Parameterized Outer Estimation of AE-Solution Sets to Parametric Interval Linear Systems

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Abstract

We consider linear algebraic equations, where the elements of the matrix and of the right-hand side vector are linear functions of interval parameters, and their parametric AE-solution sets, which are defined by applying universal and existential quantifiers to the interval parameters. Usually, interval methods find numerical interval vector that contains an AE-solution set.

In this work we propose a method that generates an outer estimate of a parametric AE-solution set in form of a linear parametric interval function, called parameterized outer solution (p-solution). Parameterized outer solution is proposed for the parametric united solution set in [L. Kolev, *Appl. Math. Comput.* 246 (2014) 229–246] and takes precedence over the classical interval solution enclosure when the latter is part of other problems involving the same parameters. The method we present generalizes the method from [L. Kolev, *J. Appl. Computat. Math.* 5(1) (2016) 294] for parametric AE-solution sets. It is also a parameterized analogue of a method from [E. Popova, M. Hladík, *Soft Computing* 17 (2013) 1403–1414] and produces the same interval enclosure as the method from the last reference.

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1. Introduction

Denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real vectors with n components and the set of real $m\times n$ matrices, respectively. Vectors are considered as one-column matrices. A real compact interval is $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^n, \mathbb{IR}^{m\times n}$ we denote the sets of interval n-vectors and interval $m \times n$ matrices, respectively. We consider systems of linear algebraic equations having linear uncertainty structure

$$\begin{aligned} A(p)x &= a(p), \quad p \in \mathbf{p}, \\ A(p) &:= A_0 + \sum_{k=1}^{K} p_k A_k, \qquad a(p) := a_0 + \sum_{k=1}^{K} p_k a_k, \end{aligned} \tag{1}$$

where $A_k \in \mathbb{R}^{n\times n}$, $a_k \in \mathbb{R}^n$, $k = 0, \ldots, K$, and the parameters $p = (p_1, \ldots, p_K)^\top$ are considered to be uncertain and varying within given intervals $\mathbf{p} = (\mathbf{p}_1, \ldots, \mathbf{p}_K)^\top$.

We consider parametric AE-solution sets of the system (1), which are defined by

$$\begin{aligned} \Sigma^p_{AE} &= \Sigma_{AE}(A(p), b(p), \mathbf{p}) \\ &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p)x = b(p))\}\,, \end{aligned} \tag{2}$$

where $\mathcal{A}$ and $\mathcal{E}$ are sets of indexes such that $\mathcal{A} \cup \mathcal{E} = \{1, \ldots, K\}$, $\mathcal{A} \cap \mathcal{E} = \emptyset$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, p_Π denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$. Particular index sets $\mathcal{A}, \mathcal{E}$ are associated to each particular AE-solution set. Some of the most studied and with more practical interest AE-solution sets are: the (parametric) united solution set

$$\Sigma^p_{\text{uni}} = \Sigma_{\text{uni}}(A(p), b(p), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\exists p \in \mathbf{p})(A(p)x = b(p))\},$$

the (parametric) tolerable solution set

$$\Sigma_{AE}(A(p_{\mathcal{A}}), b(p_{\mathcal{E}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{A}})x = b(p_{\mathcal{E}}))\},$$

and the (parametric) controllable solution set

$$\Sigma_{AE}(A(p_{\mathcal{E}}), b(p_{\mathcal{A}}), \mathbf{p}) := \{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}})(\exists p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}})(A(p_{\mathcal{E}})x = b(p_{\mathcal{A}}))\}.$$

For interpretation and applications of AE-solution sets see, e.g., [1].

Usually, when finding outer interval enclosure of a solution set, interval methods generate a numerical interval vector that contains the particular AE-solution set. For inner estimation of parametric AE-solution sets see, e.g., [2]. A new type of enclosure, called *parameterized* or *p-solution*, providing outer estimate of the parametric united solution set is proposed in [3]. The proposed p-solution is in form of a linear parametric interval function

$$x(p, l) = Lp + l, \quad p \in \mathbf{p},\ l \in \mathbf{l},$$

where L is a real $n \times K$ matrix and $\mathbf{l}$ is an n-dimensional interval vector. The parameterized solution has the property

$$\Sigma^p_{uni} \subseteq x(\mathbf{p}, \mathbf{l}),$$

where $x(\mathbf{p}, \mathbf{l})$ is the interval hull of $x(p, l)$ over $p \in \mathbf{p}$, $l \in \mathbf{l}$. For a nonempty and bounded set $\Sigma \subset \mathbb{R}^n$, its interval hull is defined by

$$\square\Sigma := \bigcap\{\mathbf{x} \in \mathbb{IR}^n \mid \Sigma \subseteq \mathbf{x}\}.$$

Some iterative methods for determining $x(p, l)$ of the parametric united solution set are proposed in [3, 4]. In order to improve their computational efficiency, a direct method for determining the p-solution $x(p, l)$ is proposed in [5]. A numerical example demonstrates in [4] that the parameterized solution is rather promising in solving some global optimization problems where the parametric linear system (1) is involved as an equality constraint.

In this work we generalize the direct method, developed in [5] for the parametric united solution set, to arbitrary parametric AE-solution sets and propose a method that generates an outer solution enclosure in a form, which is not an interval vector but a linear parametric interval function $x(p, l)$. The structure of the paper is as follows. Section 2 contains notation and basic facts about the arithmetic on proper and improper intervals [6], which will be used to simplify the proof of Theorem 6 in section 3, as well as various known results that are used as a background for the derivation of the parameterized AE-solution. The parameterized AE-solution and its interval enclosure property are proven in section 3. A numerical algorithm implementing the parameterized AE-method is presented in section 4 along with some numerical examples. The paper ends by some conclusions.

2. Preliminaries

For $\mathbf{a} = [\underline{a}, \overline{a}]$, define the mid-point $\check{a} := (\underline{a}+\overline{a})/2$, the radius $\hat{a} := (\overline{a}-\underline{a})/2$ and the absolute value (magnitude) $|\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\}$. These functions are applied to interval vectors and matrices componentwise. Inequalities are understood componentwise. The spectral radius of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\varrho(A)$. The identity matrix of appropriate dimension is denoted by I.

In order to simplify the presentation, in Section 3 we use the arithmetic on proper and improper intervals [6], called Kaucher complete arithmetic, and its properties; see also [7]. The set of proper intervals $\mathbb{IR}$ is extended in [6] by the set $\overline{\mathbb{IR}} := \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}, a_1 \geq a_2\}$ of *improper* intervals obtaining thus the set $\mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a_1, a_2] \mid a_1, a_2 \in \mathbb{R}\}$ of all ordered couples of real numbers, called also generalized intervals. The conventional interval arithmetic operations, lattice operations intersection ($\cap$) and union ($\cup$), order relations, and other interval functions are isomorphically embedded into the whole set $\mathbb{IR} \bigcup \overline{\mathbb{IR}}$ [6].

An element-to-element symmetry between proper and improper intervals is expressed by the "dual" operator, $\mathrm{dual}(\mathbf{a}) := [a_2, a_1]$ for $\mathbf{a} = [a_1, a_2] \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$. The operator `dual` is applied componentwise to vectors and matrices. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$

$$\begin{aligned} \mathrm{dual}(\mathrm{dual}(\mathbf{a})) = \mathbf{a}, \qquad & \mathrm{dual}(\mathbf{a} \circ \mathbf{b}) = \mathrm{dual}(\mathbf{a}) \circ \mathrm{dual}(\mathbf{b}),\ \circ \in \{+, -, \times, /\}, \\ \mathbf{a} \subseteq \mathbf{b} \ \Leftrightarrow \ & \mathrm{dual}(\mathbf{a}) \supseteq \mathrm{dual}(\mathbf{b}). \end{aligned} \tag{3}$$

The generalized interval arithmetic structure possesses group properties with respect to addition and multiplication operations. For $\mathbf{a}, \mathbf{b} \in \mathbb{IR} \bigcup \overline{\mathbb{IR}}$, $0 \notin \mathbf{b}$

$$\mathbf{a} - \mathrm{dual}(\mathbf{a}) = 0, \qquad \mathbf{b}/\mathrm{dual}(\mathbf{b}) = 1. \tag{4}$$

The tuple $\left(\mathbb{IR} \bigcup \overline{\mathbb{IR}}, \cap, \cup, \subseteq\right)$ is a conditionally complete lattice. Norm and metric in the generalized interval structure are very similar to norm and metric in linear spaces [6]. Mid-point, radius and absolute value are extended on generalized intervals by the same formulae.

Below we recall some known results that will be used in the derivation of the new parameterized outer AE-solution.

Definition 1. *A square interval matrix* $\mathbf{A}$ *is called* regular *if each matrix* $A \in \mathbf{A}$ *is nonsingular, and it is said to be* singular *otherwise (i.e., if it contains a singular matrix).*

Definition 2. *For a regular interval matrix* $\mathbf{A} \in \mathbb{IR}^{n\times n}$, *its* inverse interval matrix *is defined by*

$$\mathbf{A}^{-1} := \square \left\{ A^{-1} \mid A \in \mathbf{A} \right\}.$$

Theorem 1 ([8], Theorem 4.4). *Let* $\mathbf{A} = [I - \Delta, I + \Delta]$ *with* $\varrho(\Delta) < 1$. *Then the inverse interval matrix* $\mathbf{A}^{-1} = [\underline{H}, \overline{H}]$ *is given by*

$$\begin{aligned} \overline{H} = (\overline{h}_{ij}) \quad &= \quad (I - \Delta)^{-1}, \\ \underline{H} = (\underline{h}_{ij}), \qquad & \underline{h}_{ij} = \begin{cases} -\overline{h}_{ij} & \text{if } i \neq j \\ \frac{\overline{h}_{jj}}{2\overline{h}_{jj}-1} & \text{if } i = j. \end{cases} \end{aligned} \tag{5}$$

Next we recall from [5] how to obtain the parameterized solution for a parametric united solution set Σ^p_{uni}. In the presentation below some deficiencies of the presentation in [5] are corrected. Without loss of generality, [5] considers the following parametric interval linear system

$$\left(\tilde{A}_0 + \sum_{i=1}^{K} p_i \tilde{A}_i\right) x = \tilde{a}_0 + \sum_{i=1}^{K} p_i \tilde{a}_i, \qquad p_i \in [-1, 1], i = 1, \ldots, K, \tag{6}$$

where $\tilde{A}_0 = A_0 + \sum_{i=1}^{K} \check{p}_i A_i$, $\tilde{a}_0 = a_0 + \sum_{i=1}^{K} \check{p}_i a_i$, $\tilde{A}_i = \hat{p}_i A_i$, $\tilde{a}_i = \hat{p}_i a_i$, $i = 1, \ldots, K$, which has the same parametric united solution set as (1).

Theorem 2 ([5, Theorem 1]). *Let* $\tilde{A}_0$ *in (6) be nonsingular. Denote* $R = \tilde{A}_0^{-1}$, $\tilde{x} = \tilde{A}_0^{-1}\tilde{a}_0$, $F = (\tilde{a}_1 | \ldots | \tilde{a}_K)$, $G = (\tilde{A}_1\tilde{x} | \ldots | \tilde{A}_K\tilde{x})$, $B^0 = R(F - G)$ *and* $\Delta = \sum_{i=1}^{K} |R\tilde{A}_i|$. *Assume that* $\varrho(\Delta) < 1$ *is satisfied. Then*

(i) $\tilde{A}(p)$ *is regular for each* $p \in [-1, 1] \in \mathbb{IR}^K$;

(ii) the united p-solution $x(p, s)$ *of the system (6) exists and is determined by*

$$x(p, s) = \tilde{x} + Lp + s, \qquad p \in [-1, 1] \in \mathbb{IR}^K,\ s \in \mathbf{s} \in \mathbb{IR}^n,$$

where $L = \check{H}B^0$, $\mathbf{s} = [-\hat{s}, \hat{s}]$, $\hat{s} = \hat{H}|B^0|(1, \ldots, 1)^\top$, *and* $\check{H}$, $\hat{H}$ *are the midpoint and radius matrices, respectively, of the inverse interval matrix* $\mathbf{H} = [\underline{H}, \overline{H}]$ *obtained by Theorem 1 for the interval matrix* $[I - \Delta, I + \Delta]$.

It is proven in [5] that the interval hull of the parameterized solution $x(p, s)$ for $p \in [-1, 1]$, $s \in [-\hat{s}, \hat{s}]$ coincides with the interval vector obtained by the parametric single-step method from [9].

Theorem 3 ([10, Theorem 3.1]).

$$\Sigma_{AE}(A(p_{\mathcal{E}}, p_{\mathcal{A}}), a(p_{\mathcal{E}}, p_{\mathcal{A}}), \mathbf{p}) = \bigcap_{p_{\mathcal{A}} \in \mathbf{p}_{\mathcal{A}}} \bigcup_{p_{\mathcal{E}} \in \mathbf{p}_{\mathcal{E}}} \{x \in \mathbb{R}^n \mid A(p_{\mathcal{E}}, p_{\mathcal{A}})x = a(p_{\mathcal{E}}, p_{\mathcal{A}})\}.$$

Next we consider the parametric AE-solution sets as intersection of a finite number of parametric united solution sets, as it is done in [11, Section 3]. For a given index set $\mathcal{A}$, define

$$\mathcal{B}_{\mathcal{A}} := \{(\check{p}_{j_1} + \delta_{j_1}\hat{p}_{j_1}, \ldots, \check{p}_{j_{k_2}} + \delta_{j_{k_2}}\hat{p}_{j_{k_2}}) \mid \delta_{j_1}, \ldots, \delta_{j_{k_2}} \in \{-1, 1\}\}.$$

Theorem 4 ([11, Theorem 1]).

$$\Sigma^p_{AE} = \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}_{\mathcal{A}}} \Sigma(A(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), b(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), \mathbf{p}_{\mathcal{E}}).$$

Corollary 1 ([11, Corollary 1]). *For a bounded parametric AE-solution set $\Sigma^p_{AE} \neq \emptyset$ and a set $\mathcal{B}'_{\mathcal{A}}$, such that $\mathcal{B}'_{\mathcal{A}} \subseteq \mathcal{B}_{\mathcal{A}}$ and $\Sigma(A(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), b(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), \mathbf{p}_{\mathcal{E}})$ is bounded for all $\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}$, we have*

$$\square\Sigma^p_{AE} \subseteq \bigcap_{\tilde{p}_{\mathcal{A}} \in \mathcal{B}'_{\mathcal{A}}} \square\Sigma(A(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), b(p_{\mathcal{E}}, \tilde{p}_{\mathcal{A}}), \mathbf{p}_{\mathcal{E}}).$$

Corollary 1 is not suitable for parameterization since in this end-point approach each component and bound of Σ^p_{AE}, in general, may be described by a different interval linear parametric function. In section 3 below we will derive a parameterized version of the following interval AE-method.

Theorem 5 ([11, Theorem 4]). *Let $A(\check{p})$ in (1) be regular. Define*

$$C := A^{-1}(\check{p}), \qquad \tilde{x} := A^{-1}(\check{p})a(\check{p}), \qquad M := \sum_{i=1}^{K} |CA_i|\hat{p}_i.$$

If $\varrho(M) < 1$, then every $x \in \Sigma^p_{AE}$ satisfies the inequality

$$|x - \tilde{x}| \leq (I - M)^{-1} \left(\sum_{i \in \mathcal{E}} |C(A_i\tilde{x} - b_i)|\hat{p}_i - \sum_{j \in \mathcal{A}} |C(A_j\tilde{x} - b_j)|\hat{p}_j \right).$$

The numerical method based on Theorem 5 is known as AE parametric interval Bauer-Skeel method.

3. Parameterized estimate of Σ^{p}_{AE}

Lemma 1. *Let $A(p)x = a(p)$, $p \in \mathbf{p} \in \mathbb{IR}^K$, and $A(\check{p})$ be nonsingular. Then, with the notation $R = A^{-1}(\check{p})$, $\tilde{x} = Ra(\check{p})$,*

$$\Sigma_{uni}(A(p), a(p), \mathbf{p}) = \tilde{x} + \Sigma_{uni}(RA(p), R(a(p) - A(p)\tilde{x}), \mathbf{p}).$$

Proof. The proof is a part of the proof of [9, Theorem 3.2]; see also [5]. $\square$

Theorem 6. *Consider the system (1). Let $A(\check{p})$ be nonsingular. Denote*

$$R = A^{-1}(\check{p}), \qquad \tilde{x} = A^{-1}(\check{p})a(\check{p}), \quad \Delta = \sum_{i=1}^{K} |RA_i|\hat{p}_i,$$
$$F_{\mathcal{E}} = (a_{i_1}| \dots |a_{i_{k_1}}) \in \mathbb{R}^{n\times k_1}, \qquad G_{\mathcal{E}} = (A_{i_1}\tilde{x}| \dots |A_{i_{k_1}}\tilde{x}) \in \mathbb{R}^{n\times k_1},$$
$$F_{\mathcal{A}} = (a_{j_1}| \dots |a_{j_{k_2}}) \in \mathbb{R}^{n\times k_2}, \qquad G_{\mathcal{A}} = (A_{j_1}\tilde{x}| \dots |A_{j_{k_2}}\tilde{x}) \in \mathbb{R}^{n\times k_2}.$$

Assume that $\Sigma^p_{AE} \neq \emptyset$. Define

$$\hat{s} := |R(F_{\mathcal{E}} - G_{\mathcal{E}})|\,\hat{p}_{\mathcal{E}} - |R(F_{\mathcal{A}} - G_{\mathcal{A}})|\,\hat{p}_{\mathcal{A}}. \tag{7}$$

If $\varrho(\Delta) < 1$, then

(i) for each $p \in \mathbf{p}$ the matrix $A(p)$ is regular;

(ii) $\Sigma_{AE}(A(p_{\mathcal{E}}, p_{\mathcal{A}}), a(p_{\mathcal{E}}, p_{\mathcal{A}}), \mathbf{p}) \subseteq x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}])$
$\subseteq \square x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \tilde{x} + \overline{H}[-\hat{s}, \hat{s}],$
where

$$x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) := \\ \Big\{x \in \mathbb{R}^n \mid (\forall p_{\mathcal{A}} \in [-\hat{p}_{\mathcal{A}}, \hat{p}_{\mathcal{A}}])(\exists p_{\mathcal{E}} \in [-\hat{p}_{\mathcal{E}}, \hat{p}_{\mathcal{E}}], \exists s \in [-\hat{s}, \hat{s}]) \\ (x = \tilde{x} + \check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})p_{\mathcal{E}} + \check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})p_{\mathcal{A}} + \hat{H}s)\Big\}$$

and $[\underline{H}, \overline{H}] = [\check{H} - \hat{H}, \check{H} + \hat{H}]$ is the inverse interval matrix obtained by Theorem 1 for the interval matrix $[I - \Delta, I + \Delta]$.

Proof. The system (1) is equivalent to the system

$$\left(A(\check{p}) + \sum_{i=1}^{K} q_i A_i\right) x \;=\; a(\check{p}) + F_{\mathcal{E}}q_{\mathcal{E}} + F_{\mathcal{A}}q_{\mathcal{A}}, \quad q \in [-\hat{p}, \hat{p}], \tag{8}$$

with the same index sets $\mathcal{E}$, $\mathcal{A}$. Thus,

$$\Sigma_{AE}(A(p_{\mathcal{E}}, p_{\mathcal{A}}), a(p_{\mathcal{E}}, p_{\mathcal{A}}), \mathbf{p}) = \Sigma_{AE}(B(q_{\mathcal{E}}, q_{\mathcal{A}}), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}),$$

where $B(q_{\mathcal{E}}, q_{\mathcal{A}}) = A(\check{p}) + \sum_{i=1}^{K} q_i A_i$,

$$b(q_{\mathcal{E}}, q_{\mathcal{A}}) \;=\; a(\check{p}) + F_{\mathcal{E}} q_{\mathcal{E}} + F_{\mathcal{A}} q_{\mathcal{A}}, \quad \mathbf{q} = [-\hat{p}, \hat{p}].$$

It follows from Theorem 3 that if, in the parametric matrix, we replace the universally quantified parameters by existentially quantified ones varying within the same intervals, the resulted parametric AE-solution set will contain the parametric AE-solution set of the initial system. That is,

$$\Sigma_{AE}(B(q_{\mathcal{E}}, q_{\mathcal{A}}), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}) \subseteq \Sigma_{AE}(B(q_{\mathcal{E}}, t_{\mathcal{E}}), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}, \mathbf{t}_{\mathcal{E}}),\ \mathbf{t}_{\mathcal{E}} = \mathbf{q}_{\mathcal{A}}.$$

With $\mathbf{t}_{\mathcal{E}} = \mathbf{q}_{\mathcal{A}}$ denote

$$\begin{aligned}\Sigma_{AE}(B(q), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}) \;&=\; \Sigma_{AE}(B(q_{\mathcal{E}}, t_{\mathcal{E}}), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}, \mathbf{t}_{\mathcal{E}}) \\ &:=\; \{x \in \mathbb{R}^n \mid (\forall q_{\mathcal{A}} \in \mathbf{q}_{\mathcal{A}})(\exists q \in \mathbf{q})(B(q)x = b(q_{\mathcal{E}}, q_{\mathcal{A}}))\}.\end{aligned}$$

Due to Lemma 1, for every $\tilde{q}_{\mathcal{A}} \in \mathbf{q}_{\mathcal{A}}$

$$\Sigma_{uni}(B(q), b(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}) = \tilde{x} + \Sigma_{uni}(C(q), c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}), \tag{9}$$

where

$$C(q) = I + \sum_{i=1}^{K} q_i(RA_i), \quad c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}) = R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} + R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}}.$$

Consider $\Sigma_{uni}(C(q), c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q})$. According to [12, Theorem 5.6],

$$\Sigma_{uni}(C(q), c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}) \subseteq \Sigma_{uni}(\mathbf{C}, c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}), \tag{10}$$

with

$$\begin{aligned}\mathbf{C} \;&:=\; \square\{C(q) \mid q \in \mathbf{q}\} = I + \sum_{i=1}^{K} \mathbf{q}_i(RA_i) \\ &=\; I + [-\Delta, \Delta], \qquad \Delta = \sum_{i=1}^{K} \hat{p}_i |RA_i|.\end{aligned}$$

If $\varrho(\Delta) < 1$, by [8, Proposition 4.1], the interval matrix $\mathbf{C}$ is regular. According to [13, Theorem 1], regularity of $\mathbf{C}$ is equivalent to strong regularity of $A(p)$ on $\mathbf{p}$ and the latter is a sufficient condition for regularity of $A(p)$ on $\mathbf{p}$, which proves item (i) of the theorem.

Denote by $\mathbf{C}^{-1} = [\underline{H}, \overline{H}] = \check{H} + [-1,1]\hat{H}$ the inverse interval matrix obtained by Theorem 1, and $\check{H} = (\underline{H} + \overline{H})/2$, $\hat{H} = (\overline{H} - \underline{H})/2$. Thus,

$$\Sigma_{uni}(\mathbf{C}, c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}_{\mathcal{E}}) \subseteq \Sigma_{uni}(I, \mathbf{C}^{-1}R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} + \mathbf{C}^{-1}R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}}, \mathbf{q}_{\mathcal{E}}). \quad (11)$$

From (9), inclusions (10), (11) and Theorem 4, we obtain

$$\Sigma_{AE}(B(q), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}) \subseteq \tilde{x} + \bigcap_{\tilde{q}_{\mathcal{A}} \in \{\pm\hat{p}_{\mathcal{A}}\}} \Sigma_{uni}(I, \mathbf{C}^{-1}c(q_{\mathcal{E}}, \tilde{q}_{\mathcal{A}}), \mathbf{q}_{\mathcal{E}}) \subseteq$$
$$\tilde{x} + \bigcap_{\tilde{q}_{\mathcal{A}} \in \{\pm\hat{p}_{\mathcal{A}}\}} \Sigma_{uni}\left(I, \mathbf{C}^{-1}R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} + \mathbf{C}^{-1}R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}}, \mathbf{q}_{\mathcal{E}}\right), \quad (12)$$

where $\{\pm\hat{p}_{\mathcal{A}}\} = \{(\delta_1\hat{p}_{j_1}, \ldots, \delta_{k_2}\hat{p}_{j_{k_2}}) \mid \delta_1, \ldots, \delta_{k_2} \in \{-1,1\}\}$. Substituting $\mathbf{C}^{-1} = \check{H} + [-1,1]\hat{H}$ in (12), we obtain

$$\mathbf{C}^{-1}R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} \subseteq \check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} + \hat{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})\mathbf{q}_{\mathcal{E}}, \quad q_{\mathcal{E}} \in \mathbf{q}_{\mathcal{E}}$$

and

$$\bigcap_{\tilde{q}_{\mathcal{A}} \in \{\pm\hat{p}_{\mathcal{A}}\}} \mathbf{C}^{-1}R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}} = \bigcap_{\tilde{q}_{\mathcal{A}} \in \{\pm\hat{p}_{\mathcal{A}}\}} \check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}} + \hat{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})\mathrm{dual}(\mathbf{q}_{\mathcal{A}}). \quad (13)$$

Consider the following expression in Kaucher interval arithmetic

$$\begin{aligned}&\hat{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})\mathbf{q}_{\mathcal{E}} + \hat{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})\mathrm{dual}(\mathbf{q}_{\mathcal{A}}) = \\ &\hat{H}\left([-|R(F_{\mathcal{E}} - G_{\mathcal{E}})|\hat{p}_{\mathcal{E}}, |R(F_{\mathcal{E}} - G_{\mathcal{E}})|\hat{p}_{\mathcal{E}}] + [|R(F_{\mathcal{A}} - G_{\mathcal{A}})|\hat{p}_{\mathcal{A}}, -|R(F_{\mathcal{A}} - G_{\mathcal{A}})|\hat{p}_{\mathcal{A}}]\right) \\ &= \hat{H}[-\hat{s}, \hat{s}]\end{aligned}$$

with $\hat{s}$ defined in (7). Since $\Sigma^p_{AE} \neq \emptyset$, according to [10, Theorem 3.2], $\hat{s} \geq 0$ is a necessary condition for $\Sigma^p_{AE} \neq \emptyset$. This provides the existence of the interval $[-\hat{s}, \hat{s}] \in \mathbb{IR}^n$. Thus, from (12) and the considerations following it, we obtain

$$\Sigma_{AE}(A(p), a(p), \mathbf{p}) = \Sigma_{AE}(B(q), b(q_{\mathcal{E}}, q_{\mathcal{A}}), \mathbf{q}) \subseteq$$
$$\tilde{x} + \bigcap_{\tilde{q}_{\mathcal{A}} \in \{\pm \hat{p}_{\mathcal{A}}\}} \Sigma_{uni}\left(I, \check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})q_{\mathcal{E}} + \check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})\tilde{q}_{\mathcal{A}} + \hat{H}s, \mathbf{q}_{\mathcal{E}}, [-\hat{s}, \hat{s}]\right).$$

Applying again Theorem 4 we get the first inclusion in (ii). The second inclusion in (ii) is obvious. The relation

$$\Box x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \tilde{x} + \left(\check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})\right)[-\hat{p}_{\mathcal{E}}, \hat{p}_{\mathcal{E}}] + \left(\check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})\right) \mathrm{dual}([-\hat{p}_{\mathcal{A}}, \hat{p}_{\mathcal{A}}]) + \hat{H}[-\hat{s}, \hat{s}] \quad (14)$$

holds true because each component of $x(p, s)$ is a multi-linear function of $p_{\mathcal{E}}, p_{\mathcal{A}}$, s, which appear only once, $p_{\mathcal{E}} \in [-\hat{p}_{\mathcal{E}}, \hat{p}_{\mathcal{E}}]$, $p_{\mathcal{A}} \in [-\hat{p}_{\mathcal{A}}, \hat{p}_{\mathcal{A}}]$, $s \in [-\hat{s}, \hat{s}]$, and due to the relations (see, e.g., [1])

$$(\forall b \in \mathbf{b})(c \geq b) \iff c \geq \overline{b},$$
$$(\forall b \in \mathbf{b})(c \leq b) \iff c \leq \underline{b}.$$

The expression in the right-hand side of (14) should be considered in Kaucher interval arithmetic. Its evaluation gives

$$\Box x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \tilde{x} + \left[-|\check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})|\hat{p}_{\mathcal{E}}, |\check{H}R(F_{\mathcal{E}} - G_{\mathcal{E}})|\hat{p}_{\mathcal{E}}\right] + \left[|\check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})|\hat{p}_{\mathcal{A}}, -|\check{H}R(F_{\mathcal{A}} - G_{\mathcal{A}})|\hat{p}_{\mathcal{A}}\right] + \hat{H}[-s, s].$$

Since Theorem 1 implies that $\check{H}$ is a diagonal matrix with positive diagonal entries ($\check{H}_{ii} > 0$), the last expression is equivalent to

$$\Box x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \tilde{x} + \check{H}[-\hat{s}, \hat{s}] + \hat{H}[-\hat{s}, \hat{s}] = \tilde{x} + (\check{H} + \hat{H})[-\hat{s}, \hat{s}] = \tilde{x} + \overline{H}[-\hat{s}, \hat{s}].$$

Since $\Sigma_{AE}^p \neq \emptyset$ implies by [10, Theorem 3.2] that $\hat{s} \geq 0$, the last expressions are in conventional interval arithmetic. $\Box$

Theorem 6 generalizes the parameterized approach, developed in [5] for the parametric united solution set, to arbitrary parametric AE-solution sets.

On the other hand, we prove below that Theorem 6 defines a parameterized version of the interval parametric AE Bauer-Skeel method of [11, Theorem 4].

Let the interval enclosure of a Σ^p_{AE} to the system (1) obtained by the parametric Bauer-Skeel method of Theorem 5, [11, Theorem 4], be

$$\begin{aligned}
&\mathbf{u} = \tilde{x} + [-\hat{u}, \hat{u}], \\
&\hat{u} = (I - M)^{-1} \left(\sum_{i \in \mathcal{E}} |C(A_i \tilde{x} - b_i)| \hat{p}_i - \sum_{j \in \mathcal{A}} |C(A_j \tilde{x} - b_j)| \hat{p}_j \right).
\end{aligned} \tag{15}$$

Proposition 1. *The enclosure of a* Σ^p_{AE} *to the system (1), obtained by the parameterized method of Theorem 6, has the same width as the interval enclosure obtained by the method of Theorem 5 ([11, Theorem 4]), that is*

$$\Box x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \mathbf{u}.$$

Proof. Due to the difference in the notations in Theorem 6 and Theorem 3, respectively (15), we have $R = M$, $\Delta = M$ and

$$\begin{aligned}
|R(F_{\mathcal{E}} - G_{\mathcal{E}})| \hat{p}_{\mathcal{E}} &= \sum_{i \in \mathcal{E}} |C(A_i \tilde{x} - b_i)| \hat{p}_i \\
|R(F_{\mathcal{A}} - G_{\mathcal{A}})| \hat{p}_{\mathcal{A}} &= \sum_{j \in \mathcal{A}} |C(A_j \tilde{x} - b_j)| \hat{p}_j.
\end{aligned}$$

Then, the relation $\Box x(p, s, [-\hat{p}, \hat{p}], [-\hat{s}, \hat{s}]) = \tilde{x} + \overline{H}[-\hat{s}, \hat{s}]$, proven at the end of the proof of Theorem 6, and $\overline{H} = (I - M)^{-1}$ finish the proof. $\Box$

In view of the above proposition, Theorem 6 provides another derivation of the interval Bauer-Skeel method [11, Theorem 4] for parametric AE-solution sets.

In the special case when $\mathcal{A} = \emptyset$, Theorem 6 is equivalent to [5, Theorem 1]. Theorem 6 does not require an initial equivalent transformation from the general form of a parametric linear system (1) into the system (6) as [5, Theorem 1] does. Theorem 6 also corrects some notations and other deficiencies in [5].

Remark 1. *In [5], after Corollary 1 therein, Kolev writes "In proving Corollary 1, we have also shown that the assumption, adopted in [9], namely* $\mathbf{C} = [I - \Delta, I + \Delta]$ *to be an H-matrix, is superfluous.". This claim is wrong since, according to [13, Theorem 1], the three conditions: strong regularity of* $A(p)$ *in* $\mathbf{p}$, $\varrho(\Delta) < 1$ *and* $\mathbf{C}$ *being an H-matrix, are equivalent.*

4. Algorithm and examples

Here we present an algorithm implementing Theorem 6 and some numerical examples illustrating its properties.

Algorithm 1. *(Algorithm implementing Theorem 6.) Using the notation of Theorem 6 we have*

1. *If* $A^{-1}(\check{p})$ *is invertible, then compute* $R = A^{-1}(\check{p})$ *and go to 2,*

   else return the message: "The parametric matrix is not regular."; Quit.

2. *Compute* $\tilde{x} = Ra(\check{p})$; $\quad B_{\mathcal{E}} = R(F_{\mathcal{E}} - G_{\mathcal{E}})$; $\quad B_{\mathcal{A}} = R(F_{\mathcal{A}} - G_{\mathcal{A}})$;
   $\hat{s} = |B_{\mathcal{E}}|\hat{p}_{\mathcal{E}} - |B_{\mathcal{A}}|\hat{p}_{\mathcal{A}}$;

3. *If* $\hat{s}_i < 0$ *for some* $i = 1, \ldots, n$, *then return the message: "*Σ^p_{AE} *is empty"; Quit.*

4. $\Delta = \sum_{i=1}^{K} |RA_i|\hat{p}_i$; $\overline{H} = (I - \Delta)^{-1}$;

5. *If* $\overline{H}_{ij} < 0$ *for some* $i, j = 1, \ldots, n$, *then return the message: "*$A(p)$ *is not strongly regular."; Quit.*

6. *Output:*

   6.1. *either return:* $\tilde{x} + \overline{H}[-\hat{s}, \hat{s}]$

   6.2. *or compute* $\underline{H}$ *by (5);* $\check{H} = (\underline{H} + \overline{H})/2$; $\hat{H} = (\overline{H} - \underline{H})/2$;
   return: $\tilde{x}$, $\check{H}B_{\mathcal{E}}$, $\check{H}B_{\mathcal{E}}$, $\hat{H}$, $\hat{s}$.

Next examples illustrate the application of Theorem 6 and the above algorithm.

Example 1. *Find the p-solution for the parametric controllable solution set of the system with*

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad a(p_3) = \begin{pmatrix} 2p_3 \\ 2p_3 \end{pmatrix}, \quad p_1 \in [0, \frac{1}{2}], p_2 \in [1, \frac{3}{2}], p_3 \in [1, \frac{3}{2}].$$

Following the above algorithm we obtain (in rational arithmetic in order to avoid any confusion related to round-off errors)

$$A(\check{p}) = \begin{pmatrix} \frac{1}{4} & -\frac{5}{4} \\ \frac{5}{4} & \frac{1}{4} \end{pmatrix}, \qquad \tilde{x} = \left(\frac{30}{13}, -\frac{20}{13}\right)^\top$$

$$B_{\mathcal{E}} = \frac{1}{169}\begin{pmatrix} 140 & -340 \\ 340 & 140 \end{pmatrix}, \qquad B_{\mathcal{A}} = \frac{1}{13}\begin{pmatrix} 24 \\ -16 \end{pmatrix}, \quad \hat{s} = \frac{1}{169}\begin{pmatrix} 42 \\ 68 \end{pmatrix}.$$

Then, with $\Delta = \frac{1}{13}\begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$, *we obtain* $\overline{H} = \frac{1}{7}\begin{pmatrix} 10 & 3 \\ 3 & 10 \end{pmatrix}$, $\underline{H} = \begin{pmatrix} \frac{10}{13} & -\frac{3}{7} \\ -\frac{3}{7} & \frac{10}{13} \end{pmatrix}$ *and the controllable parameterized outer solution is the set*

$$x(p,s) = \left\{ (\forall p_3 \in [-\frac{1}{4}, \frac{1}{4}]), \left(\exists p_1, p_2 \in [-\frac{1}{4}, \frac{1}{4}], \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \in \begin{pmatrix} [-\frac{42}{169}, \frac{42}{169}] \\ [-\frac{68}{169}, \frac{68}{169}] \end{pmatrix}\right), \right.$$
$$\left. \begin{pmatrix} \frac{30}{13} \\ -\frac{20}{13} \end{pmatrix} + \begin{pmatrix} \frac{2000}{2197} & -\frac{34000}{15379} \\ \frac{34000}{15379} & \frac{2000}{2197} \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} \frac{2400}{1183} \\ -\frac{1600}{1183} \end{pmatrix} p_3 + \begin{pmatrix} \frac{30}{91} & \frac{3}{7} \\ \frac{3}{7} & \frac{30}{91} \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \end{pmatrix} \right\},$$

which contains the parametric controllable solution set of the initial parametric interval linear system. The interval enclosure of the parameterized controllable solution is

$$\square x(p,s) = \tilde{x} + \hat{H}[-\hat{s}, \hat{s}] = \begin{pmatrix} [\frac{162}{91}, \frac{258}{91}] \\ [-\frac{202}{91}, -\frac{6}{7}] \end{pmatrix}$$

and coincides with the interval enclosure of the parametric controllable solution set obtained by the AE-parametric Bauer-Skeel method from [11, Theorem 4].

Both methods: the interval Bauer-Skeel method from [11, Theorem 4] and its parameterized analogue form Theorem 6 have the same properties regarding the interval enclosure of parametric AE-solution sets. Most of these properties are discussed in [11].

Example 2. *Consider the parametric system from Example 1 with other intervals for the parameters* $p_1 \in \left[\frac{1}{2}, \frac{3}{2}\right]$, $p_2 \in [0,1]$, $p_3 \in [1,2]$. *The parametric matrix is regular but not strongly regular in the parameter domain.*

Therefore, the algorithm of Theorem 6 ends in step five due to

$$\overline{H} = \begin{pmatrix} -2 & -3 \\ -3 & -2 \end{pmatrix}.$$

Example 3. *We look for the parameterized controllable solution of the system from Example 1, where* $p_3 \in \left[1, \frac{5}{2}\right]$*. As seen in Example 1, the parametric matrix is strongly regular in the domain for* p_1, p_2*. However, the algorithm implementing Theorem 6 ends in step three since* $\hat{s} = \left(-\frac{66}{169}, \frac{12}{169}\right)^\top$.

Thus, by Theorem 6, we can find not only a parameterized AE-solution and its interval enclosure, but we also can sometimes prove that the parametric AE-solution set is empty.

5. Conclusion

A direct method generating parameterized outer estimate of the parametric united solution set is generalized in Theorem 6 for any parametric AE-solution set. The parameterized solution is a family of parametric interval linear functions depending on the initial K interval parameters and additional n interval parameters.

The parameterized AE-solution is a parameterized analogue of the interval AE-parametric Bauer-Skeel method from [11, Theorem 4]. The interval enclosure provided by the parameterized AE-solution coincides (up to the round-off errors) with the interval enclosure provided by the AE-parametric Bauer-Skeel method. Thus,

(i) the proof of Theorem 6 presents another derivation of the interval AE-parametric Bauer-Skeel method;

(ii) solution enclosure properties of the parameterized AE-solution are the same as the enclosure properties of the interval AE-parametric Bauer-Skeel method.

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