Interval Algebraic Approach to Equilibrium Equations in Mechanics

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Abstract The engineering demand for more realistic and accurate models involving interval uncertainties lead to a new interval model of linear equilibrium equations in mechanics, which is based on the algebraic completion of classical interval arithmetic (called Kaucher arithmetic). Interval algebraic approach consists of three parts: representation convention, computing algebraic solution and result interpretation. The proposed approach replaces straightforward a deterministic model by an interval model in terms of proper and improper intervals, fully conforms to the equilibrium principle and provides sharper enclosure of the unknown quantities than the best known methods based on classical interval arithmetic. Numerical applications described by systems of linear interval equilibrium equations where the number of the unknowns is equal to the number of the equations are considered in details.

1 Introduction

The basic principle of static (or dynamic) equilibrium under general force systems is an essential prerequisite for many branches of engineering, such as mechanical, civil, aeronautical, bioengineering, robotics, and others that address the various consequences of forces [1].

One main challenge for the models involving interval uncertainty is the overestimation of the system response. Nowadays, the most successful approaches for overestimation reduction are those that relate the dependency of interval quantities to the physics of the problem being considered, [8]. Recently, a model of a bar subjected to multiple axial external loads, where load magnitudes are represented by intervals, is considered in [3]. Although the aim at providing interval model conforming to the principle of static equilibrium is not completely achieved by the proposed model,

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the problem and its challenge are presented. A similar problem in the context of robotics is discussed in the IEEE P1788 working group on standardization of interval arithmetic, [6]. It is shown in [3], [6] that an interval model based entirely on the classical interval arithmetic, in its set-theoretic interpretation as proposed by Moore [7], cannot provide a good estimation neither of the uncertain reaction nor of the load distribution.

The engineering demand for more accurate models involving interval uncertainties lead to an interval model of linear equilibrium equations in mechanics [14], which is based on the algebraic completion of classical interval arithmetic (called also Kaucher or generalized interval arithmetic). It is proven that the proposed interval model always yields the narrowest interval enclosure and is in full conformance with the physical meaning of static equilibrium. The work [14] is focused on justification of the proposed interval model in one dimension, comparison to the approach of [3], and applications to computing resultant forces. In this paper we further develop the interval algebraic approach to models involving linear interval equilibrium equations. Considered are models of practical applications which reduce to systems of linear interval equilibrium equations where the number of the unknowns is equal to the number of the equations. The initial interval model is expanded by considering interval algebraic solution to the system of equilibrium equations, model properties are revealed and the quality of the interval algebraic solution is compared to the best interval solution enclosure obtained by classical interval arithmetic.

The structure of the paper is as follows. In the next section some basic notions and properties of the algebraic extension [4] of classical interval arithmetic are summarized. In section 3 we present the new interval model and its generalization to systems of interval equilibrium equations involving as many unknowns as is the number of the equations. Numerical applications developed in details in Section 4 illustrate the proposed interval algebraic approach, its conformance to the equilibrium principle, bring out its effectiveness and advantages over the approach based on classical interval arithmetic. The article ends by some conclusions.

2 The Algebraic Completion of $\mathbb{IR}$

The set of classical compact intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \leq a^+\}$, called also *proper* intervals, is extended in [4] by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called *generalized* (extended or Kaucher) intervals. The inclusion order relation between classical intervals[1] $\subseteq$ is generalized for $[a], [b] \in \mathbb{KR}$ by $[a] \subseteq [b] \iff b^- \leq a^-$ and $a^+ \leq b^+$. For $[a] = [a^-, a^+] \in \mathbb{KR}$ define binary variable *direction* (τ) by $\tau([a]) := \mathrm{sgn}(a^+ - a^-) = \{+ \text{ if } a^- \leq a^+, \ - \text{ if } a^- > a^+\}$. All elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called im-

[1] For a better understanding we denote the classical intervals by bold face letters and the intervals from $\mathbb{KR}$ by brackets $[a]$. Of course, $\mathbf{a} \in \mathbb{IR} \subset \mathbb{KR}$, and thus $[a] = \mathbf{a} \in \mathbb{KR}$ is a correct assignment.

proper intervals. An element-to-element symmetry between proper and improper intervals is expressed by the “Dual” operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\mathrm{Dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\mathrm{Dual}(\mathrm{Dual}([a])) = [a], \tag{1}$$

$$\mathrm{Dual}([a] \circ [b]) = \mathrm{Dual}([a]) \circ \mathrm{Dual}([b]),\ \circ \in \{+, -, \times, /\}. \tag{2}$$

Define proper projection of a generalized interval $[a]$ onto $\mathbb{IR}$ by

$$\mathrm{pro}([a]) := \begin{cases} [a] & \text{if } \tau([a]) = +, \\ \mathrm{Dual}([a]) & \text{if } \tau([a]) = -. \end{cases} \tag{3}$$

Define binary variable “sign” (σ) by $\sigma([a]) := \begin{cases} + & \text{if } \mathrm{pro}([a])^- \geq 0, \\ - & \text{otherwise.} \end{cases}$. Denote $\mathscr{T} := \{[a] \in \mathbb{KR} \mid [a] = [0,0] \text{or } a^-a^+ < 0\}$. The conventional interval arithmetic and lattice operations, as well as other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, [4]. A condensed representation of the arithmetic operations is derived in [2], Thus,

$$[a] + [b] = [a^- + b^-, a^+ + b^+] \qquad \text{for } [a], [b] \in \mathbb{KR},$$

$$[a] \times [b] = \begin{cases} [a^{-\sigma([b]}b^{-\sigma([a])}, a^{\sigma([b]}b^{\sigma([a])}] & \text{if } [a], [b] \in \mathbb{KR} \setminus \mathscr{T} \\ [a^{\sigma([a]\tau([b])}b^{-\sigma([a])}, a^{\sigma([a]\tau([b]}b^{\sigma([a])}] & \text{if } [a] \in \mathbb{KR} \setminus \mathscr{T}, [b] \in \mathscr{T} \\ [a^{-\sigma([b]}b^{\sigma([b])\tau([a])}, a^{\sigma([b])}b^{\sigma([b])\tau([a])}] & \text{if } [a] \in \mathscr{T}, [b] \in \mathbb{KR} \setminus \mathscr{T} \\ [\min\{a^-b^+, a^+b^-\}, \max\{a^-b^-, a^+b^+\}] & \text{if } [a], [b] \in \mathscr{T}, \tau([a]) = \tau([b]) \\ 0 & \text{if } [a], [b] \in \mathscr{T}, \tau([a]) \neq \tau([b]), \end{cases}$$

wherein $++ = -- = +$, $+- = -+ = -$. Interval subtraction and division can be expressed as composite operations, $[a] - [b] = [a] + (-1)[b]$ and $[a]/[b] = [a] \times (1/[b])$, where $1/[b] = [1/b^+, 1/b^-]$ if $[b] \in \mathbb{KR} \setminus \mathscr{T}$. The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space.

The generalized interval arithmetic structure possesses group properties with respect to the operations addition and multiplication. For $[a] \in \mathbb{KR}$, $[b] \in \mathbb{KR} \setminus \mathscr{T}$,

$$[a] - \mathrm{Dual}([a]) = 0, \qquad [b]/\mathrm{Dual}([b]) = 1. \tag{4}$$

The complete set of conditionally distributive relations for multiplication and addition of generalized intervals can be found in [10], [11]. Here we present only one that will be used. For $[a], [b], [s] = ([a] + [b]) \in \mathbb{KR} \setminus \mathscr{T}$, $[c] \in \mathbb{KR}$

$$([a] + [b])[c]_{\sigma([s])} = [a]_{\sigma([a])} + [b]_{\sigma([b])}, \tag{5}$$

wherein $[a]_+ = [a]$, $[a]_- = \mathrm{Dual}([a])$. Addition operation in $\mathbb{KR}$ is commutative and associative; associativity does not hold true in (interval) floating point arithmetic.

Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces, [4]. Some other properties and applications of generalized interval arithmetic can be found in [4], [2], [5], [10], [11], [15], [17] and the references given therein.

For $\mathbf{a} \in \mathbb{IR} \setminus \mathscr{T}$, define $\mathrm{Abs}(\mathbf{a}) = \{\mathbf{a} \text{ if } 0 \leq \mathbf{a}; -\mathbf{a} \text{ otherwise}\}$. Relative diameter of $\mathbf{a} \in \mathbb{IR}$ is defined as $a^+ - a^-$ if $0 \in \mathbf{a}$ and $(a^+ - a^-)/\min\{|a^-|, |a^+|\}$ otherwise.

3 Interval Model of Equilibrium Equations

In this section the algebraic approach to equilibrium equations in mechanics is derived by considering two-dimensional problems involving several forces acting on a particle. The same approach with obvious modifications is applicable to three-dimensional problems and problems whose models involve other vector physical quantities possessing magnitude and direction such as velocities, accelerations, or momenta. Such problems will be illustrated in the next section. In the text of this paper forces (and other vector quantities) are denoted by underlining the letter used to represent it. This is necessary in order to distinguish vectors from the proper intervals, which are denoted by bold-face letters, and from the real-valued scalars. The magnitude of a vector will be denoted by the corresponding italic-face letter.

In the deterministic case of two- dimensional problems involving several forces, the determination of their resultant $\underline{R}$ is best carried out by first resolving each force into rectangular components. Choosing a rectangular coordinate system (Oxy), with unit vectors $\underline{i}, \underline{j}$, any force vector $\underline{F}$ can be resolved into rectangular components $\underline{F}_x = F_x\underline{i}$, and $\underline{F}_y = F_y\underline{j}$, so that $\underline{F} = F_x\underline{i} + F_y\underline{j}$. The scalar component F_x is positive when the vector component $\underline{F}_x$ has the same direction as the unit vector $\underline{i}$ (i.e., the same direction as the positive x axis) and is negative when $\underline{F}_x$ has the opposite direction. A similar conclusion may be drawn regarding the sign of the scalar component F_y. Denoting by F the magnitude of the force $\underline{F}$ and by θ the angle between $\underline{F}$ and the axis x, measured counterclockwise from the positive axis, we may express the scalar components of $\underline{F}$ as follows: $F_x = F\cos(\theta)$ and $F_y = F\sin(\theta)$, cf. any textbook in statics, e.g., [1]. When more than one force act on a particle (or a rigid body), it is important to determine the resultant force, i.e., the single force $\underline{R}$ which has the same effect on the particle as the given forces. The resultant force $\underline{R}$ can be determined by:

1. choosing a rectangular coordinate system;
2. resolving the given forces into their rectangular components;
3. each scalar component R_x, R_y of the resultant $\underline{R}$ of several forces $\underline{F}_i$ acting on a particle is obtained by adding algebraically the corresponding scalar components of the given forces. That is, $R_x = \sum_i F_{x,i}$, $R_y = \sum_i F_{y,i}$, which gives $\underline{R} = R_x\underline{i} + R_y\underline{j}$.

Basing on the above, the one dimensional interval algebraic model for computing the resultant force (and reaction), developed in [14], can be applied to two- and three-dimensional problems involving vector physical quantities.

Theorem 1 ([14]). *Consider a bar subjected to a finite number of loads* $\underline{p}_1, \ldots, \underline{p}_k$ *that may be applied in opposite directions and have uncertain magnitude* $p_1 \in \mathbf{p_1}, \ldots, p_k \in \mathbf{p_k}$, $\mathbf{p}_i \geq 0$, $i = 1, \ldots, k$. *Assume that a coordinate system* (Ox) *is chosen. Then,*

(i) for every j, $1 \leq j \leq k$, *we have* $[N_j] = \sum_{i=1}^{j} [p_i]$, *wherein*

$$[p_i] = \begin{cases} \mathbf{p}_i & \text{if the direction of } \underline{p}_i \text{ is in the positive } x \text{ axis} \\ -\mathrm{Dual}(\mathbf{p}_i) & \text{if the direction of } \underline{p}_i \text{ is opposite to the positive } x \text{ axis}, \end{cases}$$

and $[r] = -\mathrm{Dual}([N_k]) = -\mathrm{Dual}(\sum_{i=1}^{k} [p_i])$.

(ii)The interpretation of $[N_j] \in \mathbb{KR}$, $1 \leq j \leq k$, *and similarly of* $[r]$, *is as follows.*

- *If* $[N_j] \in \mathscr{T}$, *then* $\underline{N}_j$ *may have positive or negative direction and its magnitude varies in* $pro([N_j])$.
- *If* $[N_j] \in \mathbb{KR} \setminus \mathscr{T}$, *the magnitude of* $\underline{N}_j$ *varies in* $Abs(pro([N_j]))$, *while the direction of* $\underline{N}_j$ *coincides with the sign of* $[N_j]$ *(if* $[N_j] \geq 0$ *the direction of* $\underline{N}_j$ *is the positive x axis, otherwise it is opposite to the positive x axis).*

Strong proof that Theorem 1 provides sharpest estimation of the resultant force and its reaction is given in [14] along with a detailed discussion and examples.

Now we consider the interval algebraic model of equilibrium equations from a more general perspective. Assume that there is a deterministic model described by some linear equilibrium equation(s) that involve uncertain parameters varying within given proper intervals. Clearly, the unknowns in this model will be also uncertain and we search for proper intervals that are the sharpest interval estimations of these unknowns and that conform to the physics of the problem (statics or dynamic equilibrium). Conformance to static (dynamic) equilibrium means that the intervals found for the unknowns when replaced in the equation(s) and all operations are performed results in true equality(ies).

Definition 1 ([16]). Interval *algebraic solution* to a (system of) interval equation(s) is an interval (interval vector) which substituted in the equation(s) and performing all interval operations in exact arithmetic[2] results in valid equality(ies).

Interval algebraic solutions do not exist in general in classical interval arithmetic [16]. Generalized interval arithmetic on proper and improper intervals $(\mathbb{KR}, +, \times, \subseteq)$ is the natural arithmetic for finding algebraic solutions to interval equations since it is obtained from the arithmetic for classical intervals $(IR, +, -, \times, /, \subseteq)$ via an algebraic completion. This is another justification of the proposed interval algebraic approach. Therefore, we embed the initial problem formulation in the interval space $(\mathbb{KR}, +\times, \subseteq)$, find an algebraic solution (if exists) and interpret the obtained generalized intervals back in the initial interval space $\mathbb{IR}$. This is a three steps procedure summarized below.

[2] no round-off errors

1. The **representation convention** for a model involving interval forces (and/or other physical quantities considered as vectors and possessing magnitude and direction) is:

- a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by proper interval $\mathbf{F}_x$ ($\mathbf{F}_y$, $\mathbf{F}_z$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has the same direction as the positive x (y, z) coordinate axis;
- a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by the improper interval $\mathrm{Dual}(\mathbf{F}_x)$ ($\mathrm{Dual}(\mathbf{F}_y)$, $\mathrm{Dual}(\mathbf{F}_z)$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has opposite direction to the corresponding positive x (y, z) coordinate axis.

2. Computing. Find the **algebraic solution** for the unknown(s) in $(\mathbb{KR}, +, \times, \subseteq)$. Conditions for existence of algebraic solution of interval linear equations are published in [10], [17]. Numerical methods finding the algebraic solution to an interval linear system are discussed in [5], [17]. For small systems, the approach based on *equivalent algebraic transformations* is transparent and will be used in this paper.

3. Interpretation of the obtained generalized intervals in the initial space $\mathbb{IR}$ is done according to the physics of the unknowns. If it is a force component, then Theorem 1 ii) is applied. In general the interpretation projects the generalized interval solution on $\mathbb{IR}$ by (3).

Since computing a resultant $\underline{R}$ of several forces $\underline{F}_i$ can be represented as a solution of the equilibrium equation $\sum_i \underline{F}_i - \underline{R} = 0$, Theorem 1 is a special case of the above more general interval algebraic approach.

4 Numerical Applications

Here we consider models of practical applications which reduce to systems of linear interval equilibrium equations where the number of the unknowns is equal to the number of the equations. In order to avoid many technical details that will hamper the comprehension, only two dimensional problems are considered. The numerical results presented in this section are obtained by the *Mathematica*® package `directed.m` [15]. JInterval library [9] can be used for this purpose, too.

Example 1. An 80 kg block rests on a horizontal plane, Fig. 1 a). Find the magnitude of the force $\underline{P}$ required to give the block an acceleration of 2.5 m/s^2 to the right. The coefficient of kinetic friction between the block and the plane is $\mu_k = 0.25$. Assume that the mass of the block and the angle, at which the force acts on the block, are measured with 1 % uncertainty.

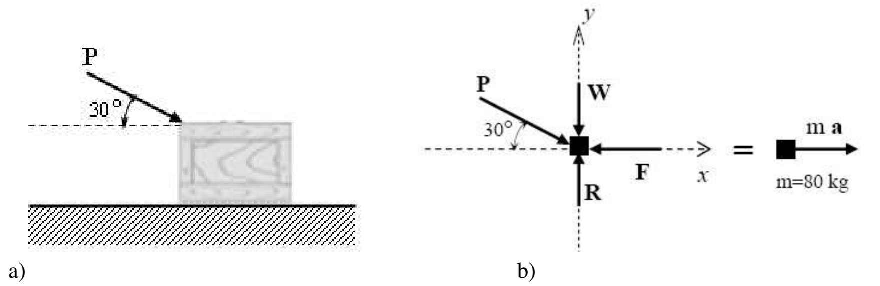


Fig. 1 a) A force acting on a block that rests on a horizontal plane; b) the free-body diagram

The chosen coordinate system is presented on the free-body diagram in Fig. 1 b). Note that $\mathbf{F} = \mu_k \mathbf{R}$. The weight of the block is[3]

$$W = mg_0 \in ([79.2, 80.8]\mathrm{kg})(9.80665\mathrm{m/s}^2) \in [776.686, 792.378]\mathrm{N}.$$

Writing Newton's second law $\sum \underline{F} = m\underline{a}$ in rectangular components and applying the representation convention, we obtain the following interval equilibrium equations

$$\mathbf{P}\cos([\theta]) - \mathrm{Dual}(0.25\mathbf{R}) = 2.5\mathbf{m} \tag{6}$$
$$\mathbf{R} - \mathrm{Dual}(\mathbf{P}\sin([\theta])) - \mathrm{Dual}(\mathbf{W}) = 0, \tag{7}$$

where $[\theta] = [29.7°, 30.3°]$. We search for proper intervals $\mathbf{P}$, $\mathbf{R}$ which satisfy these equations. Adding $\mathbf{P}\sin([\theta]) + \mathbf{W}$ to the two sides of equation (7) and applying property (4), we obtain

$$[R] = \mathbf{P}\sin([\theta]) + \mathbf{W}.$$

Replacing $[R]$ in the first equilibrium equation (6), we have

$$\mathbf{P}\cos([\theta]) - 0.25\mathrm{Dual}(\mathbf{P}\sin([\theta])) - 0.25\mathrm{Dual}(\mathbf{W}) = 2.5\mathbf{m}. \tag{8}$$

The distributive relation (5) holds true for the first two terms of (8) since

$$[s] = \cos([\theta] - 0.25\mathrm{Dual}(\sin([\theta]) \in [0.739530, 0.7425] > 0.$$

Thus, the equation (8) is equivalent to $[P][s] - 0.25\mathrm{Dual}(\mathbf{W}) = 2.5\mathbf{m}$. Adding $0.25\mathbf{W}$ to both sides of the last equation and then dividing by $\mathrm{Dual}([s])$, we obtain

[3] All computed numerical intervals are outwardly rounded to the intervals presented in the paper.

$$[P] = (2.5\mathbf{m} + 0.25\mathbf{W})/\mathrm{Dual}([s]) \in [530.297, 538.848]\mathrm{N}.$$

From the last equivalent form of equation (7), we get $[R] = [P]\sin([\theta]) + \mathbf{W} \in [1039.42, 1064.25]$N. Both $[P]$ and $[R]$ are proper intervals. Replacing them in the initial equations (6)–(7) we obtain $[-1.71 \times 10^{-13}, 1.14 \times 10^{-13}]$, $[-4.55 \times 10^{-13}, 4.55 \times 10^{-13}]$, respectively. These intervals are almost but not exactly zero due to the round-off errors and show that the equilibrium equations are completely satisfied.

Now, we compare the solution $\mathbf{P}$, $\mathbf{R}$, obtained by the discussed algebraic approach, to the solution obtained by classical interval arithmetic. In classical interval arithmetic the goal is to find the smallest interval vector enclosing the so-called united solution set[4] of the interval system

$$\begin{pmatrix} \cos([\theta]) & -0.25 \\ -\sin([\theta]) & 1 \end{pmatrix} \begin{pmatrix} P \\ R \end{pmatrix} = \begin{pmatrix} 2.5\,m \\ 9.80665\,m \end{pmatrix}, \qquad \begin{array}{c} \theta \in [29.7^\circ, 30.3^\circ], \\ m \in ([79.2, 80.8]. \end{array}$$

The smallest interval vector that encloses the united solution set of this system is $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top = ([526.56, 542.68], [1037.58, 1066.18])^\top$. The percentage by which $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top$ overestimates $(\mathbf{P}, \mathbf{R})^\top$ is $(46.9, 13.2)^\top\%$.

If we consider the same problem with 2% relative uncertainty in the angle and 1% relative uncertainty in the mass of the block, then the percentage by which $(\tilde{\mathbf{P}}, \tilde{\mathbf{R}})^\top$ overestimates $(\mathbf{P}, \mathbf{R})^\top$ is $(70.2, 20.9)^\top\%$.

Since we are looking for proper algebraic solutions of the interval equilibrium system, this restriction may not be always satisfied. The latter case is illustrated by the next example.

Example 2. A $[100 \pm 1]$ kg crate is suspended from a pulley that can roll freely on the support cable ACB and is pulled at a constant speed by cable CD, Fig. 2. If $\alpha = 30^\circ$, $\beta = 10^\circ$ and the angles are measured with 1% uncertainty, determine the tension (a) in the support cable ACB, (b) in the traction cable CD.

The chosen coordinate system is presented on the free-body diagram in Fig. 2. The deterministic equilibrium equations of force x and y components are

$$F_{\mathrm{ACB}}\cos(10^\circ) - F_{\mathrm{ACB}}\cos(30^\circ) - F_{\mathrm{CD}}\cos(30^\circ) = 0, \tag{9}$$

$$F_{\mathrm{ACB}}\sin(10^\circ) + F_{\mathrm{ACB}}\sin(30^\circ) + F_{\mathrm{CD}}\sin(30^\circ) - 100 \times 9.80665 = 0. \tag{10}$$

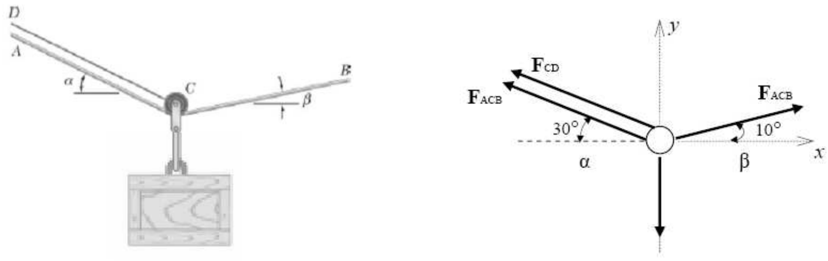
The representation convention gives the interval equilibrium equations

$$[F_{\mathrm{ACB}}]\cos([\beta]) - \mathrm{Dual}(\mathbf{F}_{\mathrm{ACB}}\cos([\alpha])) - \mathrm{Dual}(\mathbf{F}_{\mathrm{CD}}\cos([\alpha])) = 0, \tag{11}$$

$$[F_{\mathrm{ACB}}]\sin([\beta]) + [F_{\mathrm{ACB}}]\sin([\alpha]) + [F_{\mathrm{CD}}]\sin([\alpha]) - \mathrm{Dual}([99, 100] \times 9.80665) = 0, \tag{12}$$

wherein $[\alpha] = [29, 31]^\circ$, $[\beta] = [9, 11]^\circ$. We search for proper intervals $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$, that satisfy (11)–(12). First, we check the validity of the distributive relations for the first two additive terms in equations (11), (12). Since

[4] For $A(p)x = b(p)$, $p \in \mathbf{p}$, the united solution set is $\Sigma = \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\}$.



a) b)

Fig. 2 a) A crate suspended from a pulley can roll freely on the support cable ACB and is pulled at a constant speed by cable CD; b) Free-body diagram

$$[s_1] = \cos([\beta]) - \text{Dual}(\cos([\alpha])) \in [0.121107, 0.116479] > 0,$$
$$[s_2] = \sin([\beta]) + \sin([\alpha]) \in [0.667387, 0.679895] > 0,$$

by (5), the system (11)–(12) is equivalent to the system

$$[F_{\text{ACB}}][s_1] - \text{Dual}(\mathbf{F}_{\text{CD}} \cos([\alpha])) = 0,$$
$$[F_{\text{ACB}}][s_2] + [F_{\text{CD}}] \sin([\alpha]) - \text{Dual}([99, 100] \times 9.80665) = 0.$$

Remark 1. It is important that we check the distributive relations for every expression where we want to take a common interval variable out of brackets. For example, due to (5), and because $\cos([\alpha]) - \text{Dual}(\cos([\beta])) < 0$, the expression $[F_{\text{ACB}}]\cos([\alpha]) - \text{Dual}([F_{\text{ACB}}]\cos([\beta]))$ is equivalent to

$$\text{Dual}([F_{\text{ACB}}])\,(\cos([\alpha]) - \text{Dual}(\cos([\beta])))\,.$$

We add $[F_{\text{CD}}]\cos([\alpha])$ to the two sides of equation (11) and by (4) obtain the equivalent equation

$$[F_{\text{ACB}}][s_1] = [F_{\text{CD}}]\cos([\alpha]).$$

Dividing both sides of the last equation by $\text{Dual}(\cos([\alpha]))$, and due to (4), we obtain

$$[F_{\text{CD}}] = [F_{\text{ACB}}][s_1]/\text{Dual}(\cos([\alpha])). \tag{13}$$

We substitute the expression for $[F_{\text{CD}}]$ in equation (12). Since

$$[s_3] = [s_2] + \sin([\alpha])[s_1]/\text{Dual}(\cos([\alpha])) \;\in\; [5.25337, 5.57483] > 0,$$

due to the distributive relation, equation (12) is equivalent to

$$[F_{\text{ACB}}][s_3] - \text{Dual}([99, 100] \times 9.80665) = 0,$$

which is equivalent to

$$[F_{\mathrm{ACB}}] = [99,100] \times 9.80665/\mathrm{Dual}([s_3]) \ \in \ [1317.51, 1324.97]. \qquad (14)$$

Substituting (14) in (13), we obtain the second component of the algebraic solution to interval system (11)–(12)

$$[F_{\mathrm{CD}}] \in [184.806, 177.669].$$

Substituting $[F_{\mathrm{ACB}}]$ and $[F_{\mathrm{CD}}]$ into left sides of the equations (11) and (12), we obtain respectively $[-2.27 \times 10^{-13}, 2.27 \times 10^{-13}]$ and $[-5.68 \times 10^{-13}, 4.54 \times 10^{-13}]$. These intervals are almost but not exactly zero due to the round-off errors. We have to interpret $[F_{\mathrm{ACB}}]$ and $[F_{\mathrm{CD}}]$ in $\mathbb{IR}$ as the corresponding proper intervals, namely,

$$\begin{aligned} \mathbf{F}_{\mathrm{ACB}} &= \mathrm{Abs}(\mathrm{pro}([F_{\mathrm{ACB}}])) \in [1317.51, 1324.97]\,\mathrm{N}, \\ \mathbf{F}_{\mathrm{CD}} &= \mathrm{Abs}(\mathrm{pro}([F_{\mathrm{CD}}])) \in [177.669, 184.806]\,\mathrm{N}. \end{aligned}$$

However, $[F_{\mathrm{CD}}]$ is an improper interval. Therefore, substituting $\mathbf{F}_{\mathrm{ACB}}$ and $\mathbf{F}_{\mathrm{CD}}$ into left sides of the equations (11) and (12), we obtain much wider intervals involving zero, namely, $[6.16307, -6.20045]$ and $[-3.53667, 3.60141]$, respectively. The relative diameters of $\mathbf{F}_{\mathrm{ACB}}$ and $\mathbf{F}_{\mathrm{CD}}$ are 0.00565 and 0.0402, respectively.

Remark 2. Proper algebraic solution to the system (11)–(12) can be obtained if, for example, we squeeze the interval $[\alpha]$ to the interval $[30-0.1, 30+0.1]$.

Now, we compare the solution $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$, obtained by the discussed algebraic approach, to the solution obtained by classical interval arithmetic. The equations (9)–(10) are rearranged to

$$\begin{aligned} &F_{\mathrm{ACB}}\,(\cos(10^\circ) - \cos(30^\circ)) - F_{\mathrm{CD}} \cos(30^\circ) = 0, \\ &F_{\mathrm{ACB}}\,(\sin(10^\circ) + \sin(30^\circ)) + F_{\mathrm{CD}} \sin(30^\circ) = 100 \times 9.80665 \end{aligned}$$

and the corresponding interval linear system that has to be solved is

$$\begin{pmatrix} \cos([\beta]) - \cos([\alpha]), & \cos([\alpha]) \\ \sin([\beta]) + \sin([\alpha]), & \sin([\alpha]) \end{pmatrix} \begin{pmatrix} F_{\mathrm{ACB}} \\ F_{\mathrm{CD}} \end{pmatrix} = \begin{pmatrix} 0 \\ [99,100] \times 9.80665 \end{pmatrix}.$$

Since some interval parameters, e.g., $[\alpha]$, $[\beta]$, appear in more than one element of the matrix and/or the right-hand side vector, this is a parametric interval linear system. In classical interval arithmetic we search for a minimal outer interval estimation of the so-called united parametric solution set to the system. It can be proven, by method discussed in [12], that the united parametric solution set of the above system depends linearly on the interval parameters involved there. Therefore, one can find the minimal interval vector containing the united parametric solution set by finding the interval hull of the set of solutions to the point linear systems of equations obtained for the parameters taking values at all combinations of the corresponding interval end-points, the so-called combinatorial approach. Applying this approach, we found $\tilde{\mathbf{F}}_{\mathrm{ACB}} = [1293.33, 1349.74]$, $\tilde{\mathbf{F}}_{\mathrm{CD}} = [175.743, 186.773]$, whose relative diameters are respectively 0.04361 and 0.06276. Replacing $\tilde{\mathbf{F}}_{\mathrm{ACB}}$, $\tilde{\mathbf{F}}_{\mathrm{CD}}$ in the left-hand sides of the generalized interval equilibrium equations (11)–(12), we obtain much wider intervals involving zero $[4.89652, -5.02244]$, $[-20.6303, 21.4392]$. There is no inclusion relation between $\mathbf{F}_{\mathrm{ACB}}$, $\mathbf{F}_{\mathrm{CD}}$ and $\tilde{\mathbf{F}}_{\mathrm{ACB}}$, $\tilde{\mathbf{F}}_{\mathrm{CD}}$. Nevertheless, judging from the value of the relative diameters and the extent to which the interval equilibrium equations are satisfied, we conclude that the interval algebraic approach applied to the equilibrium equations provides sharper interval estimations than the traditional approach based on classical interval arithmetic.

In some deterministic models, e.g., when determine the forces in the members of a truss, in order to write the equilibrium equations one has to choose the direction of each of the unknown forces, cf. [1, Chapter 6]. It cannot be determined until the solution is completed whether the guess was correct. To do that, the value found for each of the unknowns is considered: a positive sign means that the selected direction was correct; a negative sign means that the direction is opposite to the assumed direction. This convention is transparently applicable to the corresponding interval algebraic model which delivers the correct sign together with the interval magnitude.

5 Conclusion

The engineering demand for more accurate models involving interval uncertainties, that conform to the physics of the modeled problem, lead to a new interval algebraic model of equilibrium equations in mechanics. The latter is based on the algebraic completion $(\mathbb{KR}, +, \times, \subseteq)$ of classical interval arithmetic. By a simple representation convention one can easily transform a deterministic formulation into a unique interval arithmetic formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$. Then in the same rich algebraic space one finds a sharp algebraic solution for the unknown quantities and interpret them in the original physical setting of the problem. If the algebraic solution is a proper interval (vector), it is assured that the equilibrium equations are completely satisfied and the obtained interval enclosures are the sharpest ones. It is demonstrated at the end of Example 2 that if (part of) the algebraic solution is not proper interval vector, its proper projection (3) provides narrower interval estimation for the unknowns than the best solution enclosure (the exact interval hull of the united parametric solution set) in classical interval arithmetic. Contrary to classical interval approach, the algebraic one provides satisfaction of the linear equilibrium equations even for very large parameter uncertainties. Therefore, for large uncertainties the algebraic approach is essential in obtaining sharp interval estimates.

If the deterministic model involves more unknowns that the number of equilibrium equations, other relations are obtained from the information contained in the statement of the problem. In this case a hybrid approach is necessary which will be considered in a forthcoming paper [13].

The most attractive in the interval algebraic approach to linear equilibrium equations in mechanics is its transparent application and full conformance to the deterministic model. Along with guaranteed quantification of all sources of uncertainties, the new algebraic approach provides also sharper enclosure of the unknown quantities than the best known methods based on classical interval arithmetic.

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