}})(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}$$

and the (parametric) controllable solution set

$$\Sigma(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \boldsymbol{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \boldsymbol{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \boldsymbol{p}_{\mathcal{E}})(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

For interpretation and applications of AE-solution sets see, e.g., [2].

In interval analysis we are interested in methods for finding inner or outer interval estimation of a solution set. This work is concerned with methods for inner estimation of parametric AE-solution sets that have linear shape. A parametric AE-solution set has linear shape if the boundary of the solution set consists of parts of hyperplanes. In other words, the inequalities describing a parametric AE-solution set with linear shape involve only linear functions on the coordinate variables. All non-parametric AE-solution sets have linear shape. The parametric tolerable solution sets also have linear shape [1]. Some sufficient conditions for a general parametric AE-solution set to have linear shape are proven in [3]. It is shown in [3] that parametric solution sets with linear boundary appear often in the applied domains. Methods for inner interval estimation of non-parametric solution sets are studied by several authors. A. Neumaier presented in [4] the so-called "centered" approach with most details for the non-parametric united and tolerable solution sets. S. Shary developed a formal algebraic approach for finding inner/outer estimations of the non-parametric AE-solution sets, see [2] for details. This approach is further developed by A. Goldsztejn in [5]. O. Beaumont and B. Philippe apply linear programming techniques in [6] to solve the non-parametric interval tolerance problem.

Definition 1. An interval vector $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \subseteq \Sigma$, is

- *inclusion maximal* for a solution set $\Sigma \neq \emptyset$ if for every $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{x} \subseteq \boldsymbol{y}$, $\boldsymbol{x} \neq \boldsymbol{y}$, there is $\boldsymbol{y} \not\subseteq \Sigma$.

- *size maximal* for a solution set $\Sigma \neq \emptyset$ if for every $\boldsymbol{y} \in \mathbb{R}^n$ such that $x^\Delta < y^\Delta$, there is $\boldsymbol{y} \not\subseteq \Sigma$.

Clearly, size maximal interval vector is also inclusion maximal.

A solution set may have infinitely many inclusion and size maximal inner estimations that are incomparable. See Example 1 for an illustration. It is proven that the formal algebraic approach [2] and the linear programming approach [6] provide inclusion maximal inner interval estimations for a non-parametric AE-solution set.

In [7] the centered approach of A. Neumaier was modified for inner estimation of parametric tolerable solution sets and further generalized to parametric AE-solution sets with linear shape in [8]. This approach has small computational complexity, which makes it appropriate for large parametric systems, and provides guaranteed inner interval estimation in floating point interval computations. The parametric centered approach has some advantages over the algebraic approach in inner estimation of non-parametric tolerable solution sets, which are discussed in [7] together with other practical applications. However, the centered method does not provide inclusion maximal inner estimation in general. It requires knowledge of a point $\tilde{x}$ which belongs to the interior of the solution set. Different interior points may yield inner estimations with different quality. How to choose a suitable interior point $\tilde{x}$ is mentioned as an open problem.

In this work we study a linear programming approach for finding inclusion and size maximal inner interval estimations of parametric AE-solution sets with linear shape. In order to comprehend better the methods and their application, in Section 2 we present them in most details for parametric tolerable solution sets where the right hand side of the system involves independent intervals. In that section we first show that finding a maximal inner box is an NP-hard problem. This justifies the exponential size of the linear programming problems that have to be solved for finding an inner interval box which is maximal with respect to interval inclusion or other criteria. We present also another method, which reduces to linear programming problem with polynomial size. The latter approach provides inner interval vector which is not always inclusion maximal, however it is usually larger than the interval vector resulted from the centered approach. In Section 3 we present how to apply the methods from Section 2 to general parametric AE-solution sets with linear shape. In Section 4 we discuss a hybrid numerical approach which uses the approximate optimal inner point $\tilde{x}$, obtained by particular linear program, as input of the centered approach to obtain a guaranteed inclusion maximal inner interval box for the parametric tolerable solution set. Various numerical examples in this section illustrate the discussed methods and their properties. We illustrate also the particular usefulness of the linear programming approach for (inclusion maximal) inner estimation of solution sets that are disconnected or almost disconnected. The paper ends by some conclusions.

2. Parametric Tolerable Solution Set

In this section we consider parametric linear systems with non-parametric right-hand side $\boldsymbol{b}$ and the parametric tolerable solution set $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$. The cases where the right-hand side is parameter dependent are discussed in Section 3. The parametric tolerable solution set $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is described in [1] by m inequalities

$$|A(p^c)x - b^c| + \sum_{k=1}^{K} p_k^{\Delta} |A_k x| \leq b^{\Delta}. \tag{3}$$

Let $e = (1, \ldots, 1)^\top$ be the vector of all ones; its dimension will be inferred from the application context. Denote by A_{i*} the i-th row of a matrix $A \in \mathbb{R}^{m \times n}$. Similarly, $A_{k,i*}$ denotes the i-th row of the matrix A_k. Denote by $\{\pm 1\}^m$ the set of all ± 1 vectors in $\mathbb{R}^m$, i.e., $\{\pm 1\}^m := \{x \in \mathbb{R}^m \mid |x| = (1, \ldots, 1)^\top\}$.

First we show that computing the maximal inner box in $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is NP-hard. More precisely, in order to concern a decision problem, we show that checking whether a prescribed box lies in the tolerable solution set is co-NP-hard, that is, its complement is NP-hard. Moreover, this result is valid even when we restrict on one equation only and a box equal to the cube $[-e, e]$.

Theorem 1. *Checking whether $[-e, e] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is a co-NP-hard problem on a class of problems with one equation.*

Proof. By [9], checking solvability of a system

$$|My| \leq e, \quad e^T |y| > 1$$

is NP-complete on a class of problems with a non-negative positive definite rational matrix M. Substitute $x := My$ to get

$$|x| \leq e, \quad e^T|M^{-1}x| > 1. \tag{4}$$

Now, define $A_0 = 0$, $A_k = (M^{-1})_{k*}$, $k \in \{1,\dots,K\}$, $\boldsymbol{b} := [-1,1]$, and $\boldsymbol{p} := [-e,e]$. In this case, $\Sigma_{tol}(A(p),\boldsymbol{b},\boldsymbol{p})$ is described according to (3) by the only inequality

$$e^T|M^{-1}x| \leq 1.$$

Thus, $[-e,e] \subseteq \Sigma_{tol}(A(p),\boldsymbol{b},\boldsymbol{p})$ if and only if the system (4) has no solution. □

In view of this result, we can hardly expect an efficient algorithm for determining the maximal inner box in $\Sigma_{tol}(A(p),\boldsymbol{b},\boldsymbol{p})$. In the following subsections, we therefore propose methods that are exponential in the number of parameters K and in the number of variables n, respectively. Then, we present a polynomial time heuristic that produces large, but not necessarily maximal, inner interval vector.

2.1. Exponential algorithm in K

The characterization (3) can equivalently be formulated as

$$\pm(A(p^c)x - b^c) + \sum_{k=1}^{K} \pm p_k^\Delta (A_k x) \leq b^\Delta$$

for all combinations of $\pm$ signs. More precisely, we have

$$\bigwedge_{s\in\{\pm1\}^K} \begin{cases} -A(p^c)x + \sum_{k=1}^K s_k p_k^\Delta A_k x \leq -\underline{b} & \wedge \\ A(p^c)x + \sum_{k=1}^K s_k p_k^\Delta A_k x \leq \overline{b}. \end{cases} \tag{5}$$

where $\bigwedge$, $\wedge$ denote the logical operation "And" and $s = (s_1,\dots,s_K)^\top$.

The number of linear inequalities (5) is $m2^{K+1}$. More delicate estimation can be derived as follows. Let k_i be the number of parameters that appear in i-th equation. Then the number of linear inequalities (5) after omitting the redundant ones is $\sum_{i=1}^m 2^{k_i+1}$. The reduction in the number of inequalities may be significant. Consider the example when the parameters describe symmetry of the constraint matrix of size n, and the right-hand side vector is real. Herein, $K = \frac{1}{2}n(n+1)$, so the simple estimation on the number of inequalities is $n2^{n(n+1)/2+1}$. On the other hand, each equation contains n parameters, whence $\sum_{i=1}^{n} 2^{k_i+1} = n2^{n+1}$. In any case, all numerical inequalities that appear more than once in (5) should be removed.

Denote the system (5) briefly by $Bx \leq c$. Now, it is a standard linear system of inequalities and we can apply the tolerance analysis [10, 11, 12, 13, 14] to compute tolerances for prescribed solutions. Below, we adapt some techniques to our case.

2.1.1. Symmetric boxes

Let a solution $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ be given, as well as a non-negative vector $d \in \mathbb{R}^n$ representing ratios of the side lengths of the inner box in question. The aim is to find a maximal inner box in $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ having the center in $\tilde{x}$ and the side lengths proportional to d. Formally speaking, find maximal $\delta \geq 0$ such that $[\tilde{x} - \delta d, \tilde{x} + \delta d] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$.

It is well known [12, Theorem 2.1] (or [15]) that the maximal tolerance, denoted by δ^*, is computable by the simple formula

$$\delta^* := \inf_{i:|B_{i*}|d>0} \frac{c_i - B_{i*}\tilde{x}}{|B_{i*}|d}. \tag{6}$$

Due to the symmetry of the box $[\tilde{x} - \delta^* d, \tilde{x} + \delta^* d]$ around $\tilde{x}$, it needn't be the maximal inner box in the solution set with respect to inclusion (it is maximal only over the symmetric boxes around $\tilde{x}$). That is why various techniques were developed to *extend* such symmetric boxes to the maximal ones; see [12, Section 2.3] and the references therein.

2.1.2. Variable centers

If $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is not known, it is natural to seek for such a solution, for which the corresponding tolerance δ would be maximized. This can be computed by solving the linear program

$$\max \delta;\ \ (|B|d)\delta + Bx \leq c,\ \ \delta \geq 0. \tag{7}$$

Of course, the resulting optimal tolerance is not less than (6). The linear program (7) is justified by the following observation.

Proposition 1. *We have* $[x - \delta d, x + \delta d] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ *if and only if* $(|B|d)\delta + Bx \leq c,\ \ \delta \geq 0$.

Proof. "If." Any vertex of $[x-\delta d, x+\delta d]$ has the form $x+\delta\,\mathrm{Diag}(s)d$, where $s \in \{\pm 1\}^n$. For such vertices, we have

$$B(x+\delta\,\mathrm{Diag}(s)d) = Bx + B\delta\,\mathrm{Diag}(s)d \leq Bx + \delta|B|d \leq c,$$

that is, they satisfy the inequalities (5). Due to convexity, all points of $[x-\delta d, x+\delta d]$ must fulfill the inequalities, too.

"Only if." Any vertex of $[x-\delta d, x+\delta d]$ has to satisfy

$$B(x+\delta\,\mathrm{Diag}(s)d) \leq c.$$

Its i-th inequality reads

$$B_{i*}(x+\delta\,\mathrm{Diag}(s)d) \leq c_i.$$

Define $s := \mathrm{sgn}(B_{i*})$, and for the corresponding vertex we have

$$B_{i*}x + \delta|B_{i*}|\delta d = B_{i*}(x+\delta\,\mathrm{Diag}(s)d) \leq c_i$$

which closes the proof. $\square$

As a consequence, we have the following.

Proposition 2. *Denote by δ^* and x^* the optimal value and the optimal solution of* (7)*, respectively. Below, Σ_{tol}^p stands for $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$.*

1. *If* (7) *is infeasible, then $\Sigma_{tol}^p = \emptyset$.*
2. *If $\delta^* = 0$, then $\Sigma_{tol}^p \neq \emptyset$, but it does not have a full dimension.*
3. *If $\delta^* > 0$, then $\Sigma_{tol}^p \neq \emptyset$, it has full dimension and $[x^*-\delta^* d, x^*+\delta^* d]$ is a size maximal inner box of Σ_{tol}^p having side lengths proportional to d.*

Remark 1. Some other objective functions could be used in seeking for an inclusion maximal inner box. For instance, we can compute an inner box with a maximal sum of the side lengths by employing the linear program

$$\max e^T d;\ |B|d + Bx \leq c,\ d \geq \delta^* e,$$

or to find an inner box with maximal volume by calling the nonlinear program

$$\max \prod_{i=1}^n d_i;\ |B|d + Bx \leq c,\ d \geq \delta^* e.$$

Remark 2. All linear programming approaches, discussed in Section 2 are applicable even if the point $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is on the boundary of the solution set. However, in this case the obtained inner box will be a point vector. Therefore, it is required in [4, 7, 8] that $\tilde{x}$ belongs to the topological interior of $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$.

2.2. Polynomial Heuristic Approach

By (3), $x \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ if and only if there are $y_0, \ldots, y_K \in \mathbb{R}^m$ such that

$$\begin{aligned} \sum_{i=0}^{K} y_i &\le b^{\Delta}, \\ \bigwedge_{s\in\{\pm 1\}} s(A(p^c)x - b^c) &\le y_0, \\ \bigwedge_{s_k\in\{\pm 1\}} s_k p_k^{\Delta} A_k x &\le y_k, \quad k = 1, \ldots, K. \end{aligned} \tag{8}$$

This characterization is linear and has polynomial size $m(3 + 2K)$. The characterization stands behind polynomiality of checking nonemptiness of $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ and computability of a solution of $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$. For the non-parametric interval case see [16, Section 2.16]. Denote the system (8) briefly as

$$Cx + Dy \le c, \tag{9}$$

where

$$\begin{aligned} x &= (x_1, \ldots, x_n)^{\top}, \\ y &= (y_0^{\top}, y_1^{\top}, \ldots, y_k^{\top})^{\top}, \\ C &= (0_{n,m}, A^{\top}(p^c), p_1^{\Delta} A_1^{\top}, \ldots, p_K^{\Delta} A_K^{\top}, -A^{\top}(p^c), -p_1^{\Delta} A_1^{\top}, \ldots, -p_K^{\Delta} A_K^{\top})^{\top}, \\ D &= (D_0^{\top}, -I_{m(K+1)}, -I_{m(K+1)})^{\top}, \\ c &= ((b^{\Delta})^{\top}, (b^c)^{\top}, 0_{1,mK}, (-b^c)^{\top}, 0_{1,mK})^{\top}. \end{aligned}$$

Herein, $D_0 \in \mathbb{R}^{m\times m(K+1)}$ is defined as

$$d_{0,ij} = \begin{cases} 1, & \text{for } j \in \{i, i+m, \ldots, i+mK\} \\ 0, & \text{otherwise} \end{cases} \qquad 1 \le i \le m,\ 1 \le j \le m(K+1),$$

$0_{m,n}$ denotes the $m \times n$ zero matrix, and I_n the identity matrix of size n.

Symmetric boxes

Again, we can apply various methods from tolerance analysis to the system (9). Concretely, given $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$, a symmetric inner box of the form $[\tilde{x} - \delta' d, \tilde{x} + \delta' d] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is determined by solving the linear program

$$\delta' := \max \delta; \ (|C|d)\delta + Dy \leq c - C\tilde{x}, \ \delta \geq 0, \ y \geq 0. \tag{10}$$

Feasibility of this inner box is proved as in Proposition 1, so we omit it. However, optimality cannot be established in general; the resulting inner box may not be inclusion maximal.

By eliminating the variables y in (10), we obtain an explicit formula for δ'.

Proposition 3. *For δ' in (10) we have*

$$\delta' = \inf_{i:v_i>0} \frac{u_i}{v_i}, \tag{11}$$

where

$$u := b^{\Delta} - |A(p^c)\tilde{x} - b^c| - \sum_{k=1}^{K} p_k^{\Delta} |A_k \tilde{x}|,$$
$$v := |A(p^c)|d + \sum_{k=1}^{K} p_k^{\Delta} |A_k| d.$$

Proof. The constraint $C\tilde{x} + (|C|d)\delta + Dy \leq c$ reads

$$\sum_{i=0}^{K} y_i \leq b^{\Delta},$$
$$\bigwedge_{s \in \{\pm 1\}} s(A(p^c)\tilde{x} - b^c) + |A(p^c)|d\delta \leq y_0,$$
$$\bigwedge_{s_k \in \{\pm 1\}} s_k p_k^{\Delta} A_k \tilde{x} + p_k^{\Delta} |A_k| d\delta \leq y_k, \quad k = 1, \ldots, K.$$

By eliminating the variables y we obtain

$$|A(p^c)\tilde{x} - b^c| + \sum_{k=1}^{K} p_k^{\Delta} |A_k \tilde{x}| + |A(p^c)|d\delta + \sum_{k=1}^{K} p_k^{\Delta} |A_k| d\delta \leq b^{\Delta}. \tag{12}$$

Expressing δ, we get (11). □

Proposition 4. *The optimal value δ' of* (10) *is the same as that obtained by the modified Neumaier's approach (the parametric centered approach) from [7].*

Proof. In [7], the maximal $\delta \geq 0$ is determined such that

$$A_0\tilde{x} + \sum_{k=1}^{K}(A_k\tilde{x})\boldsymbol{p}_k + \delta\left(A_0 + \sum_{k=1}^{K}A_k\boldsymbol{p}_k\right)[-d,d] \subseteq \boldsymbol{b}.$$

By using the property that two intervals satisfy $\boldsymbol{u} \subseteq \boldsymbol{v}$ iff $|u^c - v^c| + u^\Delta \leq v^\Delta$, we rewrite the above inclusion as

$$|A(p^c)\tilde{x} - b^c| + \sum_{k=1}^{K} p_k^\Delta |A_k\tilde{x}| + \delta|A(\boldsymbol{p})|d \leq b^\Delta.$$

Since for each interval $\boldsymbol{v}$ we have $|\boldsymbol{v}| = |v^c| + v^\Delta$, we can write

$$|A(\boldsymbol{p})| = |A(p^c)| + \sum_{k=1}^{K} p_k^\Delta |A_k|,$$

which completes the proof as we arrived at (12). □

Variable centers

When no $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ is known, we relax the box center $\tilde{x}$ and solve the linear program

$$\max \delta;\ (|C|d)\delta + Cx + Dy \leq c,\ \ \delta \geq 0,\ \ y \geq 0. \tag{13}$$

When δ' is the optimal value, and x' the optimal solution of (13), then $[x' - \delta'd, x' + \delta'd] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$.

Unfortunately, the above two approaches by (11) and (13) based on linear programming do not yield maximal inner boxes in general; Example 1 gives an illustration. This can be explained by our goal to compute maximal inner box $[x^* - \delta^* d, x^* + \delta^* d]$ in the projection of the convex polyhedron (9) into the x-subspace, but the embedded box $([x^* - \delta^* d, x^* + \delta^* d,], y)$ in the (x, y)-space may not be a subset of (9) itself.

In summary, we have the following polynomially computable conditions.

Proposition 5. *Denote by δ' and x' the optimal value and the optimal solution of (13), respectively.*

1. *If* (13) *is infeasible, then* $\Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p}) = \emptyset$.
2. *If* $\delta' \geq 0$*, then* $[x' - \delta' d, x' + \delta' d] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$.

In view of Proposition 4, we have the following.

Corollary 1. *The optimal value of* (13) *is greater than or equal to that one obtained by the modified Neumaier's approach (the parametric centered approach) from [7].*

2.3. Exponential algorithm in n

Even it is not very frequent, sometimes n may be substantially smaller than K. A typical example is when the parameters describe symmetry of the constraint matrix, wherein $K = \frac{1}{2}n(n+1)$. So an algorithm for the tolerance problem that is exponential with respect to n is of interest, too.

Let $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$ be given. The idea behind this algorithm is to determine the maximal span of each vertex of the inner box $[\tilde{x} - \delta d, \tilde{x} + \delta d] \subseteq \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$, and take the minimum of them. The maximal span is calculated by linear programming using the linear characterization (9), so we have to solve 2^n linear programs in total.

The algorithm works as follows. The maximum tolerance draws

$$\delta^* := \min_{s \in \{\pm 1\}^n} \delta_s^*,$$

where

$$\delta_s^* := \max \delta;\ \ (CD_s d)\delta + Dy \leq d - C\tilde{x},\ \ \delta, y \geq 0.$$

This linear program computes the maximal span in the direction of the vertex $\tilde{x} + \delta D_s d$ of the inner box $[\tilde{x} - \delta d, \tilde{x} + \delta d]$ determined by $s \in \{\pm 1\}^n$. More precisely, it calculates the maximal δ such that $\tilde{x} + \delta D_s d \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$. Due to convexity of the solution set, δ^* is an admissible tolerance.

The disadvantage of this method is that we have to know an initial solution $\tilde{x} \in \Sigma_{tol}(A(p), \boldsymbol{b}, \boldsymbol{p})$. It is not clear how to generalize this approach to variable centers.

3. Parametric *AE*-Solution Sets with Linear Shape

In this section we consider general parametric AE-solution sets (2) with linear shape. All known methods for inner estimation of parametric or non-parametric AE-solution sets require knowledge of the explicit description of the solution set. The description could be in form of inequalities or in form of equivalent interval inclusions. In general, explicit description of a parametric AE-solution set is obtained by the so-called Fourier-Motzkin elimination of existentially quantified parameters [17]. If a parametric interval linear system involves existentially quantified parameters, such that every one appears in only one equation of the initial system (1), the parametric AE-solution set is described by [18]

$$\sum_{k\in\mathcal{A}} \boldsymbol{p}_k(A_k x - b_k) \subseteq b_0 - A_0 x + \sum_{k\in\mathcal{E}} \boldsymbol{p}_k(b_k - A_k x), \tag{14}$$

equivalently

$$|A(p^c)x - b(p^c)| \ \ \le \ \ \sum_{k=1}^{K} \delta_k |A_k x - b_k| p_k^{\Delta}, \tag{15}$$

where $\delta_k := \{1 \text{ if } k \in \mathcal{E},\ -1 \text{ if } k \in \mathcal{A}\}$. Eliminating an existentially quantified parameter ($\mathcal{E}$-parameter) that appears in more than one equation of the initial parametric system generates additional (so-called "cross") inequalities, respectively inclusions. The number of cross inequalities grows exponentially with the number of E-parameters that appear in more than one equation of the initial system. The degree of the polynomials involved in the characterizing inequalities/inclusions depend, in general, on the number of $\mathcal{E}$-parameters involved in more than one row of the parametric matrix. If a parametric AE-solution set has linear shape (boundary), the generated cross inequalities/inclusions involve only multi-linear polynomials, or can be reduced to inequalities involving only multi-linear polynomials of the coordinate variables, cf. [3]. Denote by

$$\begin{array}{c} U(p) = U_0 + \sum_{k=1}^{K} p_k U_k \\ v(p) = v_0 + \sum_{k=1}^{K} p_k v_k \\ U_k \in \mathbb{R}^{q\times n}, \quad v_k \in \mathbb{R}^q,\ k = 0, \ldots, K,\ q \ge m, \end{array} \tag{16}$$

the expanded parametric matrix, resp. parametric vector, that correspond to the explicit description of a parametric AE-solution set with linear shape.

Note, that the first m rows of $U(p)$ and the first m elements of $v(p)$ are those of the initial matrix $A(p)$ and the initial vector $b(p)$ respectively. Thus, a point $x \in \mathbb{R}^n$ belongs to $\Sigma^p_{AE} = \Sigma_{AE}(A(p), b(p), \boldsymbol{p})$, if and only if

$$|U(p^c)x - v(p^c)| \leq \sum_{k=1}^{K} \delta_k p_k^{\Delta} |U_k x - v_k|, \quad \delta_k := \begin{cases} 1 & \text{if } k \in \mathcal{E}, \\ -1 & \text{if } k \in \mathcal{A} \end{cases} \tag{17}$$

equivalently

$$\sum_{k\in\mathcal{A}} \boldsymbol{p}_k(U_k x - v_k) \subseteq v_0 - U_0 x + \sum_{k\in\mathcal{E}} \boldsymbol{p}_k(v_k - U_k x). \tag{18}$$

The absolute value inequalities (17) are introduced in [19, 18] in order to reduce the number of the solution set characterizing inequalities and to unify their representation. The linear programming approach, introduced in Section 2, requires explicit description of a parametric AE-solution set in form of logical combination of linear inequalities. Therefore, one interested in applying this approach has to do the Fourier-Motzkin elimination of existentially quantified parameters without using absolute values for the coefficients of the parameters. Otherwise, any one of the equivalent representations (17), (18) can be transformed equivalently in the following logical combination of linear inequalities

$$\bigvee_{s_{\mathcal{E}}\in\{\pm1\}^{k_2}} \bigwedge_{s_{\mathcal{A}}\in\{\pm1\}^{k_1}} \begin{cases} \mu_0(x) + \mu_{\mathcal{A}}(x) - \mu_{\mathcal{E}}(x) \leq 0 & \wedge \\ -\mu_0(x) + \mu_{\mathcal{A}}(x) - \mu_{\mathcal{E}}(x) \leq 0, \end{cases} \tag{19}$$

where

$$\begin{aligned} \mu_0(x) &= U(p^c)x - v(p^c) \\ \mu_{\mathcal{A}}(x) &= \sum_{k\in\mathcal{A}} s_{\mathcal{A},k} p_k^{\Delta}(U_k x - v_k) \\ \mu_{\mathcal{E}}(x) &= \sum_{k\in\mathcal{E}} s_{\mathcal{E},k} p_k^{\Delta}(U_k x - v_k) \\ k_1 &= \mathrm{Card}(\mathcal{A}),\ k_2 = \mathrm{Card}(\mathcal{E}),\ k_1 + k_2 = K \\ s_{\mathcal{A}} &= (s_{\mathcal{A},1}, \ldots, s_{\mathcal{A},k_1})^\top \\ s_{\mathcal{E}} &= (s_{\mathcal{E},1}, \ldots, s_{\mathcal{A},k_2})^\top \\ s_{\mathcal{A},k} &= s_{\mathcal{E},k} = 1 \text{ when } U_k = 0. \end{aligned}$$

Denote by $B(s_{\mathcal{E}})$, $c(s_{\mathcal{E}})$ the numerical matrix and the numerical vector that correspond to (19) for a fixed $s_{\mathcal{E}} \in \{\pm 1\}^{k_2}$. Then we obtain for (19)

$$\bigvee_{s_{\mathcal{E}} \in \{\pm 1\}^{k_2}} B(s_{\mathcal{E}})x - c(s_{\mathcal{E}}) \leq 0. \tag{20}$$

Since the solution of $B(s_{\mathcal{E}})x - c(s_{\mathcal{E}}) \leq 0$ defines a convex polyhedron, we proved the following proposition.

Proposition 6. *Every parametric AE-solution set with linear shape is a union (20) of convex polyhedra.*

Due to the representation (20) we can apply the linear programming approach from Section 2.1.2 to every one of the polyhedra in (20).

Every parametric tolerable solution set $\Sigma(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \boldsymbol{p})$ is a convex polyhedron [1], despite the parameter dependencies in $b(p)$ and the number of additional cross inequalities/inclusions in its description (17), resp. (18). Thus, the linear programming approaches, discussed in Section 2 are applicable to and most efficient for parametric tolerable solution sets. Linear programming approach (7), applied to the corresponding explicit description (17), resp. (18), provides an inner interval box with maximal size. Example 3 is an illustration.

In general, a union of convex polyhedra is not convex. Therefore, the application of linear programming approach from Section 2.1.2 to general parametric AE-solution sets with linear shape has some pros and cons discussed in Section 4.2.

4. Examples

For practical applications it is important that an inner estimation does not contain points that do not belong to the solution set. From computational point of view inclusion and size maximal inner estimations are defined by exact numbers and can be computed by exact (e.g., rational) arithmetic. A numerical algorithm in floating point arithmetic must guarantee that the obtained inner interval estimation is contained in the solution set and in the corresponding exact maximal inner estimation. The numerical algorithms, presented in [7, 8], employ interval arithmetic in order to guarantee that their implementation in floating point environment will provide guaranteed inner estimations. Guaranteed solutions of linear programming problems in floating point computations can be provided by the verification approaches discussed in [20, 21], however, their implementations are not widely available. Alternatively, the present linear programming approach can be combined with the guaranteed parametric centered approach [7, 8] in order to obtain a guaranteed inclusion and size maximal inner estimation. Namely, the present linear programming approach (7) yields a floating point approximate point $\tilde{x}$ (and the approximate side lengths d) which is then an input for the guaranteed centered approach.

In the following examples, we use $d := e$.

4.1. Tolerable Solution Sets

Example 1 ([7, Example 1])**.** Find inner estimations for the parametric tolerable solution set of the linear system

$$\begin{pmatrix} p_1 & p_1 + 1/2 \\ -2p_2 & p_2 + 1 \end{pmatrix} x = \begin{pmatrix} [-1, 2] \\ [-3, 3] \end{pmatrix}, \qquad p_1, p_2 \in [0, 1]. \tag{21}$$

For the solution $x = (3/7, 2/7)^\top$ of $A(p^c)x = b^c$, by the parametric centered approach we obtain $\delta = 16/35$ and the interval vector $([-1/35, 31/35], [-6/35, 26/35])^\top$, which is an inner estimation of the parametric tolerable solution set presented[1] in Figure 1.

Now, we apply the approaches presented in Section 2. The explicit description of the parametric tolerable solution set is

$$\begin{aligned} |-1/2 + x_1/2 + x_2| &\leq 3/2 - |x_1 + x_2|/2 \\ |-x_1 + 3x_2/2| &\leq 3 - |-2x_1 + x_2|/2. \end{aligned}$$

Its linear form (5) after removing the repeated inequalities reads

$$\begin{aligned} -3 + 2x_1 - 2x_2 \leq 0, &\qquad -1 - x_1 - 3x_2/2 \leq 0, \\ -3 - x_2 \leq 0, &\qquad -1 - x_2/2 \leq 0, \\ -2 + x_2/2 \leq 0, &\qquad -3 + x_2 \leq 0, \\ -2 + x_1 + 3x_2/2 \leq 0, &\qquad -3 - 2x_1 + 2x_2 \leq 0, \end{aligned}$$

from where we get the matrix B and the vector c. For $\tilde{x} = (3/7, 2/7)^\top$ and using (6), the maximal symmetric tolerance $\delta^* = 16/35$ is obtained as by the

[1] The explicit description of a parametric AE-solution set by inequalities is used also for its visualization.

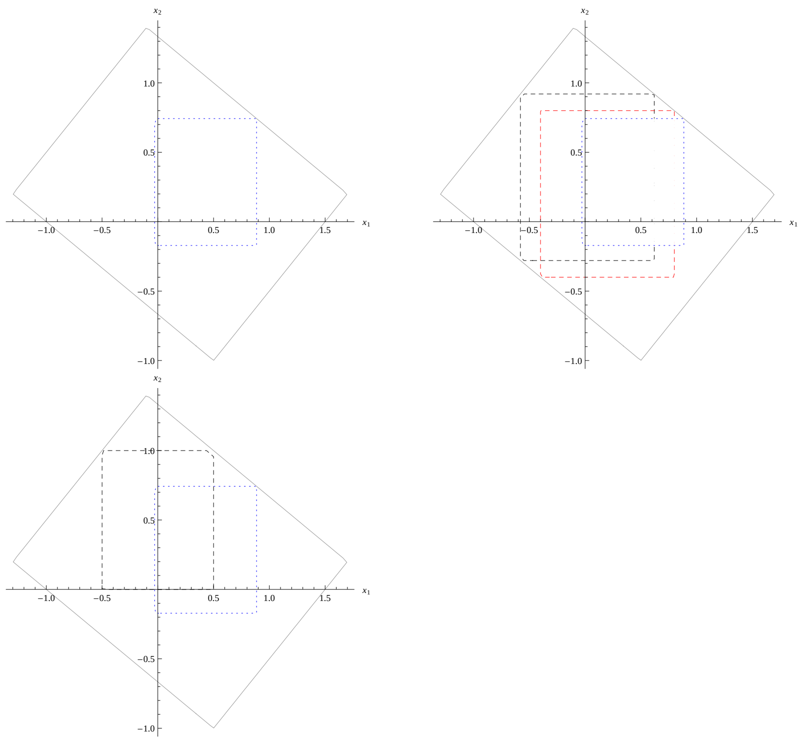

Figure 1: The parametric tolerable solution set for the system from Example 1 together with: a) its inner estimation (dotted, blue) found by the centered approach; b) the size maximal inner estimations found by (7); c) the inner estimations obtained by the heuristic approach from Section 2.2 and by the parametric centered approach.

parametric centered approach. Relaxing the center $\tilde{x}$, the linear program (7) computed by the *Mathematica*® function `Maximize`[2] yields a better center $x^* = (1/5, 1/5)^\top$ with the largest tolerance $\delta^* = 3/5$, which gives the size maximal inner box $([-0.4, 0.8], [-0.4, 0.8])^\top$. Computing the linear program (7) by the *Mathematica*® function `NMaximize` yields another (now, approximate) center $x^* = (0.02, 0.32)^\top$ with the largest tolerance $\delta^* = 0.6$, which gives another also size maximal inner box $([-0.58, 0.62], [-0.28, 0.92])^\top$. Both size maximal inner boxes are presented in Figure 1 b). `NMaximize` with

[2]The *Mathematica*® function `Maximize` returns exact results if all input data are exact. The numerical function `NMaximize` returns approximate values and supports four numerical methods. Both functions can always find global maxima if the function and the constraints are linear.

the method “RandomSearch” returns a slightly different approximate center $x^* = (0.0261442, 0.315904)^\top$.

As discussed in the beginning of this section, exact optimal solutions, respectively inclusion maximal inner boxes, can be found for small systems. For large problems we use the linear programming approach to find approximate optimal points $\tilde{x}$ which are used as initial data by the implementation of the parametric centered method to obtain guaranteed inner estimation close to the inclusion maximal inner interval vector in the parametric solution set.

The polynomial heuristic from Section 2.2 works as follows. The corresponding linear program (13), cannot be solved by the *Mathematica*® function `Maximize` in reasonable time, probably due to the many $(n + m(K + 1))$ variables involved. The numerical function `NMaximize` yields an approximate optimal value $\delta' = 0.5$ and an approximate optimal solution $\tilde{x} = (2.77556 * 10^{-17}, 0.5)^\top$, which gives the approximate inner box $([-0.5, 0.5], [0, 1])^\top$. Since we obtain approximate numerical values, we have to combine the obtained approximate optimal solution $\tilde{x}$ with guaranteed computations by the parametric centered approach in order to obtain a guaranteed inner box in the solution set. The obtained optimal value $\delta' = 0.5$ is slightly bigger than that obtained by the parametric centered approach $\delta = 16/35 \approx 0.457143$. However, the inner interval vector obtained by the heuristic linear programming approach is not inclusion maximal, as seen in Figure 1 c).

The symmetric inner box, obtained by the centered approach for the midpoint solution $\tilde{x} = (3/7, 2/7)^\top$, can be *expanded* to an inclusion maximal one (nonsymmetric around $\tilde{x}$) by applying the algorithms discussed in [12, Section 2.3].

Example 2 (Example 6.3 in [5], see also [7])**.** Consider the nonparametric tolerable solution set $\Sigma_{tol}(\boldsymbol{A}, \boldsymbol{b})$, where $\boldsymbol{A} = A(\boldsymbol{p})$ and

$$A(p) := \begin{pmatrix} p_1 & p_2 & 0 & 0 & 0 & 0 \\ p_1 & p_2 & p_3 & 0 & 0 & 0 \\ 0 & p_2 & p_3 & p_4 & 0 & 0 \\ 0 & 0 & p_3 & p_4 & p_5 & 0 \\ 0 & 0 & 0 & p_4 & p_5 & p_6 \\ 0 & 0 & 0 & 0 & p_5 & p_6 \end{pmatrix}, \quad [b] = \begin{pmatrix} [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, -0.9] \\ [0.9, 1.1] \\ [-1.1, -0.9] \end{pmatrix},$$

$p_1, \ldots, p_6 \in [0.999, 1.001]$. Since $\Sigma_{tol}(\boldsymbol{A}, \boldsymbol{b}) = \Sigma_{tol}(A(p)\boldsymbol{b}, \boldsymbol{p})$, the parametric centered approach from [7] gives the following inner interval vector

$$\begin{aligned}(&[-0.0316, 0.0316], [0.9684, 1.0316], [-2.0316, -1.9684],\\ &[1.9684, 2.0316], [-1.0316, -0.9684], [-0.0316, 0.0316])^\top .\end{aligned}$$

An inner estimation of $\Sigma_{tol}(\boldsymbol{A}, \boldsymbol{b})$ in the form of a parallelepiped was found in [5], while a straightforward application of the formal algebraic approach considered in [2] fails. After several trials to squeeze the components of $\boldsymbol{b}$, as recommended by [2, Theorem 6.4], one may come to a system whose formal solution is an inner estimation. For $b_1, b_3, b_5 \in [0.95, 1.1]$, S. Shary has found

$$\begin{aligned}(&[-0.0499, 0.0499], [1.001, 1.0489], [-2.0479, -2.002],\\ &[2.002, 2.0479], [-1.0489, -1.001], [-0.0499, 0.0499])^\top .\end{aligned}$$

Applying the variable center method (7) we obtain $\delta^* \approx 0.0316917$ and the size maximal inner box

$$\begin{aligned}(&[-0.0672410, -0.00385768], [0.968277, 1.03166], [-1.99415, -1.93077],\\ &[1.93077, 1.99415], [-1.03166, -0.968277], [0.00385768, 0.06724108])^\top .\end{aligned}$$

4.2. AE-Solution Sets

Example 3 ([7, Example 2])**.** Find inner estimations for the parametric tolerable solution set of the linear system

$$\begin{pmatrix} p_1 & p_2 \\ -2p_1 & p_2 + 1/2 \end{pmatrix} x = \begin{pmatrix} q_1 \\ q_2 - q_1 \end{pmatrix},$$

where $p_1 \in [0, 1]$, $p_2 \in [1/2, 3/2]$, $q_1, q_2 \in [-1, 2]$.

The set of interval inclusions describing the parametric tolerable solution set is

$$\begin{aligned} [p_1]x_1 + [p_2]x_2 &\subseteq [q_1] \\ \frac{1}{2}x_2 - 2[p_1]x_1 + [p_2]x_2 &\subseteq [q_2] - [q_1] \\ -\frac{1}{2}x_2 + [p_1]x_1 - 2[p_2]x_2 &\subseteq -[q_2], \end{aligned} \tag{22}$$

from where we get the matrix $U(p)$ and the vector $v(q)$. For the solution $x = (3/7, 2/7)^\top$ of the midpoint system $U(\check{p})x = v(\check{q})$, the parametric centered approach, respectively (11), gives $\delta = 2/9$. Thus the interval vector

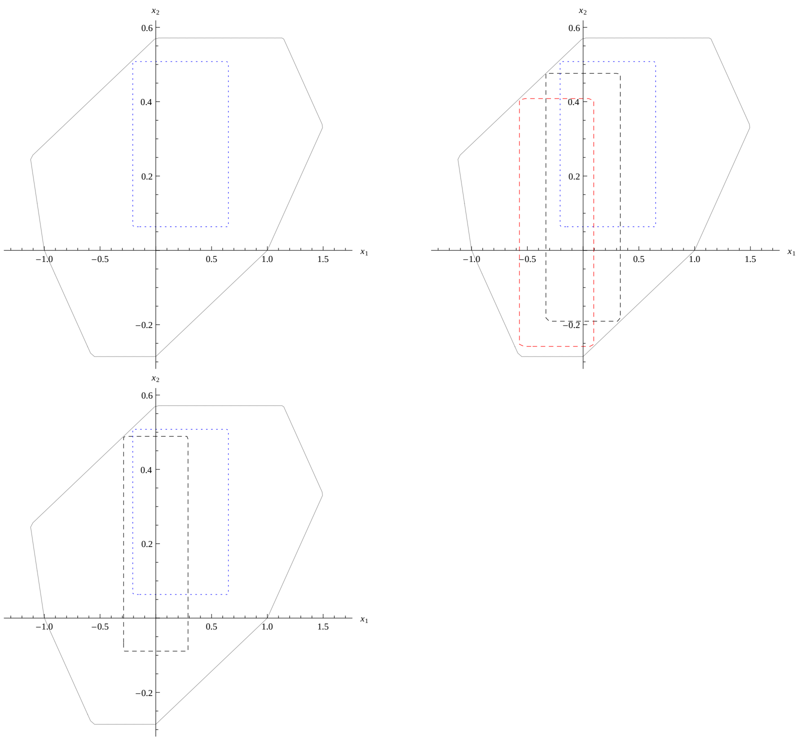

Figure 2: The parametric tolerable solution set for the system from Example 3 together with: a) its inner estimation (dotted, blue) found by the centered approach; b) the size maximal inner estimations found by (7); c) the inner estimations obtained by the heuristic approach from Section 2.2 and by the parametric centered approach.

$([13/63, 41/63], [4/63, 32/63])^\top$ presents an inner estimation of the parametric tolerable solution set, as seen in Figure 2.

Now, we apply the approaches presented in Section 2. The linear form (5) that corresponds to (22) involves 22 inequalities after removing the repeated ones. Relaxing the center $\tilde{x}$, the linear program (7) computed by the *Mathematica*® function `Maximize` yields a better center $x^* = (-19/80, 3/40)^\top$ with the largest tolerance $\delta^* = 1/3$, which gives the size maximal inner box $([-139/240, 23/240], [-31/120, 49/120])^\top$. Computing the linear program (7) by the *Mathematica*® function `NMaximize` yields another (now, approximate) center $x^* = (0, 1/7 \approx 0.142857)^\top$ with the $\delta^* = 0.333333$, which gives another also size maximal inner box $([-0.333333, 0.333333], [-0.190476, 0.047619])^\top$. Both size maximal inner boxes are presented in Figure 2 b).

The polynomial heuristic from Section 2.2 works as follows. Relaxing the center, (13) gives the center $x' = (0, 0.2)^T$ and the tolerance $\delta' = 13/45 \approx 0.288889$.

The next two examples illustrate the advantage of the linear programming approach for inner estimation of disconnected or almost disconnected solution sets.

Example 4 ([4])**.** Find an inner estimation of the united solution set of the following non-parametric linear system

$$\begin{pmatrix} [2,4] & [-1,1] \\ [-1,1] & [2,4] \end{pmatrix} x = \begin{pmatrix} [-3,3] \\ 0 \end{pmatrix},$$

whose united solution set has butterfly-shaped form drawn in Figure 3 b). Since all intervals in the system are considered as independent the explicit description of the united solution set is defined by (14), resp. (15). The corresponding expanded description (20) is union of four convex sets, presented in Figure 3 a) in red and blue. The application of (7) to every one of these four convex sets gives size maximal inner estimations, presented in Figure 3 a) by dash lines. Since two by two of these inner interval boxes intersect, we take the interval hull of the intersecting boxes and obtain two larger inner estimations

$$([-1.4, -0.8], [-0.4, 0.4])^\top \quad \bigcup \quad ([0.8, 1.4], [-0.4, 0.4])^\top,$$

presented in Figure 3 b). Although we obtained quite good inner estimations of the two almost disconnected parts of the united solution set, these are not inclusion maximal, as seen in Figure 3. For a discussion on the inner estimation of almost disconnected solution sets and obtaining inclusion maximal inner estimation of non-parametric AE-solution set see [2, Section 6.3].

Often, especially when the parametric AE-solution set is disconnected, the approximate midpoint solution does not belong to the interior of the solution set which hampers the application of the parametric centered approach.

Example 5. Consider the parametric linear system

$$\begin{pmatrix} p_1 & p_1 + 1 \\ p_2 + 1 & -2p_4 \end{pmatrix} x = \begin{pmatrix} p_3 \\ -3p_2 + 1 \end{pmatrix},$$

where $p_1, p_2 \in [0, 1]$, $p_3, p_4 \in [-1, 1]$. The parametric AE-solution set

$$\Sigma_{\forall p_2 \exists p_1 p_3 p_4} = \{x \in \mathbb{R}^2 \mid (\forall p_2 \in \boldsymbol{p}_2))(\exists p_1 \in \boldsymbol{p}_1, p_3 \in \boldsymbol{p}_3, p_4 \in \boldsymbol{p}_4))(A(p)x = b(p))\}$$

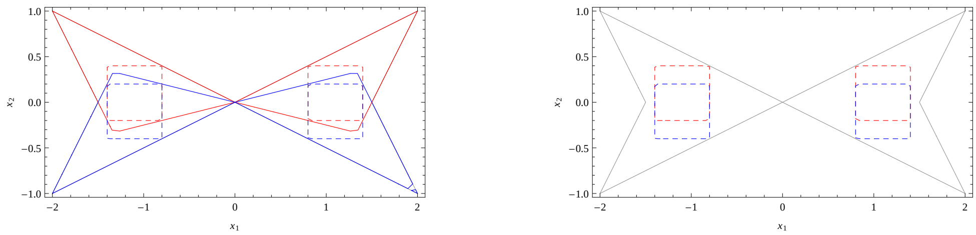

Figure 3: The united solution set for the system from Example 4 consisting of four convex sets (solid lines in a)) and their size maximal inner estimations (dash lines in a) and b)).

is described completely by (14), resp. (15). The solution set is nonempty, bounded and disconnected, as seen in Figure 4. Notice that the midpoint solution $\tilde{x} = (-\frac{1}{3}, \frac{1}{9})^\top$ does not belong to the solution set. For the point $\tilde{x} = (-\frac{1}{3}, -\frac{8}{9})^\top$, which is in the interior of the bottom triangle of the solution set, the parametric centered approach yields the inner interval vector $([-10/27, -8/27], [-25/27, -23/27])^\top$.

The expanded linear description (20) of the solution set consists of four convex sets. One of them is empty, another one corresponds to the line segment in Figure 4 and the remaining two correspond to the two disconnected triangles. The application of (7) to every one of these four convex sets yields the corresponding result described by Proposition 2. Thus, the linear programming approach from Section 3 yields a size maximal inner estimation (dash line in Figure 4)

$$([-1/2, -1/4], [3/4, 1])^\top \quad \bigcup \quad ([-1/2, -1/4], [-1, -3/4])^\top$$

for the disconnected parametric AE-solution set $\Sigma_{\forall p_2 \exists p_1 p_3 p_4}$.

Example 6. We look for an inner estimation of the united solution set Σ^g_{uni} to the parametric system modeling a resistive electrical network, presented in Figure 5, the so-called Okumura's problem [22]. The resistive network consists of two current sources J_1 and J_2 and nine resistors. The problem of finding the voltages $v_1, \ldots, v_5$, when the voltage of each conductance g_i, $i = 1, \ldots, 9$ varies independently in prescribed bounds $\boldsymbol{g}_i$, leads to a parametric linear system.

The explicit description of Σ^q_{uni} by Fourier-Motzkin-like elimination of the

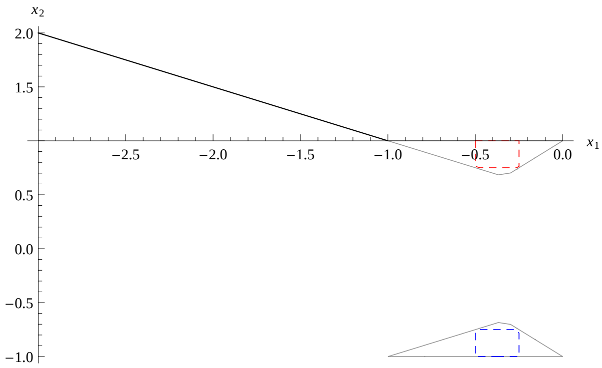

Figure 4: The parametric AE-solution set $\Sigma_{\forall p_2 \exists p_1 p_3 p_4}$ for the system from Example 5 and its size maximal inner estimation (dash line).

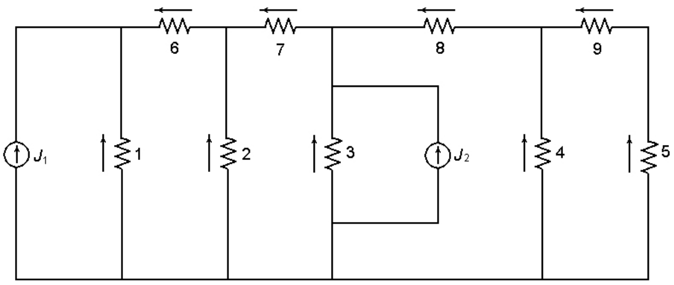

Figure 5: A resistive electrical network after [22].

parameters leads to

$$U(g) = \begin{pmatrix} g_1+g_6 & -g_6 & 0 & 0 & 0 \\ -g_6 & g_2+g_6+g_7 & -g_7 & 0 & 0 \\ 0 & -g_7 & g_3+g_7+g_8 & -g_8 & 0 \\ 0 & 0 & -g_8 & g_4+g_8+g_9 & -g_9 \\ 0 & 0 & 0 & -g_9 & g_5+g_9 \\ -g_1 & -g_2-g_7 & g_7 & 0 & 0 \\ g_6 & -g_2-g_6 & -g_3-g_8 & g_8 & 0 \\ g_1 & g_2 & g_3+g_8 & -g_8 & 0 \\ 0 & g_7 & -g_3-g_7 & -g_4-g_9 & g_9 \\ -g_6 & g_2+g_6 & g_3 & g_4+g_9 & -g_9 \\ -g_1 & -g_2 & -g_3-2g_8 & g_4+2g_8+g_9 & -g_9 \\ 0 & 0 & g_8 & -g_4-g_8 & -g_5 \\ 0 & -g_7 & g_3+g_7 & g_4 & g_5 \\ g_6 & -g_2-g_6 & -g_3 & -g_4-2g_9 & g_5+2g_9 \\ g_1 & g_2 & g_3+2g_8 & -g_4-2g_8-2g_9 & g_5+2g_9 \end{pmatrix}, \quad v(g) = \begin{pmatrix} 10 \\ 0 \\ 10 \\ 0 \\ 0 \\ -10 \\ -10 \\ 20 \\ -10 \\ 10 \\ -20 \\ 0 \\ 10 \\ -10 \\ 20 \end{pmatrix}.$$

Note that $U_i(g) = A_i(g)$ and $v_i(g) = b_i(g)$ for $i = 1, \ldots, 5$. The parameters vary within $g_i \in [0.99, 1.01]$, $i = 1, \ldots, 9$, as in [8]. The approximate solution of $U(g^c)x = v(g^c)$, found by least squares, is an interior point for the parametric united solution set. Then, the parametric centered approach from [8] gives the following (rounded inward) guaranteed inner estimation

$$([7.084, 7.098], [4.175, 4.189], [5.448, 5.461], [2.175, 2.189], [1.084, 1.098])^\top.$$

The expanded linear description (20) of the united parametric solution set consists of 512 convex sets. The application of (7) to every one of these convex sets resulted in six δ^* that were significantly greater than zero. Two couples of the obtained size maximal inner estimations were intersecting and we take the interval hull of the intersecting boxes. Here we present the larger one (rounded inward)

$$\begin{aligned}([7.1208, 7.1353], [4.1992, 4.2153], [5.462, 5.4766],\\ [2.1817, 2.2066], [1.0865, 1.1069])^\top,\end{aligned}$$

which is also slightly larger than the above obtained by the parametric centered approach. Two dimensional projections of the parametric united solution set are presented in [23, Example 5.3].

5. Conclusion

In this work we proposed and studied finding inner interval estimations[3] of a parametric AE-solution set with linear shape by solving some linear programming problems. Finding inner boxes that are maximal with respect to their size or with respect to the interval inclusion relation is of particular interest. We proved that the latter is an NP-hard problem.

The linear programming approach, defined by Proposition 2, answers the question about feasibility of the inner estimation and provides a size maximal inner interval box for any parametric tolerable solution set (the latter being a convex polyhedron). For such solution sets, the obtained approximate optimal solution $\tilde{x}$ could be used as input for the verified centered approach in order to obtain a guaranteed maximal inner box in floating point computations. The other linear programming approaches, discussed in Section 2, provide large but not maximal inner interval estimations.

[3] interval boxes that are completely contained in the solution set

Since every parametric AE-solution set with linear shape can be represented as an union of convex polyhedra (Proposition 6), the linear programming approaches from Section 2 could be applied to every one of these convex polyhedral sets in order to obtain a large but, in general, not maximal inner interval estimation for the whole solution set. The estimation is maximal for every one of the polyhedral sets in (20) but not for the parametric solution set which is their union, see Example 4. We demonstrated the advantage of this approach when the solution set is disconnected or almost disconnected.

All the available methods for inner estimation of a parametric AE-solution set require knowledge of the explicit description of the solution set by means of interval inclusions or equivalent inequalities on the coordinate variables. An explicit description, in general, involves exponential number of inclusions/inequalities. How to obtain the minimal set of inclusions/inequalities describing a parametric solution set in general is still an open problem.

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