Outer bounds for the parametric controllable solution set with linear shape*

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Abstract. We consider linear algebraic equations, where the elements of the matrix and of the right-hand side vector are linear functions of interval parameters, and their parametric AE-solution sets, which are defined by universal and existential quantifiers for the parameters. We present how some sufficient conditions for a parametric AE-solution set to have linear boundary can be exploited for obtaining sharp outer bounds of that parametric AE-solution set. For a parametric controllable solution set having linear boundary we present a numerical method for outer interval enclosure of the solution set. The new method has better properties than some other methods available so far.

Keywords: interval linear systems, parameter dependencies, AE-solution set, controllable solution set, solution enclosure, iteration method

1 Introduction

Consider linear algebraic systems involving linear dependencies between a number of interval parameters $p = (p_1, \ldots, p_K)^\top \in \mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_K)^\top$

$$\begin{aligned} A(p)x \;&=\; b(p) \\ A(p) := A_0 + \sum_{k=1}^{K} p_k A_k, &\qquad b(p) := b_0 + \sum_{k=1}^{K} p_k b_k, \end{aligned} \tag{1}$$

where $A_k \in \mathbb{R}^{m\times n}$, $b_k \in \mathbb{R}^m$, $k = 0, \ldots, K$, and $\mathbb{R}^{m\times n}$ is the set of real $m \times n$ matrices, $\mathbb{R}^m := \mathbb{R}^{m\times 1}$ denotes the set of real vectors with m components. A real compact interval is $\mathbf{a} = [a_1, a_2] := \{a \in \mathbb{R} \mid a_1 \le a \le a_2; a_1, a_2 \in \mathbb{R}\}$. By $\mathbb{IR}^m, \mathbb{IR}^{m\times n}$ we denote the sets of interval m-vectors and interval $m\times n$ matrices, respectively.

We consider the parametric AE-solution sets of the system (1)

$$\Sigma^p_{AE} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p)x = b(p))\}\,, \tag{2}$$

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for the sets of indexes $\mathcal{A}$, $\mathcal{E}$ such that $\mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}$, $\mathcal{A} \cap \mathcal{E} = \emptyset$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$. Among the AE-solution sets most studied and of particular practical interest are: the (parametric) united solution set

$$\Sigma_{\text{uni}}(A(p), b(p), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\}$$

and the (parametric) controllable solution set

$$\Sigma(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

A parametric solution set is usually smaller [4, Theorem 5.6] than the solution set of the corresponding nonparametric interval linear system. Therefore, the former describes more precisely a physical phenomenon, whose model is a linear algebraic system involving dependencies between interval model parameters; for practical examples see [3], [5] and the references given therein.

A nonempty parametric AE-solution set, in general, has a complicated structure, see [4]. Its boundary is defined by parts of polynomials that may have arbitrary high degree. It is proven in [4] that the universally quantified parameters contribute linearly to the boundary of a parametric AE-solution set and therefore only the existentially quantified parameters determine the shape (boundary) of a nonempty parametric AE-solution set. On the other hand, the existentially quantified parameters can be classified in two groups: parameters which contribute linearly to the boundary of a solution set and parameters which determine the nonlinear boundary of a parametric AE-solution set. Recently, in [5], some sufficient conditions for an existentially quantified parameter to contribute linearly to the boundary of a parametric united solution set were proven. Based on these conditions, the scope of applicability of an efficient interval method [3] finding outer bounds for the parametric united solution set with linear shape was greatly expanded.

The goal of the present work is two-fold:

a) to present the applicability of the above sufficient conditions for obtaining sharp outer bounds of a parametric AE-solution set;
b) to generalize the interval method, proposed in [3], for parametric controllable solution sets.

The paper is organized as follows. Section 2 introduces some notions that will be used. Section 3 discusses the parametric AE-solution sets with linear shape and the goal a). A new interval method for outer enclosure of the parametric controllable solution set with linear shape is presented in Section 4 together with some illustrative examples. The paper ends by some conclusions.

2 Theoretical background

For $\mathbf{a} = [a_1, a_2]$, define mid-point $\check{\mathbf{a}} := (a_1 + a_2)/2$, radius $\hat{\mathbf{a}} := (a_2 - a_1)/2$, width (diameter) $\omega(\mathbf{a}) := 2\hat{\mathbf{a}}$, magnitude (absolute value) $|\mathbf{a}|$ and mignitude $\langle\mathbf{a}\rangle$ by

$$
\begin{aligned}
|\mathbf{a}| &:= \max\{|a_1|, |a_2|\} \\
\langle\mathbf{a}\rangle &:= \min\{|a_1|, |a_2|\} \text{ if } 0 \notin \mathbf{a},\ \langle\mathbf{a}\rangle := 0 \text{ otherwise.}
\end{aligned}
$$

These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation of the parameter dependencies, we assume that $\hat{p}_k > 0$ for all $1 \leq k \leq K$. For a bounded $\Sigma^p_{AE} \neq \emptyset$, $\square\Sigma^p_{AE} := \bigcap\{\mathbf{x} \in \mathbb{IR}^n \mid \mathbf{x} \supseteq \Sigma^p_{AE}\}$.

In order to simplify the presentation, in Section 4 we use the arithmetic on proper and improper intervals [1], [2], called Kaucher complete arithmetic or modal interval arithmetic, and its properties. The set of proper intervals $\mathbb{IR}$ is extended in [2] by the set $\overline{\mathbb{IR}} := \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}, a_1 \geq a_2\}$ of *improper* intervals obtaining thus the set $\mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}\}$ of all ordered couples of real numbers called also generalized intervals. The conventional interval arithmetic and lattice operations, order relations and other interval functions are isomorphically extended onto the whole set $\mathbb{IR} \bigcup \overline{\mathbb{IR}}$, [2]. *Modal interval analysis* imposes a logical-semantic background on generalized intervals (considered there as modal intervals) and allows giving a logical meaning to the interval results, see [1] for more details.

An element-to-element symmetry between proper and improper intervals is expressed by the "Dual" operator, $\mathrm{Dual}(\mathbf{a}) := [a_2, a_1]$ for $\mathbf{a} = [a_1, a_2] \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$. `Dual` is applied componentwise to vectors and matrices. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$

$$
\begin{aligned}
\mathrm{Dual}(\mathrm{Dual}(\mathbf{a})) = \mathbf{a}, \quad \mathrm{Dual}(\mathbf{a} \circ \mathbf{b}) = \mathrm{Dual}(\mathbf{a}) \circ \mathrm{Dual}(\mathbf{b}),\ \circ \in \{+, -, \times, /\}, \\
\mathbf{a} \subseteq \mathbf{b} \Leftrightarrow \mathrm{Dual}(\mathbf{a}) \supseteq \mathrm{Dual}(\mathbf{b}).
\end{aligned} \tag{3}
$$

The generalized interval arithmetic structure possesses group properties with respect to the operations addition and multiplication of intervals that do not involve zero. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$, $0 \notin \mathbf{b}$

$$
\mathbf{a} - \mathrm{Dual}(\mathbf{a}) = 0, \qquad \mathbf{b}/\mathrm{Dual}(\mathbf{b}) = 1. \tag{4}
$$

Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces [2]. Mid-point, radius, absolute value and mignitude are extended on generalized intervals by the same formulae. For $\mathbf{a} \in \overline{\mathbb{IR}}$, $\omega(\mathbf{a}) := 2|\hat{\mathbf{a}}|$. For

$$
\mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}, \quad \mathbf{b} \neq 0, \quad 0 \notin \text{interior of} \begin{cases} \mathbf{b} & \text{if } \mathbf{b} \in \mathbb{IR} \\ \mathrm{Dual}(\mathbf{b}) & \text{if } \mathbf{b} \in \overline{\mathbb{IR}}, \end{cases} \tag{5}
$$

$\mathrm{sgn}(\mathbf{b}) := \{1 \text{ if } b_1, b_2 \geq 0, -1 \text{ if } b_1, b_2 \leq 0\}$. For $\mathbf{a} \in \mathbb{IR}$, $0 \in \mathbf{a}$, and $\mathbf{b} \in \overline{\mathbb{IR}}$ satisfying (5),

$$
\mathbf{a} * \mathbf{b} = \mathrm{sgn}(\mathbf{b})\langle\mathbf{b}\rangle\mathbf{a}. \tag{6}
$$

3 Parametric *AE*-solution sets with linear shape

Since the universally quantified parameters contribute linearly to the boundary of a parametric AE-solution set, cf. [4], the theory about parametric united solution sets with linear shape can be generalized to parametric AE-solution sets with linear shape; the latter are also called polyhedral AE-solution sets, cf. [9]. Theorem 2 and Lemma 1 from [5] imply the following theorem.

Theorem 1. *A parameter p_k, $k \in \mathcal{E}$, contributes linearly to the boundary of a parametric AE-solution set if some of the following three equivalent conditions holds true*

(i) the nonzero elements of $A_k x - b_k$ are linearly dependent
(ii) if $A_k = 0$ or the polynomial greatest common divisor (GCD) of the elements of $A_k x - b_k$ is a nonconstant linear polynomial of $x_1, \ldots, x_n$
(iii) rank$((A_k|b_k)) = 1$, where $(A_k|b_k) \in \mathbb{R}^{m\times(n+1)}$ is the matrix obtained by augmenting the columns of A_k with the vector b_k.

The following theorem follows from the property of the universally quantified parameters mentioned above and [5, Theorem 3].

Theorem 2. *Let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ be such that $A_k \neq 0$ for $k \in \mathcal{E}_1$. Denote $k_1 := \mathrm{Card}(\mathcal{E}_1)$, $k_2 := \mathrm{Card}(\mathcal{E}_2)$. Denote by $g_k(x)$ the GCD of the elements of $A_k x$ and let $g_k(x)$ be a nonconstant linear polynomial for every $k \in \mathcal{E}_1$. Define*

$$L := (l_1|\ldots|l_{k_1}) \in \mathbb{R}^{m\times k_1}, \;\; \textit{where} \;\; l_k := A_k x / g_k(x) \in \mathbb{R}^m$$
$$R := (r_1|\ldots|r_{k_1})^\top \in \mathbb{R}^{k_1\times n}, \;\; \textit{where} \;\; r_k := \left(\frac{\partial g_k(x)}{\partial x_1}, \ldots, \frac{\partial g_k(x)}{\partial x_n}\right)^\top \in \mathbb{R}^n.$$

If there exists $t_k \in \mathbb{R}$ such that $t_k l_k = b_k := \partial b(p)/\partial p_k$ for every $k \in \mathcal{E}_1$, then

$$A_0 x - b_0 + \sum_{k\in\mathcal{E}\cup\mathcal{A}} p_k(A_k x - b_k) \;=\; LDRx - LDt - F(p_1, \ldots, p_{k_2})^\top + A_0 x - b_0 + \sum_{k\in\mathcal{A}} p_k(A_k x - b_k),$$

where $F := (b_1|\ldots|b_{k_2}) \in \mathbb{R}^{m\times k_2}$, $t = (t_1, \ldots, t_{k_1})^\top$ and $D = Diag(p_1, \ldots, p_{k_1})$.

Theorem 2 contains Theorem 3 of [5] as a special case. The following corollary is important for finding sharp outer bounds of parametric AE-solution sets.

Corollary 1 ([5], Corollary 4). *For a bounded $\Sigma^p_{AE} \neq \emptyset$,*

$$\inf\{\Box\Sigma^p_{AE}\}_i \quad \textit{and} \quad \sup\{\Box\Sigma^p_{AE}\}_i, \qquad i = 1, \ldots, n,$$

are attained at particular end-points of the intervals for the parameters that contribute linearly to the boundary of the solution set.

For a given index set $\mathcal{A}$, define the set $\mathcal{B}_\mathcal{A}$ of all end-points (vertices) of $\mathbf{p}_\mathcal{A}$.

Proposition 1 ([6], **Corollary 1**). *For a bounded parametric AE-solution set $\Sigma^p_{AE} \neq \emptyset$ and a set $\mathcal{B}'_{\mathcal{A}}$, such that $\mathcal{B}'_{\mathcal{A}} \subseteq \mathcal{B}_{\mathcal{A}}$ and $\Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \mathbf{p}_{\mathcal{E}})$ is bounded for every $\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}$, we have*

$$\square \Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \square \Sigma(A(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), b(\tilde{p}_{\mathcal{A}}, p_{\mathcal{E}}), \mathbf{p}_{\mathcal{E}}).$$

Proposition 1 shows that outer bounds of a parametric AE-solution set can be found by bounding only parametric united solution sets. Then, the methodology for finding sharp bounds of parametric united solution sets with linear shape applies to parametric AE-solution set via Proposition 1.

Let $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$, $\mathcal{E}_1 \cap \mathcal{E}_2 = \emptyset$ be such that p_k, $k \in \mathcal{E}_1$, satisfy Theorem 1. Then,

$$\square \Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \square \left(\bigcup_{\tilde{p}_{\mathcal{E}_1} \in \mathcal{B}_{\mathcal{E}_1}} \square \Sigma(A(\tilde{p}_{\mathcal{A}}, \tilde{p}_{\mathcal{E}_1}, p_{\mathcal{E}_2}), b(\tilde{p}_{\mathcal{A}}, \tilde{p}_{\mathcal{E}_1}, p_{\mathcal{E}_2}), \mathbf{p}_{\mathcal{E}_2}) \right). \tag{7}$$

Note, that the above methodology can be applied to parametric AE-solution sets such that not all existentially quantified parameters satisfy Theorem 1 or contribute linearly to the boundary of the parametric AE-solution set. A larger discussion is contained in [5].

4 Enclosure of $\boldsymbol{\Sigma_{ctrl}(A(p), b(q), \mathrm{p}, \mathrm{q})}$ with linear shape

Consider a parametric interval algebraic system

$$\begin{aligned} A(p)x &= b(q), \qquad p \in \mathbf{p} \in \mathbb{IR}^{K_p}, q \in \mathbf{q} \in \mathbb{IR}^{K_q}, \\ A(p) &:= A_0 + \sum_{k=1}^{K_p} p_k A_k, \quad b(q) := b_0 + \sum_{k=1}^{K_q} q_k b_k. \end{aligned} \tag{8}$$

We assume that the structure of the dependencies between the parameters p is such that the conditions defined in Theorem 1 hold true for all the parameters and, therefore, the system (8) has the equivalent representation

$$(A_0 + L\mathrm{Diag}(p)R)\, x = b_0 + Fq, \quad p \in \mathbf{p} \in \mathbb{IR}^{K_p}, q \in \mathbf{q} \in \mathbb{IR}^{K_q}, \tag{9}$$

with suitable numerical matrices L, R, F found by Theorem 2.

We search for an outer interval enclosure of the parametric controllable solution set

$$\begin{aligned} \Sigma^p_{ctrl} &= \Sigma_{ctrl}(A(p), b(q), \mathbf{p}, \mathbf{q}) \\ &:= \{x \in \mathbb{R}^n \mid (\forall q \in \mathbf{q})(\exists p \in \mathbf{p})(A(p)x = b(q))\}. \end{aligned}$$

4.1 Iteration method

Theorem 3. *If $x \in \Sigma_{ctrl}^{p} \neq \emptyset$ for the system (9), $D_0 \in \mathrm{Diag}(\mathbf{p})$ and $A_0 + LD_0R$ is invertible, then $x \in (A_0 + LD_0R)^{-1}(b_0 + F\mathrm{Dual}(\mathbf{q}) + L\mathbf{d})$, wherein*

$$\mathbf{d} := (D_0 - \mathrm{Diag}(\mathbf{p}))Rx, \tag{10}$$

and $b_0 + F\mathrm{Dual}(\mathbf{q}) + L\mathbf{d}$ is a proper interval vector in Kaucher interval arithmetic.

Proof. If $x \in \Sigma_{ctrl}^{p} \neq \emptyset$, according to [7, Theorem 3.2] we have

$$(A_0 + L\mathrm{Diag}(\mathbf{p})R)\, x \supseteq b_0 + F\mathbf{q}. \tag{11}$$

We apply the Dual operator to the above inclusion relation, the relation (3), and the distributivity of multiplication by a point vector x. Then, we add $-L\mathrm{Diag}(\mathbf{p})Rx$ to both sides of the obtained inclusion. Due to (4) we obtain

$$A_0x \subseteq b_0 + F\mathrm{Dual}(\mathbf{q}) - L\mathrm{Diag}(\mathbf{p})Rx. \tag{12}$$

The relation (10) implies $-\mathrm{Diag}(\mathbf{p})Rx = \mathbf{d} - D_0Rx$ which we substitute in (12) and obtain

$$(A_0 + LD_0R)x \subseteq b_0 + F\mathrm{Dual}(\mathbf{q}) + L\mathbf{d}.$$

The inclusion (11), which holds true for $x \in \Sigma_{ctrl}^{p} \neq \emptyset$, implies $\omega(F\mathbf{q}) \leq \omega(L\mathbf{d})$. The latter implies that $b_0 + F\mathrm{Dual}(\mathbf{q}) + L\mathbf{d}$ is a proper interval vector since $L\mathbf{d}$ is a proper interval vector and $F\mathrm{Dual}(\mathbf{q})$ is an improper one. Due to invertibility of $A_0 + LD_0R$, we obtain $x \in (A_0 + LD_0R)^{-1}(b_0 + F\mathrm{Dual}(\mathbf{q}) + L\mathbf{d})$.

Theorem 4. *Let $D_0 \in \mathbb{R}^{K_p \times K_p}$, $D_0 \in \mathrm{Diag}(\mathbf{p}) = \mathbf{D}$ be such that $A_0 + LD_0R$ is invertible and put*

$$C := (A_0 + LD_0R)^{-1}.$$

Define

$$w' := w - |D_0 - \mathbf{D}|\,|RCL|w, \tag{13}$$

$$w'' := |D_0 - \mathbf{D}|\langle RCb_0 + (RCF)\mathrm{Dual}(\mathbf{q})\rangle, \tag{14}$$

for some vector $w \geq 0$, and

$$\mathbf{u} := [-\alpha w, \alpha w], \quad \alpha = \max_i \frac{w''_i}{w'_i}. \tag{15}$$

(i) $x = x(p, q) \in \Sigma_{ctrl}^{p} \neq \emptyset$ for (9) is related to $y = Rx(p, q)$ by the inclusions

$$x \in Cb_0 + (CF)\mathrm{Dual}(\mathbf{q}) + CL\mathbf{d}, \tag{16}$$

$$y \in RCb_0 + (RCF)\mathrm{Dual}(\mathbf{q}) + (RCL)\mathbf{d}, \tag{17}$$

where

$$\mathbf{d} = (D_0 - \mathbf{D})y. \tag{18}$$

(ii) If $w' > 0$ and $0 \notin b_0 + F\mathbf{q}$, then $\mathbf{d} \subseteq \mathbf{u}$.

(iii) If $\mathbf{x} := Cb_0 + (CF)\mathrm{Dual}(\mathbf{q}) + (CL)\mathbf{u}$ is a proper interval vector, then every $x \in \Sigma_{ctrl}^{p}$ satisfies $x \in \mathbf{x}$.

Proof. (i) follows from Theorem 3.

(ii) Since $w' > 0$ we put

$$\beta = \max_i |\mathbf{d}_i|/w_i$$

and note that $|\mathbf{d}| \le \beta w$, with equality in some component i. The definition of α and $0 \notin b_0 + F\mathbf{q}$ imply $0 \le w'' \le \alpha w'$. From (16)–(18) and a subdistributive law in Kaucher arithmetic we have

$$\begin{aligned}\mathbf{d} &\subseteq (D_0 - \mathbf{D})(RCb_0 + (RCF)\mathrm{Dual}(\mathbf{q}) + (RCL)\mathbf{d})\\ &\subseteq (D_0 - \mathbf{D})(RCb_0 + (RCF)\mathrm{Dual}(\mathbf{q})) + (D_0 - \mathbf{D})(RCL)\mathbf{d}.\end{aligned}$$

Then, formula (6) implies

$$\begin{aligned}|\mathbf{d}| &\le |D_0 - \mathbf{D}|\langle RCb_0 + (RCF)\mathrm{Dual}(\mathbf{q})\rangle + |D_0 - \mathbf{D}||RCL|\beta w\\ &\overset{(14),(13)}{\le} w'' + \beta(w - w') \le \alpha w' + \beta(w - w').\end{aligned}$$

Thus $\beta w_i = |d_i| \le \alpha w'_i + \beta(w_i - w'_i)$, hence $\beta w'_i \le \alpha w'_i$. As $w' > 0$, we conclude that $\beta \le \alpha$, and $\mathbf{d} \subseteq \mathbf{u}$ follows.

(iii) follows by (ii) and Theorem 3.

In the computations we take D_0 as the midpoint of $\mathbf{D}$, and w, e.g., as the vector with all entries one. In order to provide guaranteed enclosures, w' should be rounded downward, w'' and α should be rounded upward. If $w' \le 0$ in (13), we may apply the approach proposed in [3] to compute the largest eigenvalue ϱ (= the spectral radius) of the nonnegative matrix

$$M := |D_0 - \mathbf{D}||RCL|.$$

If $\varrho < 1$, any $w > 0$ sufficiently close to an associated eigenvector makes $w' > 0$. In practice, one could run a Lanczos iteration and stop as soon as an intermediate eigenvector approximation $w > 0$ satisfies $Mw < w$, [3].

The computed initial interval enclosure $\mathbf{u}$ of $\mathbf{d}$ can be further improved by iterating and intersecting with the previously computed enclosures. It is sufficient to iterate the enclosures for y and d, and compute the enclosures for x when the intersected results no longer improve significantly. Thus we iterate

$$\begin{aligned}&\mathbf{y} = \{(RCb_0) + (RCF)\mathrm{Dual}(\mathbf{q}) + (RCL)\mathbf{u})\} \cap \mathbf{y},\\ &\mathbf{u} = \{(D_0 - \mathbf{D})\mathbf{y}\} \cap \mathbf{u}\end{aligned}$$

until some stopping test holds, and then get the enclosure

$$\mathbf{x} := (Cb_0) + (CF)\mathrm{Dual}(\mathbf{q}) + (CL)\mathbf{u}$$

for all x that belong to Σ^p_{ctrl} of (9). In the implementation of the method we used the stopping criterion proposed in [3], namely, the iteration stops when the sum of widths of the components of $\mathbf{u}$ does not improve by a factor of 0.999 or after at most 10 iterations.

The method presented here can be considered as an extension of the so-called formal (algebraic) approach, cf. [10], for enclosing nonparametric *AE*-solution sets to parametric controllable solution sets.

4.2 Numerical examples

Here we illustrate the advantages of the above parametric iterative method by some numerical examples and compare this method to the only discussed by now methods for outer enclosure of the parametric controllable solution set presented in [6]. The implementations and the numerical computations are done in the environment of *Mathematica*® using the package `directed.m` [8]. The latter package supports the arithmetic of proper and improper intervals and provides compatibility with the conventional interval arithmetic supported by the *Mathematica*® kernel. The numerical computations are done exactly if all input data are represented exactly (e.g., by rational numbers) or by appropriate directed rounding in floating-point if some input data are in floating point arithmetic. This software environment and the implementation provide obtaining numerical interval vectors which are guaranteed to contain the considered parametric solution set.

The iteration method, proposed in Section 4.1, can be implemented in any software environment which does not support the arithmetic of proper and improper intervals if the lower and upper bounds of the corresponding intervals are computed separately applying the corresponding formulae for the arithmetic operations and applying correct directed rounding in floating-point arithmetic. The iteration method from [3] is implemented in the environment of Matlab, see [3], and in C-XSC, see [11].

Explicit representation of any parametric controllable solution set, considered below, is obtained by methods from [4] as a system of real inequalities in the coordinate variables, which is then solved by suitable *Mathematica*® functions.

Example 1. Find an enclosure of the parametric controllable solution set to the system

$$\begin{pmatrix} p_1+p_2, \, p_1-p_2 \\ p_1-p_2, \, p_1+p_2 \end{pmatrix} x = \begin{pmatrix} 3/2+q \\ q \end{pmatrix},$$

where $p_1, p_2 \in [1/2, 3/2]$, $q \in [-1/10, 1/10]$. According to Theorem 1 the parametric controllable solution set has linear shape and the application of Theorem 4 with an iterative refinement gives the following (rounded outward) interval enclosure

$$\begin{pmatrix} [0.099951, 1.400049] \\ [-0.650049, 0.650049] \end{pmatrix}.$$

Since the parametric matrix is not strongly regular, the parametric Bauer-Skeel method from [6, Corollary 6] fails. Due to the same reason Proposition 1 cannot be applied together with the parametric Bauer-Skeel method while it can be applied together with Theorem 4.

Example 2. Find an enclosure of the parametric controllable solution set to

$$\begin{pmatrix} p_1+p_2, & p_1-2p_2 \\ p_1-p_2/2, & p_1+p_2 \end{pmatrix} x = \begin{pmatrix} 3/2+q/3 \\ q/2 \end{pmatrix},$$

where $p_1 \in [1, 3/2]$, $p_2 \in [-1, -1/2]$, $q \in [9/10, 11/10]$. According to Theorem 1 the parametric controllable solution set has linear shape and the application of Theorem 4 with an iterative refinement gives the interval enclosure

$$([-0.17779, 0.39507],\ [0.39875, 0.89507])^\top,$$

while the parametric Bauer-Skeel method from [6] gives the enclosure

$$([-0.32840, 0.54568],\ [0.29135, 0.1.00247])^\top.$$

The former interval enclosure overestimates the exact interval hull

$$([-\frac{59}{405}, \frac{3}{10}], [\frac{421}{810}, \frac{239}{270}])^\top$$

of the parametric controllable solution set by $(22.2, 26.4)^\top\%$, while the latter enclosure overestimates the hull by $(49, 48.6)^\top\%$. In general, the percentage of overestimation depends on the particular problem (parameter dependencies), the problem size, the number of the parameters, and the width of the parameter intervals. The parametric Bauer-Skeel method can be applied to the above system, where the parameter intervals $\mathbf{p}_1, \mathbf{p}_2$ are with enlarged radius from $r = 1/4$ (the intervals considered above) to a radius $r = 0.481$ which still provides strong regularity of the parametric matrix. In the latter case the overestimation of the corresponding exact interval hull is $(99.81, 99.79)^\top\%$. The method from Theorem 4 is applicable to the above system where the parameter intervals $\mathbf{p}_1, \mathbf{p}_2$ have radius $r = 0.749$ still providing regularity of the parametric matrix. The overestimation of the corresponding exact interval hull for these intervals is $(35.86, 37.94)^\top\%$.

Further improvement of an enclosure obtained by some applicable enclosure method (or by some method for obtaining the exact interval hull of Σ^p_{uni} with linear shape) could be obtained by Proposition 1, respectively (7), at the expense of a bigger computational effort. The application of (7) to Example 2 with the initial parameter intervals gives an interval enclosure which overestimates the exact interval hull of the parametric controllable solution set by $(0, 5.7)^\top\%$.

In the following example, we consider the behaviour of the proposed method on parametric controllable solution sets that are empty sets or unbounded.

Example 3. Consider the parametric linear system

$$\begin{pmatrix} p_1 - p_2, & p_2 \\ p_1 + p_2, & -p_2 \end{pmatrix} x = \begin{pmatrix} 2q \\ 2q \end{pmatrix}.$$

a) For $p_1 \in [3/4, 5/4]$, $p_2 \in [0, 1]$, $q \in [1, 2]$, $\Sigma^p_{ctrl} = \emptyset$. The application of the method considered above yields $w' < 0$ and the largest eigenvalue (= the spectral radius) of the matrix $|D_0 - \mathbf{D}||RCL|$ equal to 1.

b) With the same data as in a) but twice bigger radius of p_1, $p_1 \in [1/2, 3/2]$, the corresponding Σ^p_{ctrl} is unbounded and defined by $8/3 \leq x_1 \leq 4$, $x_2 \in \mathbb{R}$. The application of the method proposed in this paper yields the same output as in a).

The method, proposed in this paper, for bounding parametric controllable solution sets with linear shape possesses the same scalability property as the methods from [3], [5] for bounding parametric united solution sets. Examples of large parametric linear systems with over 5000 variables and over 10 000 parameters which appear in finite element analysis of uncertain truss structures can be found in [3], while [5] presents some examples coming from modeling of electrical circuits and models in biology.

5 Conclusion

We presented a new interval method for outer enclosure of a class of nonempty and bounded parametric controllable solution sets with linear shape. Contrary to other available so far interval methods for bounding general parametric controllable solution sets, which require strong regularity of the parametric matrix, the new method does not have such a restriction. Furthermore, the method is applicable to parametric linear systems of high dimensions that involve many parameters, see [3], and when the parameter intervals are large, see the end of Example 2. Further improvement of the solution enclosure may be achieved by methods presented in Section 3 and [5, Section 2].

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