Interval Model of Equilibrium Equations in Mechanics

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Abstract. The engineering demand for more realistic and accurate models involving interval uncertainties lead to a new interval model of equilibrium equations in mechanics, which is based on the algebraic completion of classical interval arithmetic called Kaucher arithmetic. The proposed approach replaces straightforward a deterministic model by an interval model in terms of proper and improper intervals, fully conforms to the equilibrium principle and provides sharper enclosure of the unknown quantities than the best known methods based on classical interval arithmetic. The paper presents the interval algebraic approach to equilibrium equations and demonstrates its applications to various practical problems with emphasis on modeling of structures.

Keywords: force equilibrium, modeling structures, interval arithmetic, proper and improper intervals

1. Introduction

The basic principle of static (or dynamic) equilibrium under general force systems is an essential prerequisite for many branches of engineering, such as mechanical, civil, aeronautical, bioengineering, robotics, and others that address the various consequences of forces, (Beer et al., 2010).

One main challenge for the models involving interval uncertainty is the overestimation of the system response. Nowadays, the most successful approaches for overestimation reduction are those that relate the dependency of interval quantities to the physics of the problem being considered, (Muhanna et al., 2013). Recently, a model of a bar subjected to multiple axial external loads, where load magnitudes are represented by intervals, is considered in (Elishakoff et al., 2015). Although the aim at providing interval model conforming to the principle of static equilibrium is not completely achieved by the proposed model, the paper demonstrates what are the challenges in this non-trivial task. A similar problem in the context of robotics is discussed in the IEEE P1788 working group on standardization of interval arithmetic, (Mazandarani, 2015). It is shown in (Elishakoff et al., 2015), (Mazandarani, 2015) that an interval model based entirely on the classical interval arithmetic, in its set-theoretic interpretation as proposed by Moore (Moore, 1966), cannot provide a good estimation of the unknown quantities involved in interval equilibrium equations.

The demand for more accurate models involving interval uncertainties lead to an interval model of equilibrium equations in mechanics (Popova, 2016), which is based on the algebraic completion of classical interval arithmetic, called also Kaucher or generalized interval arithmetic. It is proven that the proposed interval model always yields the narrowest interval enclosure and is in full conformance with the physical meaning of static equilibrium. The work (Popova, 2016) is focused on justification of the proposed interval model in one dimension, comparison to the approach of (Elishakoff et al., 2015), and applications to computing resultant forces. A next paper (Popova, 2016) further develops the interval algebraic approach to models involving interval equilibrium equations by considering models of practical applications which reduce to systems of interval equilibrium equations where the number of the unknowns is equal to the number of the equations. The initial interval model is expanded by considering interval algebraic solution to the system of equilibrium equations, model properties are revealed and the quality of the interval algebraic solution is compared to the best interval solution enclosure obtained by classical interval arithmetic. The present paper summarizes all achieved by now and continues the applications of the model to problems where the number of equilibrium equations is less than the number of the involved unknown quantities.

The structure of the present paper is as follows. In the next section some basic notions and properties of the algebraic extension (Kaucher, 1980) of classical interval arithmetic are summarized. In section 3 we present the new interval model, its generalization to systems of interval equilibrium equations involving as many unknowns as is the number of the equations, and a methodology how to apply the interval algebraic approach when the number of equilibrium equations is less than the number of the involved unknown quantities. Numerical applications developed in details in Section 4 illustrate the proposed interval algebraic approach, its conformance to the equilibrium principle, bring out its effectiveness and advantages over the approach based on classical interval arithmetic. The article ends by some conclusions.

2. The Algebraic Completion of $\mathbb{IR}$

The set of classical compact intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \le a^+\}$, called also *proper* intervals, is extended in (Kaucher, 1980) by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \ge a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called *generalized* (extended or Kaucher) intervals. For a better understanding we denote the classical intervals by bold face letters (e.g., $\mathbf{a}$) and the intervals from $\mathbb{KR}$ by brackets (e.g., $[a]$). Of course, $\mathbf{a} \in \mathbb{IR} \subset \mathbb{KR}$, and thus $[b] = \mathbf{a} \in \mathbb{KR}$ is a correct assignment. The inclusion order relation between classical intervals $\subseteq$, $\le$ is generalized for $[a], [b] \in \mathbb{KR}$ by $[a] \subseteq [b] \iff b^- \le a^-$ and $\mathrm{a}^+ \le \mathrm{b}^+$. Denote $\mathcal{T} := \{[a] \in \mathbb{KR} \mid [a] = [0, 0] \text{ or } \mathrm{a}^-\mathrm{a}^+ < 0\}$. For $[a] = [a^-, a^+] \in \mathbb{KR}$ and $[b] \in \mathbb{KR} \setminus \mathcal{T}$, define binary variables *direction* (τ) and "sign" (σ) by

$$\tau([a]) := \begin{cases} + & \text{if } \mathrm{a}^- \le \mathrm{a}^+, \\ - & \text{if } \mathrm{a}^- > \mathrm{a}^+; \end{cases} \qquad \sigma([a]) := \begin{cases} + & \text{if } \operatorname{pro}([\mathrm{a}])^- \ge 0, \\ - & \text{otherwise.} \end{cases}$$

All elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called improper intervals. An element-to-element symmetry between proper and improper intervals is expressed by the "Dual" operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\mathrm{Dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\begin{aligned} \mathrm{Dual}(\mathrm{Dual}([a])) &= [a], && (1) \\ \mathrm{Dual}([a] \circ [b]) &= \mathrm{Dual}([a]) \circ \mathrm{Dual}([b]), \ \circ \in \{+, -, \times, /\}. && (2) \end{aligned}$$

Define proper projection of a generalized interval $[a]$ onto $\mathbb{IR}$ by

$$\mathrm{pro}([a]) := \begin{cases} [a] & \text{if } \tau(\text{[a]}) = +, \\ \mathrm{Dual}([a]) & \text{if } \tau(\text{[a]}) = -. \end{cases}$$

The conventional interval arithmetic and lattice operations, as well as other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, (Kaucher, 1980). Thus,

$$\begin{aligned} [a] + [b] \;&=\; [a^- + b^-, a^+ + b^+] \qquad \text{for [a], [b]} \in \mathbb{KR}, \\ [a] \times [b] \;&=\; \begin{cases} [a^{-\sigma([b]} b^{-\sigma([a])}, a^{\sigma([b]} b^{\sigma([a])}] & \text{if [a], [b]} \in \mathbb{KR} \setminus \mathcal{T} \\ [a^{\sigma([a]\tau([b])} b^{-\sigma([a])}, a^{\sigma([a]\tau([b]} b^{\sigma([a])}] & \text{if [a]} \in \mathbb{KR} \setminus \mathcal{T}, \text{[b]} \in \mathcal{T} \\ [a^{-\sigma([b]} b^{\sigma([b])\tau([a])}, a^{\sigma([b])} b^{\sigma([b])\tau([a])}] & \text{if [a]} \in \mathcal{T}, \text{[b]} \in \mathbb{KR} \setminus \mathcal{T} \\ [\min\{a^- b^+, a^+ b^-\}, \max\{a^- b^-, a^+ b^+\}] & \text{if [a], [b]} \in \mathcal{T}, \tau(\text{[a]}) = \tau(\text{[b]}) \\ 0 & \text{if [a], [b]} \in \mathcal{T}, \tau(\text{[a]}) \neq \tau(\text{[b]}), \end{cases} \end{aligned}$$

wherein $++ = -- = +$, $+- = -+ = -$. Interval subtraction and division can be expressed as composite operations, $[a]-[b] = [a]+(-1)[b]$ and $[a]/[b] = [a]\times(1/[b])$, where $1/[b] = [1/b^+, 1/b^-]$ if $[b] \in \mathbb{KR} \setminus \mathcal{T}$. The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space.

The generalized interval arithmetic structure possesses group properties with respect to the operations addition and multiplication. For $[a] \in \mathbb{KR}$, $[b] \in \mathbb{KR} \setminus \mathcal{T}$,

$$[a] - \mathrm{Dual}([a]) = 0, \qquad [b]/\mathrm{Dual}([b]) = 1. \tag{3}$$

The complete set of conditionally distributive relations for multiplication and addition of generalized intervals can be found in (Popova, 1998), (Popova, 2001). Here we present only one that will be used. For $[a], [b], [s] = ([a] + [b]) \in \mathbb{KR} \setminus \mathcal{T}$, $[c] \in \mathbb{KR}$

$$([a] + [b])[c]_{\sigma([s])} = [a]_{\sigma([a])} + [b]_{\sigma([b])}, \tag{4}$$

wherein $[a]_+ = [a]$, $[a]_- = \mathrm{Dual}([a])$. Addition operation in $\mathbb{KR}$ is commutative and associative; associativity does not hold true in (interval) floating point arithmetic. Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces, (Kaucher, 1980). Some other properties and applications of generalized interval arithmetic can be found in (Kaucher, 1980), (Markov et al., 1996), (Popova, 1998), (Popova, 2001), (Popova and Ullrich, 1996), (Shary, 2002) and the references given therein.

For $\mathbf{a} \in \mathbb{IR} \setminus \mathcal{T}$, define Abs $(\mathbf{a}) = \{\mathbf{a}$ if $0 \leq \mathbf{a}; -\mathbf{a}$ otherwise $\}$. Relative diameter of $\mathbf{a} \in \mathbb{IR}$ is defined as $a^+ - a^-$ if $0 \in \mathbf{a}$ and $(a^+ - a^-)/\min\{|a^-|, |a^+|\}$ otherwise. For $\mathbf{a} \subseteq \mathbf{b}$, the percentage by which $\mathbf{b}$ overestimates $\mathbf{a}$ is defined by

$$100(1 - \omega(\mathbf{a}))/\omega(\mathbf{b}), \qquad \omega(\mathbf{a}) := a^+ - a^-.$$

3. Interval Model of Equilibrium Equations

In this section the algebraic approach to equilibrium equations in mechanics is derived by considering two-dimensional problems involving several forces acting on a particle. The same approach with obvious modifications is applicable to three-dimensional problems and problems whose models involve other vector physical quantities possessing magnitude and direction such as velocities, accelerations, or momenta. Such problems will be illustrated in the next section. In the text of this paper forces (and other vector quantities) are denoted by underlining the letter used to represent it. This is necessary in order to distinguish vectors from the proper intervals, which are denoted by bold-face letters, and from the real-valued scalars. The magnitude of a vector will be denoted by the corresponding italic-face letter.

In the deterministic case of two- dimensional problems involving several forces, the determination of their resultant $\underline{R}$ is best carried out by first resolving each force into rectangular components. Choosing a rectangular coordinate system (Oxy), with unit vectors $\underline{i}, \underline{j}$, any force vector $\underline{F}$ can be resolved into rectangular components $\underline{F}_x = F_x\underline{i}$, and $\underline{F}_y = F_y\underline{j}$, so that $\underline{F} = F_x\underline{i} + F_y\underline{j}$. The scalar component F_x is positive when the vector component $\underline{F}_x$ has the same direction as the unit vector $\underline{i}$ (i.e., the same direction as the positive x axis) and is negative when $\underline{F}_x$ has the opposite direction. A similar conclusion may be drawn regarding the sign of the scalar component F_y. Denoting by F the magnitude of the force $\underline{F}$ and by θ the angle between $\underline{F}$ and the axis x, measured counterclockwise from the positive axis, we may express the scalar components of $\underline{F}$ as follows: $F_x = F\cos(\theta)$ and $F_y = F\sin(\theta)$, cf. any textbook in statics, e.g., (Beer et al., 2010). When more than one force act on a particle (or a rigid body), it is important to determine the resultant force, i.e., the single force $\underline{R}$ which has the same effect on the particle as the given forces. The resultant force $\underline{R}$ can be determined by:

1. choosing a rectangular coordinate system;
2. resolving the given forces into their rectangular components;
3. each scalar component R_x, R_y of the resultant $\underline{R}$ of several forces $\underline{F}_i$ acting on a particle is obtained by adding algebraically the corresponding scalar components of the given forces. That is, $R_x = \sum_i F_{x,i}$, $R_y = \sum_i F_{y,i}$, which gives $\underline{R} = R_x\underline{i} + R_y\underline{j}$.

Basing on the above, the one dimensional interval algebraic model for computing the resultant force (and reaction), developed in (Popova, 2016), can be applied to two- and three-dimensional problems involving vector physical quantities.

*Theorem 1. (*Popova, 2016*)* Consider a bar subjected to a finite number of loads $\underline{p}_1, \ldots, \underline{p}_k$ that may be applied in opposite directions and have uncertain magnitude $p_1 \in \mathbf{p_1}, \ldots, p_k \in \mathbf{p_k}$, $\mathbf{p}_i \geq 0$, $i = 1, \ldots, k$. Assume that a coordinate system (Ox) is chosen. Then,

(i) for every j, $1 \leq j \leq k$, we have $[N_j] = \sum_{i=1}^{j}[p_i]$, wherein

$$[p_i] = \begin{cases} \mathbf{p}_i & \text{if the direction of } \underline{p}_i \text{ is in the positive } x \text{ axis} \\ -\mathrm{Dual}(\mathbf{p}_i) & \text{if the direction of } \underline{p}_i \text{ is opposite to the positive } x \text{ axis} \end{cases},$$

and $[r] = -\mathrm{Dual}([N_k]) = -\mathrm{Dual}(\sum_{i=1}^{k}[p_i])$.

(ii) The interpretation of $[N_j] \in \mathbb{KR}$, $1 \leq j \leq k$, and similarly of $[r]$, is as follows.

- If $[N_j] \in \mathcal{T}$, then $\underline{N}_j$ may have positive or negative direction and its magnitude varies in $\mathrm{pro}([N_j])$.
- If $[N_j] \in \mathbb{KR} \setminus \mathcal{T}$, the magnitude of $\underline{N}_j$ varies in Abs(pro ([Nj])), while the direction of $\underline{N}_j$ coincides with the sign of $[N_j]$ (if $[N_j] \geq 0$ the direction of $\underline{N}_j$ is the positive x axis, otherwise it is opposite to the positive x axis).

Strong proof that Theorem 1 provides sharpest estimation of the resultant force and its reaction is given in (Popova, 2016) along with a detailed discussion and examples.

Now we consider the interval algebraic model of equilibrium equations from a more general perspective. Assume that there is a deterministic model described by some equilibrium equation(s) that involve uncertain parameters varying within given proper intervals. Clearly, the unknowns in this model will be also uncertain and we search for proper intervals that are the sharpest interval estimations of these unknowns and that conform to the physics of the problem (statics or dynamic equilibrium). Conformance to static (dynamic) equilibrium means that the intervals found for the unknowns when replaced in the equation(s) and all operations are performed results in true equality(ies).

*Definition 1. (*Ratschek and Sauer*, *1982*)* Interval *algebraic solution* to a (system of) interval equation(s) is an interval (interval vector) which substituted in the equation(s) and performing all interval operations in exact arithmetic[1] results in valid equality(ies).

Interval algebraic solutions do not exist in general in classical interval arithmetic (Ratschek and Sauer, 1982). Generalized interval arithmetic on proper and improper intervals $(\mathbb{KR}, +, \times, \subseteq)$ is the natural arithmetic for finding algebraic solutions to interval equations since it is obtained from the arithmetic for classical intervals $(IR, +, -, \times, /, \subseteq)$ via an algebraic completion. This is another justification of the proposed interval algebraic approach. Therefore, we embed the initial problem formulation in the interval space $(\mathbb{KR}, +\times, \subseteq)$, find an algebraic solution (if exists) and interpret the obtained generalized intervals back in the initial interval space $\mathbb{IR}$. This is a three steps procedure summarized below.

1. The **representation convention** for a model involving interval forces (and/or other physical quantities considered as vectors and possessing magnitude and direction) is:

   – a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by proper interval $\mathbf{F}_x$ ($\mathbf{F}_y$, $\mathbf{F}_z$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has the same direction as the positive x (y, z) coordinate axis;

   – a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by the improper interval $\mathrm{Dual}(\mathbf{F}_x)$ ($\mathrm{Dual}(\mathbf{F}_y)$, $\mathrm{Dual}(\mathbf{F}_z)$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has opposite direction to the corresponding positive x (y, z) coordinate axis.

[1] no round-off errors

2. **Computing**. Find the **algebraic solution** for the unknown(s) in $(\mathbb{KR}, +, \times, \subseteq)$. Conditions for existence of algebraic solution of interval linear equations are published in (Popova, 1998), (Shary, 2002). Numerical methods finding the algebraic solution to an interval linear system are discussed in (Markov et al., 1996), (Shary, 2002). For small systems, the approach based on *equivalent algebraic transformations* is transparent and will be used in this paper.

3. **Interpretation** of the obtained generalized intervals in the initial space $\mathbb{IR}$ is done according to the physics of the unknowns. If it is a force component, then Theorem 1 ii) is applied. In general the interpretation projects the generalized interval solution on $\mathbb{IR}$.

Since computing a resultant $\underline{R}$ of several forces $\underline{F}_i$ can be represented as a solution of the equilibrium equation $\sum_i \underline{F}_i - \underline{R} = 0$, Theorem 1 is a special case of the above more general interval algebraic approach.

If the deterministic model involves more unknowns than the number of equilibrium equations, other relations are obtained from the information contained in the statement of the problem. In this case the following hybrid approach should be applied. Let the number of the equilibrium equations be k and the number of the unknown quantities be n, $n > k$. From the statement and the physics of the problem we find $n - k$ additional relations involving (some of) the unknowns.

a. If the $n - k$ additional relations involve $n - k$ of the n unknowns, by methods of classical interval analysis find interval estimations of these $n - k$ unknowns. Then replace the obtained interval estimations in the interval model of the equilibrium equations and find the algebraic solution with respect to the remaining k unknowns.

b. Let $n - k$ additional relations involve $n - k + q$ of the unknowns. We consider q of the k equilibrium equations together with the $n - k$ additional relations in a way that the system involves $n - k + q$ unknowns. Then the process continues as in a. above.

This approach ensures that the unknown uncertain quantities are estimated in a way that the equilibrium equations are satisfied to a highest extent that corresponds to the initial uncertainties. Next section illustrates the proposed approach.

4. Numerical Applications

Here we consider models of practical applications which illustrate the application of interval algebraic approach to equilibrium equations and its properties. In order to avoid many technical details that will hamper the comprehension, no more than two dimensional problems are considered. The numerical results presented in this section are obtained by the *Mathematica*® package `directed.m` (Popova and Ullrich, 1996). JInterval library (Nadezhin and Zhilin, 2014) can be used for this purpose, too.

Example 1. Three horizontal forces are applied, as shown in Fig. 1, to a vertical cast iron arm. Assume that the distances shown in Fig. 1 are measured with 1% uncertainty. Determine the

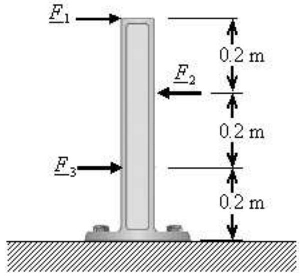

Figure 1. Horizontal forces applied to a vertical cast iron arm

resultant of the forces, which are measured with 5% uncertainty, and the distance from the ground to its line of action when $F_2 = 700$N, $F_3 = 150$N, and a) $F_1 = 300$N, b) $F_1 = 550$N, c) $F_1 = 1200$N.

With a standard coordinate system, applying the representation convention, for the resultant force $\underline{R}$ we have

$$[R] = \mathbf{F}_1 - \mathrm{Dual}(\mathbf{F}_2) + \mathbf{F}_3.$$

The computation results in[2]

$$[R] = \begin{cases} [-237.5001, -262.4999] & \text{if a)} \\ [-2.27 \times 10^{-13}, 2.27 \times 10^{-13}] & \text{if b)} \\ [617.5000, 682.5001] & \text{if c).} \end{cases}$$

According to the interpretation convention the resultant force has magnitude $\mathbf{R} = \mathrm{Abs}(\mathrm{pro}([R]))$ and its direction is determined by the sign of $[R]$. That is, in case a) the magnitude varies in $[237.5001, 262.4999]$N, the direction is opposite to the direction of $\underline{F}_1$; in case b) the tiny fluctuations around zero are due to round-off errors and therefore the forces are in equilibrium ($R = 0$N); in case c) the magnitude varies in $[617.5000, 682.5001]$N and the direction of the resultant force coincides with the direction of $\underline{F}_1$.

For the moment $\underline{M}$ (and positive direction pointing the positive y coordinate axis) the representation convention gives

$$[M] = -\mathrm{Dual}(\mathbf{F}_1\mathbf{d}_1) + \mathbf{F}_2\mathbf{d}_2 - \mathrm{Dual}(\mathbf{F}_3\mathbf{d}_3),$$

wherein $\mathbf{d}_1 \in [0.6 \mp 6 \times 10^{-3}]$, $\mathbf{d}_2 \in [0.4 \mp 4 \times 10^{-3}]$, $\mathbf{d}_3 \in [0.2 \mp 2 \times 10^{-3}]$. The computation results in

$$[M] = \begin{cases} [65.8349, 74.2351] & \text{if a)} \\ [-75.2340, -84.8400] & \text{if b)} \\ [-442.0351, -498.4349] & \text{if c).} \end{cases}$$

According to the interpretation convention the moment $\underline{M}$ has the following magnitude $\mathbf{M}$ and direction. In case a) the magnitude is $[65.8349, 74.2351]$N.m and the direction points to the positive y axis; in case b) the magnitude is $[75.2340, 84.8400]$N.m and direction pointing opposite to the positive y axis; in case c) $\mathbf{M} = [442.0351, 498.4349]$N.m and the direction points opposite to the positive y axis.

[2] All computed numerical intervals are outwardly rounded to the intervals presented in the paper.

Then, the distance is $\mathbf{d} = \mathbf{M}/\mathbf{R}$ which gives $d_a \in [0.2508, 0.3126]$m in case a); $d_b = \infty$ in case b); and $d_c \in [0.6476, 0.8072]$m in case c).

Example 2. An 80 kg block rests on a horizontal plane, Fig. 2 a). Find the magnitude of the force $\underline{P}$ required to give the block an acceleration of 2.5 m/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$. Assume that the mass of the block and the angle, at which the force acts on the block, are measured with 1 % uncertainty.

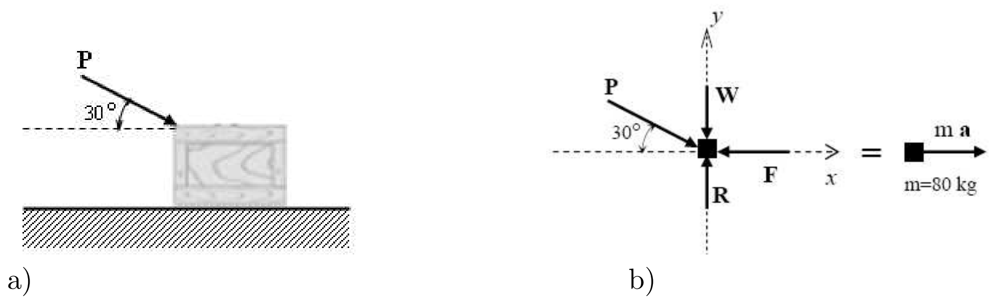

Figure 2. a) A force acting on a block that rests on a horizontal plane; b) the free-body diagram

The chosen coordinate system is presented on the free-body diagram in Fig. 2 b). Note that $\mathbf{F} = \mu_k \mathbf{R}$. The weight of the block is

$$W = mg_0 \in ([79.2, 80.8]\text{kg})(9.80665\text{m/s}^2) \in [776.686, 792.378]\text{N}.$$

Writing Newton's second law $\sum \underline{F} = m\underline{a}$ in rectangular components and applying the representation convention, we obtain the following interval equilibrium equations

$$\mathbf{P}\cos([\theta]) - \text{Dual}(0.25\mathbf{R}) = 2.5\mathbf{m} \tag{5}$$

$$\mathbf{R} - \text{Dual}(\mathbf{P}\sin([\theta])) - \text{Dual}(\mathbf{W}) = 0, \tag{6}$$

where $[\theta] = [29.7°, 30.3°]$. We search for proper intervals $\mathbf{P}$, $\mathbf{R}$ which satisfy these equations. Adding $\mathbf{P}\sin([\theta]) + \mathbf{W}$ to the two sides of equation (6) and applying property (3), we obtain

$$[R] = \mathbf{P}\sin([\theta]) + \mathbf{W}.$$

Replacing $[R]$ in the first equilibrium equation (5), we have

$$\mathbf{P}\cos([\theta]) - 0.25\text{Dual}(\mathbf{P}\sin([\theta])) - 0.25\text{Dual}(\mathbf{W}) = 2.5\mathbf{m}. \tag{7}$$

The distributive relation (4) holds true for the first two terms of (7) since

$$[s] = \cos([\theta] - 0.25\text{Dual}(\sin([\theta]) \in [0.739530, 0.7425] > 0.$$

Thus, the equation (7) is equivalent to $[P][s] - 0.25\text{Dual}(\mathbf{W}) = 2.5\mathbf{m}$. Adding $0.25\mathbf{W}$ to both sides of the last equation and then dividing by Dual$([s])$, we obtain

$$[P] = (2.5\mathbf{m} + 0.25\mathbf{W})/\text{Dual}([s]) \in [530.297, 538.848]\text{N}.$$

From the last equivalent form of equation (6), we get $[R] = [P]\sin([\theta]) + \mathbf{W} \in [1039.42, 1064.25]$N. Both $[P]$ and $[R]$ are proper intervals. Replacing them in the initial equations (5)–(6) we obtain $[-1.71\times 10^{-13}, 1.14\times 10^{-13}]$, $[-4.55\times 10^{-13}, 4.55\times 10^{-13}]$, respectively. These intervals are almost but not exactly zero due to the round-off errors and show that the equilibrium equations are completely satisfied.

Now, we compare the solution $\mathbf{P}$, $\mathbf{R}$, obtained by the discussed algebraic approach, to the solution obtained by classical interval arithmetic. In classical interval arithmetic the goal is to find the smallest interval vector enclosing the so-called united solution set[3] of the interval system

$$\begin{pmatrix} \cos([\theta]) & -0.25 \\ -\sin([\theta]) & 1 \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} = \begin{pmatrix} 2.5\ m \\ 9.80665\ m \end{pmatrix}, \qquad \begin{array}{l} \theta \in [29.7^\circ, 30.3^\circ], \\ m \in ([79.2, 80.8]. \end{array}$$

The smallest interval vector that encloses the united solution set of this system is $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top = ([526.56, 542.68], [1037.58, 1066.18])^\top$. The percentage by which $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top$ overestimates $(\mathbf{P}, \mathbf{R})^\top$ is $(46.9, 13.2)^\top$%.

If we consider the same problem with 2% relative uncertainty in the angle and 1% relative uncertainty in the mass of the block, then the percentage by which $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top$ overestimates $(\mathbf{P}, \mathbf{R})^\top$ is $(70.2, 20.9)^\top$%.

Since we are looking for proper algebraic solutions of the interval equilibrium system, this restriction may not be always satisfied. The latter case is illustrated by the next example.

Example 3. A $[100 \pm 1]$ kg crate is suspended from a pulley that can roll freely on the support cable ACB and is pulled at a constant speed by cable CD, as shown in Fig. 3. If $\alpha = 30^\circ$, $\beta = 10^\circ$ and the angles are measured with 1% uncertainty, determine the tension (a) in the support cable ACB, (b) in the traction cable CD.

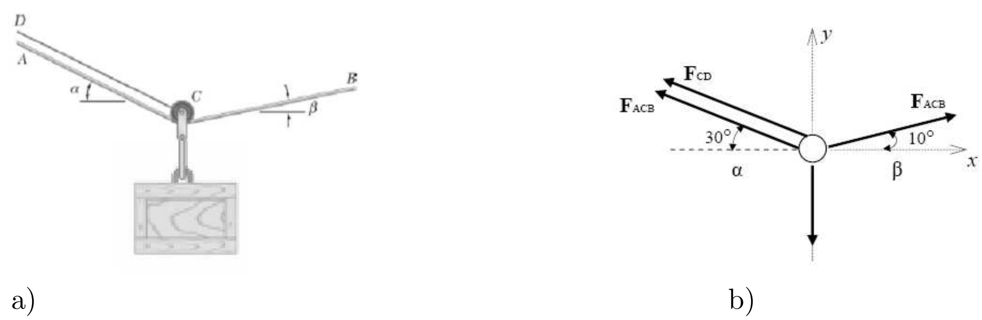

Figure 3. a) A crate suspended from a pulley can roll freely on the support cable ACB and is pulled at a constant speed by cable CD; b) Free-body diagram

The chosen coordinate system is presented on the free-body diagram in Fig. 3. The deterministic equilibrium equations of force x and y components are

$$F_{\text{ACB}}\cos(10^\circ) - F_{\text{ACB}}\cos(30^\circ) - F_{\text{CD}}\cos(30^\circ) = 0, \tag{8}$$
$$F_{\text{ACB}}\sin(10^\circ) + F_{\text{ACB}}\sin(30^\circ) + F_{\text{CD}}\sin(30^\circ) - 100\times 9.80665 = 0. \tag{9}$$

[3] For $A(p)x = b(p)$, $p \in \mathbf{p}$, the united solution set is $\Sigma = \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\}$.

The representation convention gives the interval equilibrium equations

$$[F_{\mathrm{ACB}}]\cos([\beta]) - \mathrm{Dual}(\mathbf{F}_{\mathrm{ACB}}\cos([\alpha])) - \mathrm{Dual}(\mathbf{F}_{\mathrm{CD}}\cos([\alpha])) = 0, \tag{10}$$
$$[F_{\mathrm{ACB}}]\sin([\beta]) + [F_{\mathrm{ACB}}]\sin([\alpha]) + [F_{\mathrm{CD}}]\sin([\alpha]) - \mathrm{Dual}([99,100]\times 9.80665) = 0, \tag{11}$$

wherein $[\alpha] = [29,31]^\circ$, $[\beta] = [9,11]^\circ$. We search for proper intervals $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$, that satisfy (10)–(11). First, we check the validity of the distributive relations for the first two additive terms in equations (10), (11). Since

$$\begin{aligned}
[s_1] &= \cos([\beta]) - \mathrm{Dual}(\cos([\alpha])) \in [0.121107, 0.116479] > 0,\\
[s_2] &= \sin([\beta]) + \sin([\alpha]) \in [0.667387, 0.679895] > 0,
\end{aligned}$$

by (4), the system (10)–(11) is equivalent to the system

$$\begin{aligned}
&[F_{\mathrm{ACB}}][s_1] - \mathrm{Dual}(\mathbf{F}_{\mathrm{CD}}\cos([\alpha])) = 0,\\
&[F_{\mathrm{ACB}}][s_2] + [F_{\mathrm{CD}}]\sin([\alpha]) - \mathrm{Dual}([99,100]\times 9.80665) = 0.
\end{aligned}$$

Remark 1. It is important that we check the distributive relations for every expression where we want to take a common interval variable out of brackets. For example, due to (4), and because $\cos([\alpha]) - \mathrm{Dual}(\cos([\beta])) < 0$, the expression $[F_{\mathrm{ACB}}]\cos([\alpha]) - \mathrm{Dual}([F_{\mathrm{ACB}}]\cos([\beta]))$ is equivalent to

$$\mathrm{Dual}([F_{\mathrm{ACB}}])\left(\cos([\alpha]) - \mathrm{Dual}(\cos([\beta]))\right).$$

We add $[F_{\mathrm{CD}}]\cos([\alpha])$ to the two sides of equation (10) and by (3) obtain the equivalent equation

$$[F_{\mathrm{ACB}}][s_1] = [F_{\mathrm{CD}}]\cos([\alpha]).$$

Dividing both sides of the last equation by $\mathrm{Dual}(\cos([\alpha]))$, and due to (3), we obtain

$$[F_{\mathrm{CD}}] = [F_{\mathrm{ACB}}][s_1]/\mathrm{Dual}(\cos([\alpha])). \tag{12}$$

We substitute the expression for $[F_{\mathrm{CD}}]$ in equation (11). Since

$$[s_3] = [s_2] + \sin([\alpha])[s_1]/\mathrm{Dual}(\cos([\alpha])) \ \in\ [5.25337, 5.57483] > 0,$$

due to the distributive relation, equation (11) is equivalent to

$$[F_{\mathrm{ACB}}][s_3] - \mathrm{Dual}([99,100]\times 9.80665) = 0,$$

which is equivalent to

$$[F_{\mathrm{ACB}}] = [99,100]\times 9.80665/\mathrm{Dual}([s_3]) \ \in\ [1317.51, 1324.97]. \tag{13}$$

Substituting (13) in (12), we obtain the second component of the algebraic solution to interval system (10)–(11)

$$[F_{\mathrm{CD}}] \in [184.806, 177.669].$$

Substituting $[F_{\mathrm{ACB}}]$ and $[F_{\mathrm{CD}}]$ into left sides of the equations (10) and (11), we obtain respectively $[-2.27\times 10^{-13}, 2.27\times 10^{-13}]$ and $[-5.68\times 10^{-13}, 4.54\times 10^{-13}]$. These intervals are almost but not exactly zero due to the round-off errors.

We have to interpret $[F_{\mathrm{ACB}}]$ and $[F_{\mathrm{CD}}]$ in $\mathbb{IR}$ as the corresponding proper intervals, namely,

$$\begin{aligned}\mathbf{F}_{\mathrm{ACB}} &= \mathrm{Abs}(\mathrm{pro}([F_{\mathrm{ACB}}])) \in [1317.51, 1324.97]\ \mathrm{N},\\ \mathbf{F}_{\mathrm{CD}} &= \mathrm{Abs}(\mathrm{pro}([F_{\mathrm{CD}}])) \in [177.669, 184.806]\ \mathrm{N}.\end{aligned}$$

Since $[F_{\mathrm{CD}}]$ is an improper interval, substituting $\mathbf{F}_{\mathrm{ACB}}$ and $\mathbf{F}_{\mathrm{CD}}$ into left sides of the equations (10) and (11), we obtain much wider intervals involving zero, namely, $[6.16307, -6.20045]$ and $[-3.53667, 3.60141]$, respectively. The relative diameters of $\mathbf{F}_{\mathrm{ACB}}$ and $\mathbf{F}_{\mathrm{CD}}$ are 0.00565 and 0.0402, respectively.

Remark 2. Proper algebraic solution to the system (10)–(11) can be obtained if, for example, we squeeze the interval $[\alpha]$ to the interval $[30-0.1, 30+0.1]$.

Now, we compare the solution $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$, obtained by the discussed algebraic approach, to the solution obtained by classical interval arithmetic. The equations (8)–(9) are rearranged to

$$\begin{aligned}&F_{\mathrm{ACB}}\,(\cos(10^\circ)-\cos(30^\circ)) - F_{\mathrm{CD}}\cos(30^\circ) = 0,\\ &F_{\mathrm{ACB}}\,(\sin(10^\circ)+\sin(30^\circ)) + F_{\mathrm{CD}}\sin(30^\circ) = 100\times 9.80665\end{aligned}$$

and the corresponding interval linear system that has to be solved is

$$\begin{pmatrix}\cos([\beta])-\cos([\alpha]), & \cos([\alpha])\\ \sin([\beta])+\sin([\alpha]), & \sin([\alpha])\end{pmatrix}\begin{pmatrix}F_{\mathrm{ACB}}\\ F_{\mathrm{CD}}\end{pmatrix} = \begin{pmatrix}0\\ [99,100]\times 9.80665\end{pmatrix}.$$

Since some interval parameters, e.g., $[\alpha]$, $[\beta]$, appear in more than one element of the matrix and/or the right-hand side vector, this is a parametric interval linear system. In classical interval arithmetic we search for a minimal outer interval estimation of the so-called united parametric solution set to the system. It can be proven, by method discussed in (Popova, 2006), that the united parametric solution set of the above system depends linearly on the interval parameters involved there. Therefore, one can find the minimal interval vector containing the united parametric solution set by finding the interval hull of the set of solutions to the point linear systems of equations obtained for the parameters taking values at all combinations of the corresponding interval end-points, the so-called combinatorial approach. Applying this approach, we found $\tilde{\mathbf{F}}_{\mathrm{ACB}} = [1293.33, 1349.74]$, $\tilde{\mathbf{F}}_{\mathrm{CD}} = [175.743, 186.773]$, whose relative diameters are respectively 0.04361 and 0.06276. Replacing $\tilde{\mathbf{F}}_{\mathrm{ACB}}$, $\tilde{\mathbf{F}}_{\mathrm{CD}}$ in the left-hand sides of the generalized interval equilibrium equations (10)–(11), we obtain much wider intervals involving zero $[4.89652, -5.02244]$, $[-20.6303, 21.4392]$. There is no inclusion relation between $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$ and $\tilde{\mathbf{F}}_{\mathrm{ACB}}$, $\tilde{\mathbf{F}}_{\mathrm{CD}}$. Nevertheless, judging from the value of the relative diameters and the extent to which the interval equilibrium equations are satisfied, we conclude that the interval algebraic approach applied to the equilibrium equations provides sharper interval estimations than the traditional approach based on classical interval arithmetic.

In some deterministic models, e.g., when determine the forces in the members of a truss, in order to write the equilibrium equations one has to choose the direction of each of the unknown forces, cf.

(Beer et al., 2010, Chapter 6). It cannot be determined until the solution is completed whether the guess was correct. To do that, the value found for each of the unknowns is considered: a positive sign means that the selected direction was correct; a negative sign means that the direction is opposite to the assumed direction. This convention is transparently applicable to the corresponding interval algebraic model which delivers the correct sign together with the interval magnitude.

5. Impact on Models of Structures

Example 4. After (Kulpa et al., 1998) consider a simple planar frame with three types of support and an external load distributed uniformly along the beam as shown in Figure 4 a).

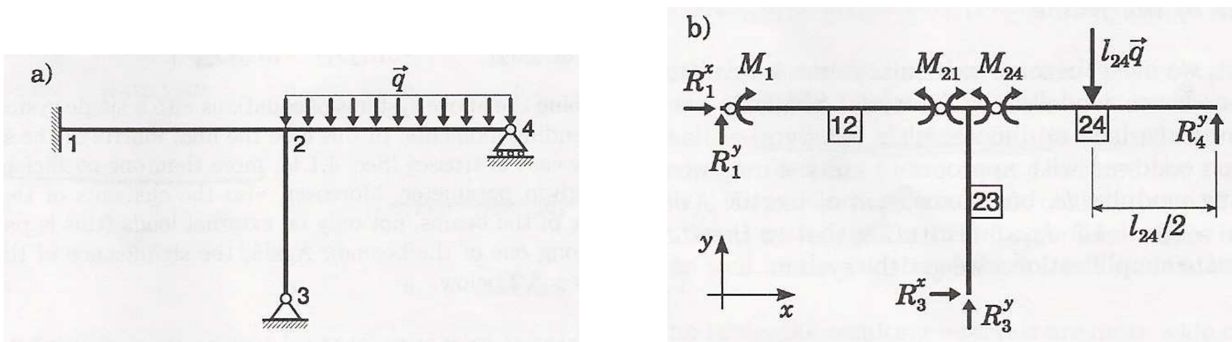

Figure 4. Planar frame (a) and its fundamental system of internal parameters (b), after (Kulpa et al., 1998).

Assuming small displacements and linear elastic material law, and using the method of forces, the frame is described in (Kulpa et al., 1998) by the following set of equilibrium equations for forces and bending moments, see Figure 4 (b).

$$R_1^x + R_3^x = 0, \tag{14}$$

$$R_1^y + R_3^y + R_4^y - ql_{24} = 0, \tag{15}$$

$$-M_1 + R_4^y(l_{12} + l_{24}) + R_3^y l_{12} + R_3^x l_{23} - ql_{24}(l_{12} + \frac{1}{2}l_{24}) = 0, \tag{16}$$

$$-R_1^y l_{12} - M_1 + M_{21} = 0, \tag{17}$$

$$R_4^y l_{24} - \frac{1}{2}ql_{24}^2 - M_{24} = 0. \tag{18}$$

The equilibrium equations involve more unknowns than the number of the equations. Then, the three canonical equations linking bending moments with material properties (Young modulus E and momentum of inertia J of the beam cross-section) of the beams are given by

$$\begin{pmatrix} \frac{l_{12}}{3E_{12}J_{12}} & \frac{l_{12}}{6E_{12}J_{12}} & 0 \\ \frac{l_{12}}{6E_{12}J_{12}} & \frac{l_{12}}{3E_{12}J_{12}} + \frac{l_{23}}{3E_{23}J_{23}} & \frac{-l_{23}}{3E_{23}J_{23}} \\ 0 & \frac{-l_{23}}{3E_{23}J_{23}} & \frac{l_{24}}{3E_{12}J_{12}} + \frac{l_{23}}{3E_{23}J_{23}} \end{pmatrix} \begin{pmatrix} M_1 \\ M_{21} \\ M_{24} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{-ql_{24}^3}{24E_{24}J_{24}} \end{pmatrix}. \tag{19}$$

The parameters of this frame will be given as dimensionless numbers. It is assumed that all the beams have the same Young modulus E and momentum of inertia J of the beam cross-sections are related by the formula $J_{12} = J_{23} = 1.5J_{24}$. The lengths of the beams and the load are considered to be uncertain with the following nominal values $l_{12} = l_{24} = 1$, $l_{23} = 0.75$, and $q = 10$. Substituting these into the equations (14)–(19) for the frame and making appropriate simplifications, a parametric linear system described by the following relations is obtained

$$\frac{1}{2}a_{11} = a_{12} = a_{21} = a_{65} = -a_{74} = l_{12}$$
$$a_{22} = 2l_{12} + 2l_{23},\ a_{33} = 3l_{24} + 2l_{23},\ a_{66} = l_{12} + l_{24},\ a_{23} = a_{32} = -2l_{23},\ a_{68} = l_{23},\ a_{86} = l_{24}$$
$$a_{47} = a_{48} = a_{54} = a_{55} = a_{56} = -a_{61} = -a_{71} = a_{72} = -a_{83} = 1$$
$$b = (0,\ 0,\ -\frac{3}{8}q\,l_{24}^3,\ 0,\ q\,l_{24},\ q\,l_{24}(l_{12} + \frac{1}{2}l_{24}),\ 0,\ \frac{1}{2}q\,l_{24}^2)^\top.$$

It is assumed that there is no prestressing of the structure due to inexact dimensions of the beams. For that, the uncertainties are considered either as errors of measurements of the elements of the already existing structure, or else assume the structure will be assembled from inexact elements, but in a way that does not lead to prestressing (e.g., by slightly moving appropriate supports when necessary).

Usually, interval estimations for the unknown reactions and moments are found by bounding the united parametric solution set of the last system. In (Popova, 2006) it is proven that the united parametric solution set of the system (14)–(19) depends linearly on the interval parameters. In (Popova, 2005, Table 8, column 2) the sharpest interval enclosing the solution set is reported for the parameters

$$l_{12} \in [0.995, 1.005],\ l_{24} \in [0.995, 1.005],\ l_{23} \in [0.74625, 0.75375], q \in [9.95, 10.05]. \qquad (20)$$

The obtained enclosure is

Table I. Solutions for moments and reactions of the planar frame system with 0.5% uncertain parameters.

$\mathbf{M}_1$	$\mathbf{M}_{21}$	$\mathbf{M}_{24}$	$\mathbf{R}_1^x$
[.24479, .25530]	[-.51059, -.48958]	[-1.0171, -.98309]	[-.68421, -.64953]

$\mathbf{R}_1^y$	$\mathbf{R}_3^y$	$\mathbf{R}_4^y$	$\mathbf{R}_3^x$
[-0.76973, -0.73072]	[6.6698, 6.8309]	[3.9600, 4.0401]	[0.64953, 0.68421]

However, $\mathbf{R}_1^y + \mathbf{R}_3^y + \mathbf{R}_4^y - \mathbf{ql}_{24} \in [-0.24, 0.24]$ and the equilibrium equation (15) is not satisfied.

In (Popova, 2005, Table 10, column 2) the sharpest intervals enclosing the solution set is reported for the the planar frame system with 1% uncertain lengths and 15 % uncertain load. In this case, the obtained interval estimations are such that $\mathbf{R}_1^y + \mathbf{R}_3^y + \mathbf{R}_4^y - \mathbf{ql}_{24} \in [3.47, 3.48]$ and the equilibrium equation (15) is not satisfied, too.

In order to obtain more realistic interval estimations that satisfy the equilibrium equations (14)–(18), we will apply the proposed interval algebraic approach. First, we find the exact interval hull of the united parametric solution set of the system (19) for the parameters (20) as shown in Table II.

Table II. Solutions for moments of the system (19) with 0.5% uncertain parameters.

$\mathbf{M}_1$	$\mathbf{M}_{21}$	$\mathbf{M}_{24}$
[.2452, .2548]	[-.5096, -.4905]	[-1.0171, -.98309]

Then, we replace the obtained intervals for $\mathbf{M}_1$, $\mathbf{M}_{21}$ and $\mathbf{M}_{24}$ in the proposed interval model of the equilibrium equations (14)–(18) and find the algebraic solution for $\mathbf{R}^y_{1,3,4}$, $\mathbf{R}^x_{1,3}$, as follows. From equation (17)

$$[R^y_1] = (-\mathbf{M}_1 + \mathbf{M}_{21})/\mathrm{Dual}(\mathbf{l}_{12}) = -3\mathbf{M}_1/\mathrm{Dual}(\mathbf{l}_{12}) \in [-0.76055, -0.73954].$$

From equation (18)

$$[R^y_4] = (\mathbf{M}_{24} + \mathbf{q}\mathbf{l}^2_{24}/2)/\mathrm{Dual}(\mathbf{l}_{24}) \in [3.9279, 4.07192].$$

From equation (15)

$$[R^y_3] = \mathbf{q}\mathbf{l}_{24} - [R^y_1] - \mathrm{Dual}([R^y_4]) \in [6.7118, 6.7889].$$

If the computations are done in exact (rational) arithmetic, so that there are no round-off errors, we will have

$$\mathrm{Dual}([R^y_1]) + [R^y_3] + [R^y_4] - \mathrm{Dual}(\mathbf{q}\mathbf{l}_{24}) = [0, 0].$$

From equation (16)

$$[R^x_3] = (\mathbf{q}\mathbf{l}_{24}(\mathbf{l}_{12} + \mathbf{l}_{24}/2) + \mathbf{M}_1 - \mathrm{Dual}([R^y_4](\mathbf{l}_{12} + \mathbf{l}_{24})) - \mathrm{Dual}([R^y_3]\mathbf{l}_{12}))/\mathrm{Dual}(\mathbf{l}_{23}) \in [.70557, .62824].$$

And from equation (14)

$$[R^x_1] = -\mathrm{Dual}([R^x_3]) \in [-.70557, -.62824].$$

$[R^x_3]$ is improper interval, so it should be interpreted as the corresponding proper one.

Denote by $\tilde{\mathbf{R}}^y_{1,3,4}$, $\tilde{\mathbf{R}}^x_{1,3}$ the sharpest enclosures obtained by classical interval approach and presented in Table I. Comparing the estimations of the reactions, got by the interval equilibrium model, to the reaction estimations $\tilde{\mathbf{R}}^y_{1,3,4}$, $\tilde{\mathbf{R}}^x_{1,3}$ we obtain

$$\begin{array}{ll} \tilde{\mathbf{R}}^y_1 \text{overdetermines } \mathbf{R}^y_1 & \text{by } 46.2\% \\ \tilde{\mathbf{R}}^y_3 \text{overdetermines } \mathbf{R}^y_3 & \text{by } 52.2\% \\ \mathbf{R}^y_4 \text{overdetermines } \tilde{\mathbf{R}}^y_4 & \text{by } 44.3\% \\ \mathbf{R}^x_{1,3} \text{overdetermine } \tilde{\mathbf{R}}^x_{1,3} & \text{by } 55.2\%. \end{array}$$

Note that the last two lines, showing that the classical interval estimates *underdetermine* the variation in reaction magnitude, are especially dangerous.

Remark 3. Interval forces in the interval equilibrium model are like connected vessels — expanding some interval estimations shrinks the estimation of others, so that the equilibrium equations are always satisfied. This property is particularly important because not always we can obtain the sharpest interval estimations of the unknowns involved in the additional relations. To illustrate this property we take the values of $\mathbf{M}_{1,21,24}$ from Table I and round them outwardly to the second place after the decimal point as follows

$$\mathbf{M}_1 \in [24/100, 26/100], \quad \mathbf{M}_{21} \in [-52/100, -48/100], \quad \mathbf{M}_{24} \in [-102/100, -98/100]. \tag{21}$$

These intervals overestimate the intervals presented in Table II by $52.5, 52.5, 93.5\%$, respectively. Now, we repeat the above computations in the interval model of equilibrium equations and obtain slightly different intervals for $\mathbf{R}^y_{1,3,4}$, $\mathbf{R}^x_{1,3}$ which also satisfy the equilibrium equations.

Remark 4. Expanding the uncertainty cannot be unlimited. We illustrate this by the following example. Assume that in the input of interval estimates we have a typing bug so that $\mathbf{M}_{24} \in [-102/100, -50/100]$. Then the computations in the interval model of equilibrium equations results in $[R^y_3] = [144/199, -16/597]$, $[R^x_1] = [-144/199, 16/597]$, both involving zero. The latter means that we cannot determine the direction of both reactions. Also the first equilibrium equation is not satisfied, $\mathbf{R}^x_1 + \mathbf{R}^x_3 \approx [-0.75, 0.75]$. Thus, if the interval model of equilibrium equations results in interval containing zero for some reaction magnitude, this might be due to wrong model or overestimation of some uncertain quantities.

Remark 5. The algebraic interval approach to equilibrium equations should be applied to all interval models (parametric interval systems of equations) involving equilibrium equations, for example to systems for both primary and derived variables (**?**). Keeping the equilibrium equations involving the same number of unknowns out of the parametric system that is solved by classical interval methods reduces the number of both equations and interval parameters in the latter, which additionally helps reducing the overestimation of the unknowns in the latter system.

6. Conclusion

The engineering demand for more accurate models involving interval uncertainties, that conform to the physics of the modeled problem, lead to a new interval algebraic model of equilibrium equations in mechanics. The latter is based on the algebraic completion $(\mathbb{KR}, +, \times, \subseteq)$ of classical interval arithmetic. By a simple representation convention one can easily transform a deterministic formulation into a unique interval arithmetic formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$. Then in the same rich algebraic space one finds a sharp algebraic solution for the unknown quantities and interpret them in the original physical setting of the problem. If the algebraic solution is a proper interval (vector), it is assured that the equilibrium equations are completely satisfied and the obtained interval enclosures are the sharpest ones. It was demonstrated by some examples that if even (part of) the algebraic solution is not proper interval vector, its proper projection provides narrower interval estimation for the unknowns than the best solution enclosure in classical interval arithmetic. Contrary to classical interval approach, the algebraic one provides satisfaction of the equilibrium equations even for very large parameter uncertainties. Therefore, for large uncertainties the algebraic approach is essential in obtaining sharp interval estimates.

The most attractive in the interval algebraic approach to equilibrium equations in mechanics is its transparent application and full conformance to the deterministic model. Along with guaranteed quantification of all sources of uncertainties, the new algebraic approach provides also sharper enclosure of the unknown quantities than the best known methods based on classical interval arithmetic.

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