Equilibrium equations in interval models of structures

Evgenija D. Popova

Institute of Mathematics and Informatics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Block 8,
1113 Sofia, Bulgaria
Email: epopova@math.bas.bg

Abstract: A new interval model of linear equilibrium equations in mechanics is based on the algebraic completion of classical interval arithmetic and provides more realistic and accurate models involving interval uncertainties. It replaces straightforwardly a linear deterministic model by an interval model in terms of proper and improper intervals, fully conforms to the equilibrium principle, and provides sharper enclosure of a resultant force than the methods based on classical interval arithmetic. The present paper presents the interval algebraic approach to linear equilibrium equations and discusses its applications to some interval models of structures.

Keywords: static equilibrium; structures; interval arithmetic; proper and improper intervals; overestimation.

Biographical notes: Evgenija D. Popova is working as a Professor in Mathematical Modelling and Application of Mathematics at the Institute of Mathematics and Informatics (IMI) of the Bulgarian Academy of Sciences (BAS), Sofia, Bulgaria. She received her PhD degree in Informatics at IMI-BAS. She has published over 100 research articles and has internationally recognised results in interval algebra, numerical interval methods and software tools with result verification, applications in engineering and biology. She was leading researcher in 16 national and 15 international research projects including a NATO linkage grant, projects funded by DFG, Swiss NSF, TUBITAK, etc. Her recent research activities are in the area of parameter dependent interval linear systems and their application to uncertainty modelling, as well as in applications of interval analysis to structural mechanics. She is a member of the IEEE Interval Standard Working Group P1788.

This paper is a revised and expanded version of a paper entitled 'Interval Model of Equilibrium Equations in Mechanics' presented at the '7th International Workshop on Reliable Engineering Computing (REC2016)', Ruhr University Bochum, Germany, 15–17 June 2016.

1   Introduction

Interval variables are widely used to represent non-probabilistic uncertainties in model parameters of various engineering problems, cf. Moens and Hanss (2011). It is already well understood that the dependencies between model parameters in the deterministic problem formulation should be present in and be considered by the corresponding interval model. If not accounted for, the interval dependencies may cause a large overestimation of the true system response. Nowadays, the most successful approaches for overestimation reduction are those that relate the dependency of interval quantities to the physics of the problem being considered (Muhanna et al., 2013).

The basic principle of static (or dynamic) equilibrium under general force systems is an essential prerequisite for many branches of engineering, such as mechanical, civil, aeronautical, bioengineering, robotics, and others that address various consequences of forces (Beer et al., 2010). Recently, it was discussed by several authors that interval models of static equilibrium based only on classical interval arithmetic do not conform to the problem physics. A model of a bar subjected to multiple axial external loads, with different directions and uncertain load magnitudes varying within given intervals, is considered in Elishakoff et al. (2015). Although the aim of providing interval model conforming to the principle of static equilibrium is not completely achieved by the proposed model and the proposed new interval operation, the paper demonstrates what are the challenges in this non-trivial task. A similar problem in the context of robotics is discussed in the IEEE P1788 working group on standardisation of interval arithmetic, (Mazandarani, 2015). Since classical interval arithmetic, in its set-theoretic interpretation as developed in Moore (1966), does not possess group properties with respect to addition and multiplication operations, we need a more general interval structure in order to provide a good interval model of equilibrium equations.

The engineering demand for more accurate models involving interval uncertainties has led to a new interval model of linear equilibrium equations in mechanics, proposed in Popova (2017a). This model is based on the algebraic completion of classical interval arithmetic, called also Kaucher or generalised interval arithmetic. Modelling summation of vectors with different directions and interval magnitude, the proposed interval model is in full conformance with the physical meaning of static equilibrium. It is proven that this interval model always yields the narrowest interval enclosure of a resultant force. The work of Popova (2017a) is focused on justification of the proposed interval model in one dimension, comparison to the approach of Elishakoff et al. (2015), and basic applications in computing resultant forces. Another study by Popova (2016) further develops the interval algebraic approach to models involving interval equilibrium equations by considering models of practical applications which reduce to systems of interval equilibrium equations where the number of the unknowns is equal to the number of the equations. The initial interval model is expanded by considering interval algebraic solution to the system of equilibrium equations, model properties are revealed and the quality of the interval algebraic solution is compared to the best interval solution enclosure obtained by classical interval arithmetic. The goal of the present paper is to summarise all achieved by now and to further illustrate the applicability of the interval model of linear equilibrium equations to models of statically indeterminate structures involving equilibrium equations with unknown forces and momenta. The solution of the planar frame model from Popova (2016) is completely rewritten here following two different goals.

The structure of the paper is as follows. Section 2 presents some basic notions and properties of the algebraic extension of classical interval arithmetic proposed in Kaucher (1980). In section 3 we present the new interval model of linear equilibrium equations, the algebraic approach to systems of interval equilibrium equations involving as many unknowns as the number of the equations, and a methodology how to apply the interval algebraic approach when the number of equilibrium equations is less than the number of the involved unknown forces and momenta. Numerical applications to models of structures are developed in details in section 4. They illustrate the new interval model of linear equilibrium equations, its conformance to the equilibrium principle and the interval algebraic approach applied to equilibrium equations with unknown forces. Subsection 4.1 considers a statically determinate structure and illustrates a basic application of the interval algebraic approach to provide realistic and sharp interval enclosure of the unknown forces and/or momenta to be used in the subsequent computations. Subsection 4.2 considers two examples of a statically indeterminate structure. The first example demonstrates that the new interval model of linear equilibrium equations provides a realistic enclosure and should be used (instead of the equations in classical interval arithmetic) for verifying the quality of the interval enclosures of the unknowns obtained by any (also the classical) interval approach. The second example demonstrates the applicability of a hybrid interval approach to a model of statically indeterminate frame structure involving equilibrium equations with unknown forces and momenta. While the unknown momenta in this model are enclosed by the classical interval approach from the equations connecting the momenta and the material properties of the beams, the remaining unknown forces are estimated from the new interval equilibrium equations by the interval algebraic approach. This way both are achieved: sharp realistic enclosures of the unknown forces and momenta and their full conformance to the physics of static equilibrium. The article ends by some conclusions.

2   Algebraic completion of interval arithmetic

We assume that the reader is familiar with classical interval arithmetic (Moore, 1966; Moore et al., 2009), and its properties.

The set of classical compact intervals $\mathbb{IR} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \leq a^+\}$, called also proper intervals, is extended in Kaucher (1980) by the set $\overline{\mathbb{IR}} := \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{KR} = \mathbb{IR} \bigcup \overline{\mathbb{IR}} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called *generalised* (extended or Kaucher) intervals. For a better understanding we denote the classical intervals by bold face letters (e.g., $\mathbf{a}$) and the intervals from $\mathbb{KR}$ by brackets (e.g., $[a]$). Of course, $\mathbf{a} \in \mathbb{IR} \subset \mathbb{KR}$, and thus $[b] = \mathbf{a} \in \mathbb{KR}$ is a correct assignment. The inclusion order relation ($\subseteq$) between classical intervals is generalised for $[a], [b] \in \mathbb{KR}$ by

$$[a] \subseteq [b] \Leftrightarrow b^- \leq a^- \text{ and } a^+ \leq b^+.$$

For $[a] = [a^-, a^+] \in \mathbb{KR}$, define a binary variable *direction* (τ) by

$$\tau([a]) := \begin{cases} + \text{if } a^- \leq a^+, \\ - \text{if } a^- \geq a^+. \end{cases}$$

All elements of $\mathbb{KR}$ with positive direction are called proper intervals and the elements with negative direction are called improper intervals.

An element-to-element symmetry between proper and improper intervals is expressed by the "Dual" operator. For $[a] = [a^-, a^+] \in \mathbb{KR}$, $\text{Dual}([a]) := [a^+, a^-]$. For $[a], [b] \in \mathbb{KR}$,

$$\text{Dual}(\text{Dual}([a])) = [a], \tag{1}$$

$$\text{Dual}([a] \circ [b]) = \text{Dual}([a]) \circ \text{Dual}([b]), \circ \in \{+, -, \times, /\}. \tag{2}$$

Define proper projection of a generalised interval $[a]$ onto $\mathbb{IR}$ by

$$\text{pro}([a]) := \begin{cases} [a] & \text{if } \tau([a]) = +, \\ \text{Dual}([a]) & \text{if } \tau([a]) = -. \end{cases}$$

Denote $\mathcal{T} := \{[a] \in \mathbb{KR} \mid [a] = [0,0] \text{ or } a^- a^+ < 0\}$. For $[a] = [a^-, a^+] \in \mathbb{KR} \setminus \mathcal{T}$ define "sign" (σ) by

$$\sigma([a]) := \begin{cases} + \text{ if } \text{pro}([a])^- \geq 0, \\ - \text{ otherwise.} \end{cases}$$

Multiplication of two binary variables $\lambda, \mu \in \{+, -\}$ is defined by $\lambda\mu = ++ = -- = +$, $\lambda\mu = +- = -+ = -$. The conventional interval arithmetic and lattice operations, as well as other interval functions are isomorphically extended onto the whole set $\mathbb{KR}$, cf. Kaucher (1980). A condensed representation of the arithmetic operations is derived in Dimitrova et al. (1992). Thus,

$$[a] + [b] = [a^- + b^-, a^+ + b^+] \qquad \text{for } [a], [b] \in \mathbb{KR},$$

$$[a] \times [b] = \begin{cases} \left[a^{-\sigma([b])} b^{-\sigma([a])}, a^{\sigma([b])} b^{\sigma([a])} \right] & \text{if } [a], [b] \in \mathbb{KR} \setminus \mathcal{T} \\ \left[a^{\sigma([a]\tau([b]))} b^{-\sigma([a])}, a^{\sigma([a]\tau([b]))} b^{\sigma([a])} \right] & \text{if } [a] \in \mathbb{KR} \setminus \mathcal{T}, [b] \in \mathcal{T} \\ \left[a^{-\sigma([b])} b^{\sigma([b])\tau([a])}, a^{\sigma([b])} b^{\sigma([b])\tau([a])} \right] & \text{if } [a] \in \mathcal{T}, [b] \in \mathbb{KR} \setminus \mathcal{T} \\ \left[\min\{a^- b^+, a^+ b^-\}, \max\{a^- b^-, a^+ b^+\} \right] & \text{if } [a], [b] \in \mathcal{T}, \tau([a]) = \tau([b]) \\ 0 & \text{if } [a], [b] \in \mathcal{T}, \tau([a]) \neq \tau([b]). \end{cases}$$

Interval subtraction and division can be expressed as composite operations, $[a] - [b] = [a] + (-1)[b]$ and $[a]/[b] = [a] \times (1/[b])$, where $1/[b] = [1/b^+, 1/b^-]$ if $[b] \in \mathbb{KR} \setminus \mathcal{T}$. The restrictions of the arithmetic operations to proper intervals produce the familiar operations in the conventional interval space.

The generalised interval arithmetic structure possesses group properties with respect to the operations addition and multiplication. For $[a] \in \mathbb{KR}$, $[b] \in \mathbb{KR} \setminus \mathcal{T}$,

$$[a] - \text{Dual}([a]) = 0, \qquad [b]/\text{Dual}([b]) = 1. \tag{3}$$

The complete set of conditionally distributive relations for multiplication and addition of generalised intervals can be found in Popova (1998, 2001). Here we present only one that is usually used. For $[a],[b],[s] = ([a]+[b]) \in \mathbb{KR} \setminus \mathcal{T}$, $[c] \in \mathbb{KR}$

$$([a]+[b])[c]_{\sigma([s])} = [a] \times [c]_{\sigma([a])} + [b] \times [c]_{\sigma([b])}, \tag{4}$$

wherein $[a]_+ = [a]$, $[a]_- = \text{Dual}([a])$. Addition operation in $\mathbb{KR}$ is commutative and associative; associativity does not hold true in (interval) floating point arithmetic. Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces (Kaucher, 1980). Some other properties and applications of generalised interval arithmetic can be found in Kaucher (1980), Markov et al. (1996), Popova (1998), Popova (2001), Popova and Ullrich (1996), Shary (2002) and the references given therein.

For $\mathbf{a} \in \mathbb{IR} \setminus \mathcal{T}$, define

$$\text{Abs}(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } 0 \le \mathbf{a}, \\ -\mathbf{a} & \text{otherwise.} \end{cases}$$

For $\mathbf{a} \subseteq \mathbf{b}$, the percentage by which $\mathbf{b}$ overestimates $\mathbf{a}$ is defined by

$$100(1 - \omega(\mathbf{a})) / \omega(\mathbf{b}), \qquad \omega(\mathbf{a}) := a^+ - a^-.$$

3   Interval model of linear equilibrium equations

We denote forces and other vector quantities by underlining the letter used to represent it. This is necessary in order to distinguish vectors from the proper intervals, which are denoted by bold-face letters, and from the real-valued scalars. The magnitude of a vector will be denoted by the corresponding italic-face letter.

The interval model of linear equilibrium equations in mechanics models summation of vectors with different directions and interval magnitude. The following theorem presents the model in one dimension.

Theorem 1 *(Popova, 2017a) Consider a bar subjected to a finite number of loads* $\underline{p}_1, \dots, \underline{p}_k$ *that may be applied in opposite directions and have uncertain magnitude* $p_1 \in \mathbf{p_1}, \dots, p_k \in \mathbf{p_k}$, $\mathbf{p}_i \ge 0$, $i = 1, \dots, k$, *(Figure 1). Assume that a coordinate system* (Ox) *is chosen. Then,*

Figure 1 Bar subjected to a number of loads with opposite directions
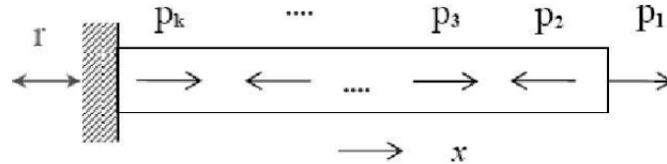
(i) for every j, $1 \leq j \leq k$, we have $[N_j] = \sum_{i=1}^{j} [p_i]$, wherein

$$[p_i] = \begin{cases} \mathbf{p}_i & \text{if the direction of } \underline{p}_i \text{ is in the positive } x \text{ axis} \\ -\text{Dual}(\mathbf{p}_i) & \text{if the direction of } \underline{p}_i \text{ is opposite to the positive } x \text{ axis,} \end{cases}$$

and $[r] = -\text{Dual}([N_k]) = -\text{Dual}\left(\sum_{i=1}^{k}[p_i]\right)$.

(ii) The interpretation of $[N_j] \in \mathbb{KR}$, $1 \leq j \leq k$, and similarly of $[r]$, is as follows.

- *If $[N_j] \in \mathcal{T}$, then $\underline{N}_j$ may have positive or negative direction and its magnitude varies in $pro([N_j])$.*

- *If $[N_j] \in \mathbb{KR} \setminus \mathcal{T}$, the magnitude of $\underline{N}_j$ varies in $Abs(pro([N_j]))$, while the direction of $\underline{N}_j$ coincides with the sign of $[N_j]$ (if $[N_j] \geq 0$ the direction of $\underline{N}_j$ is the positive x axis, otherwise it is opposite to the positive x axis).*

Strong proof that Theorem 1 provides sharpest enclosure of a resultant force and its reaction is given in Popova (2017a) along with a detailed discussion and examples.

In the deterministic case of two- (or three-) dimensional problems involving several forces, the determination of their resultant $\underline{R}$ is best carried out by first resolving each force into rectangular components (Beer et al., 2010). Choosing a rectangular coordinate system (Oxy), with unit vectors $\underline{i}, \underline{j}$, any force vector $\underline{F}$ can be resolved into rectangular components $\underline{F}_x = F_x \underline{i}$, and $\underline{F}_y = F_y \underline{j}$, so that $\underline{F} = F_x \underline{i} + F_y \underline{j}$. The scalar component F_x is positive when the vector component $\underline{F}_x$ has the same direction as the unit vector $\underline{i}$ (i.e., the same direction as the positive x axis) and is negative when $\underline{F}_x$ has the opposite direction. A similar conclusion may be drawn regarding the sign of the scalar component F_y. When more than one force act on a particle (or a rigid body), it is important to determine the resultant force, i.e., the single force $\underline{R}$ which has the same effect on the particle as the given forces. The resultant force $\underline{R}$ is determined by:

1 choosing a rectangular coordinate system;

2 resolving the given forces into their rectangular components;

3 each scalar component R_x, R_y of the resultant $\underline{R}$ of several forces $\underline{F}_i$ acting on a particle is obtained by adding algebraically the corresponding scalar components of the given forces. That is, $R_x = \sum_i F_{x,i}$, $R_y = \sum_i F_{y,i}$, which gives $\underline{R} = R_x \underline{i} + R_y \underline{j}$.

Basing on the above deterministic setting, the one dimensional interval algebraic model for computing a resultant force (and its reaction), developed in Popova (2017a), can be applied transparently to two- and three-dimensional problems involving vector physical quantities.

The interval model of linear equilibrium equations can be considered from a more general perspective. Assume that there is a deterministic model described by some equilibrium equation(s) that involve uncertain vector parameters varying within given

proper intervals. Clearly, the unknowns in this model will be also uncertain and we search for proper intervals that are the sharpest interval enclosures of these unknowns and that conform to the physics of the problem (statics or dynamic equilibrium). Conformance to static (dynamic) equilibrium means that the intervals found for the unknowns when replaced in the equation(s) and all operations are performed results in true equality(ies).

Definition 3.1 (Ratschek and Sauer, 1982): Interval *algebraic solution* to a (system of) interval equation(s) is an interval (interval vector) which substituted in the equation(s) and performing all interval operations in exact arithmetic (without round-off errors) results in valid equality(ies).

Interval algebraic solutions do not exist in general in classical interval arithmetic (Ratschek and Sauer, 1982). Generalised interval arithmetic on proper and improper intervals $(\mathbb{KR}, +, \times, \subseteq)$ is the natural arithmetic for finding algebraic solutions to interval equations since it is obtained from the arithmetic for classical intervals $(IR, +, -, \times, /, \subseteq)$ via an algebraic completion. This is another justification of the proposed interval algebraic approach. Therefore, we embed the initial problem formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$, find an algebraic solution (if exists) and interpret the obtained generalised intervals back in the initial interval space $\mathbb{IR}$. This is a three steps procedure summarised below.

1. The **representation convention** for a model involving interval forces (and/or other physical quantities considered as vectors and possessing magnitude and direction) is:

   - a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by proper interval $\mathbf{F}_x$ ($\mathbf{F}_y$, $\mathbf{F}_z$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has the same direction as the positive x (y, z) coordinate axis;

   - a scalar force component F_x (F_y, F_z) involving any kind of uncertainty is represented by the improper interval $\text{Dual}(\mathbf{F}_x)$ ($\text{Dual}(\mathbf{F}_y)$, $\text{Dual}(\mathbf{F}_z)$) if the force component $\underline{F}_x$ ($\underline{F}_y$, $\underline{F}_z$) has opposite direction to the corresponding positive x (y, z) coordinate axis.

2. **Computing**. Find the **algebraic solution** for the unknown(s) in $(\mathbb{KR}, +, \times, \subseteq)$. Conditions for existence of algebraic solution of interval linear equations are published in Markov (1999), Popova (1998), Shary (2002). Numerical methods finding the algebraic solution to an interval linear system are discussed in Markov et al. (1996), Markov (1999), Shary (2002), Sainz et al. (2014). For small systems, the approach based on *equivalent algebraic transformations* is transparent and will be used in this paper.

3. **Interpretation** of the obtained generalised intervals in the initial space $\mathbb{IR}$ is done according to the physics of the unknowns. If it is a force component, then Theorem 1 ii) is applied. In general the interpretation projects the generalised interval solution on $\mathbb{IR}$.

Since computing a resultant of several forces $\underline{F}_i$ (or its reaction $\underline{R}$) can be represented as a solution of the equilibrium equation $\sum_i \underline{F}_i - \underline{R} = 0$, Theorem 1 is a special case of the above more general interval algebraic approach.

If the deterministic model involves more unknowns than the number of equilibrium equations, other relations are obtained from the information contained in the statement of the problem. In this case (illustrated by the Example 3 below) the following hybrid approach should be applied. Let the number of the equilibrium equations be k and the number of the unknown quantities be n, $n > k$. From the statement of the problem we find $n - k$ additional relations involving (some of) the unknowns.

a. If the $n - k$ additional relations involve $n - k$ of the n unknowns, by methods of classical interval analysis find interval enclosures of these $n - k$ unknowns. Then replace the obtained interval enclosures in the interval model of the equilibrium equations and find the algebraic solution with respect to the remaining k unknowns.

b. Let $n - k$ additional relations involve $n - k + q$ of the unknowns. We consider q of the k equilibrium equations together with the $n - k$ additional relations in a way that the system involves $n - k + q$ unknowns. Then the process continues as in a. above.

This approach ensures that the unknown uncertain quantities are estimated in a way that the equilibrium equations are satisfied to a highest extent that corresponds to the initial uncertainties. The next section illustrates the proposed interval algebraic approach.

4   Numerical applications

In this section we illustrate the new model of linear interval equilibrium equations and the interval algebraic approach to some models of structures. In order to avoid many technical details that will hamper the comprehension, no more than two dimensional problems are considered. The computations presented in this section are done by the *Mathematica*® package `directed.m` (Popova and Ullrich, 1996). JInterval library (Nadezhin and Zhilin, 2014), can be used for this purpose, too. All numerical computations are done in exact (rational) interval arithmetic and then rounded outwardly to the presented floating point numbers. This is done on purpose so that round-off floating point errors do not obscure the full conformance of the new interval equilibrium equations to the problem physics, that is the interval equilibrium equations evaluate exactly to zero.

4.1   Statically determinate structures

Example 1 Consider a truss with two types of support and two external loads as shown in Figure 2. Each horizontal panel has length 2.4m, the length of each vertical panels is 1.8m and the loads applied to joints E and G are $F_E = F_G = 12$ kN. Assuming that the magnitude of the loads is uncertain so that $\mathbf{F}_E \in [10.8, 13.2]$, $\mathbf{F}_G \in [10.8, 13.2]$, determine the force in the member FG.

Figure 2   A truss with two types of support and two external loads
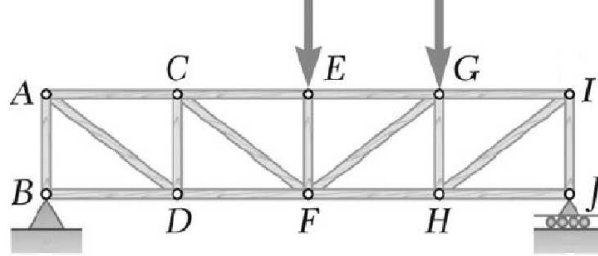
Assuming clockwise direction as positive direction for the momenta and the standard positive y-axis for the y-components of forces, and applying the representation convention from Section 3, the interval equilibrium equations for the entire truss are

$$\mathbf{F}_E \, l_{BF} + \mathbf{F}_G \, l_{HJ} - \text{Dual}(\mathbf{R}_B \, l_{BJ}) = 0,$$

$$\mathbf{R}_B - \text{Dual}(\mathbf{F}_E) - \text{Dual}(\mathbf{F}_G) + \mathbf{R}_J = 0, \tag{5}$$

where l_{BF}, l_{HJ}, l_{BJ} are the horizontal distances between the corresponding nodes. From these equations by algebraic transformations and by applying property (3) for addition in Kaucher interval arithmetic, we obtain the y-components $\mathbf{R}_B$, $\mathbf{R}_J$ of the reactions at the supports as

$$\mathbf{R}_B = [R_B] = (\mathbf{F}_E \, l_{BF} + \mathbf{F}_G \, l_{HJ}) / l_{BJ} = [8.1, 9.9]\text{kN},$$

$$\mathbf{R}_J = [R_J] = -\text{Dual}(\mathbf{R}_B) + \mathbf{F}_E + \mathbf{F}_G = [13.5, 16.5]\text{kN}.$$

Then from the interval equilibrium equation

$$-\text{Dual}\left(\frac{3}{5}\mathbf{F}_{FG}\right) - \text{Dual}(\mathbf{F}_G) + \mathbf{R}_J = 0 \tag{6}$$

for the y-coordinate of the forces at the joint G we obtain

$$\mathbf{F}_{FG} = \frac{5}{3}(-\text{Dual}(\mathbf{F}_G) + \mathbf{R}_J) = [4.5, 5.5]\text{kN}.$$

With the obtained $\mathbf{R}_B$, $\mathbf{R}_J$, $\mathbf{F}_{FG}$ the left-hand sides of all the considered interval equilibrium equations evaluate to zero.

If we had applied classical interval arithmetic approach to the deterministic equilibrium equations above, we would obtain the same interval $\tilde{\mathbf{R}}_B = \mathbf{R}_B$ and

$$\tilde{\mathbf{R}}_J = -\tilde{\mathbf{R}}_B + \mathbf{F}_E + \mathbf{F}_G = [11.7, 18.3],$$

$$\tilde{\mathbf{F}}_{FG} = \frac{5}{3}(-\mathbf{F}_G + \tilde{\mathbf{R}}_J) = [-2.5, 12.5].$$

In order to obtain a sharper enclosure of F_{FG} in classical interval arithmetic one can solve symbolically the classical setting of the equations obtaining

$$\frac{3}{5} F_{FG} = \mathbf{F}_E \left(1 - \frac{l_{BF}}{l_{BJ}} \right) - \mathbf{F}_G \frac{l_{HJ}}{l_{BJ}}$$

and in classical interval arithmetic $\tilde{\tilde{\mathbf{F}}}_{FG} = [3.5, 6.5]$. Although $\tilde{\mathbf{R}}_B$, $\tilde{\mathbf{R}}_J$, $\tilde{\tilde{\mathbf{F}}}_{FG}$ provide the exact solution to the classical interval model, this solution satisfies neither the classical interval equations of equilibrium nor the new interval algebraic model (5), (6) of the equilibrium equations.

4.2 Statically indeterminate structures

Here we present with more details and actual computations the example of a planar frame considered in Popova (2016). The solution of the planar frame model is completely rewritten here in Example 2 and Example 3 following two different goals.

Example 2 After Kulpa et al. (1998) consider a simple planar frame with three types of support and an external load distributed uniformly along the beam as shown in Figure 3(a).

Figure 3 Planar frame (a) and its fundamental system of internal parameters (b), after Kulpa et al. (1998)
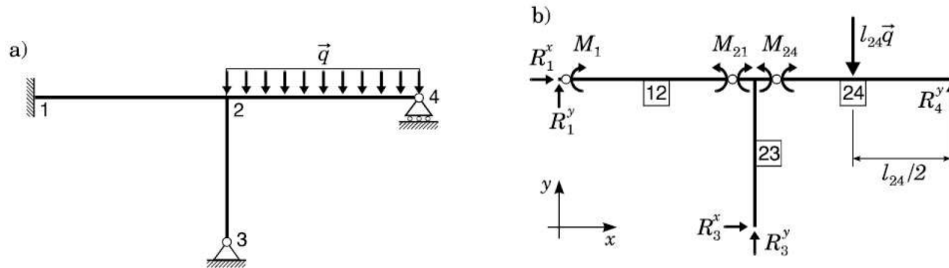
Assuming small displacements and linear elastic material law, and using the method of forces, the frame is described in Kulpa et al. (1998) by the following set of equilibrium equations for forces and bending moments, see Figure 3(b).

$$R_1^x + R_3^x = 0, \tag{7}$$

$$R_1^y + R_3^y + R_4^y - q l_{24} = 0, \tag{8}$$

$$-M_1 + R_4^y (l_{12} + l_{24}) + R_3^y l_{12} + R_3^x l_{23} - q l_{24} \left(l_{12} + \frac{1}{2} l_{24} \right) = 0, \tag{9}$$

$$-R_1^y l_{12} - M_1 + M_{21} = 0, \tag{10}$$

$$R_4^y l_{24} - \frac{1}{2} q l_{24}^2 - M_{24} = 0. \tag{11}$$

The equilibrium equations involve more unknowns than the number of the equations. Then, the three canonical equations linking bending moments with material properties (Young modulus E and momentum of inertia J of the beam cross-section) of the beams are given by

$$
\begin{pmatrix}
\dfrac{l_{12}}{3E_{12}J_{12}} & \dfrac{l_{12}}{6E_{12}J_{12}} & 0 \\[2ex]
\dfrac{l_{12}}{6E_{12}J_{12}} & \dfrac{l_{12}}{3E_{12}J_{12}}+\dfrac{l_{23}}{3E_{23}J_{23}} & \dfrac{-l_{23}}{3E_{23}J_{23}} \\[2ex]
0 & \dfrac{-l_{23}}{3E_{23}J_{23}} & \dfrac{l_{24}}{3E_{24}J_{24}}+\dfrac{l_{23}}{3E_{23}J_{23}}
\end{pmatrix}
\begin{pmatrix} M_1 \\ M_{21} \\ M_{24} \end{pmatrix}
=
\begin{pmatrix} 0 \\ 0 \\ \dfrac{-ql_{24}^3}{24E_{24}J_{24}} \end{pmatrix}.
\tag{12}
$$

The parameters of this frame are given as dimensionless numbers. It is assumed that all the beams have the same Young modulus $E=1$ and the momentums of inertia J of the beam cross-sections are related by the formula $J_{12}=J_{23}=1.5J_{24}=1.5$. The lengths of the beams and the load are considered to be uncertain. Substituting the known values into the equations (12), (7)–(11) for the frame and making appropriate simplifications, a parametric interval linear system

$$
A(l_{12},l_{23},l_{24})x=b(q,l_{12},l_{24}), \quad q\in\mathbf{q}, l_{12}\in\mathbf{l}_{12}, l_{23}\in\mathbf{l}_{23}, l_{24}\in\mathbf{l}_{24}
\tag{13}
$$

is obtained. In this system the elements a_{ij}, $i,j=1,\dots,8$, of the matrix $A(l_{12},l_{23},l_{24})$ are functions of the uncertain parameters l_{12},l_{23},l_{24} and specified by the following relations

$$
\frac{1}{2}a_{11}=a_{12}=a_{21}=a_{65}=-a_{74}=l_{12}, a_{22}=2l_{12}+2l_{23},
$$

$$
a_{33}=3l_{24}+2l_{23}, a_{66}=l_{12}+l_{24}, a_{23}=a_{32}=-2l_{23}, a_{68}=l_{23}, a_{86}=l_{24},
$$

$$
a_{47}=a_{48}=a_{54}=a_{55}=a_{56}=-a_{61}=-a_{71}=a_{72}=-a_{83}=1.
$$

The vector $b(q,l_{12},l_{24})$ is

$$
\left(0, 0, -\frac{3}{8}ql_{24}^3, 0, ql_{24}, ql_{24}\left(l_{12}+\frac{1}{2}l_{24}\right), 0, \frac{1}{2}ql_{24}^2\right)^{\top}
$$

and $x=(M_1,M_{21},M_{24},R_1^y,R_3^y,R_4^y,R_1^x,R_3^x)^{\top}$. The nominal values of the uncertain parameters are $l_{12}=l_{24}=1$, $l_{23}=0.75$, and $q=10$. It is assumed that there is no prestressing of the structure due to inexact dimensions of the beams. For that, the uncertainties are considered either as errors of measurements of the elements of the already existing structure, or else assume the structure will be assembled from inexact elements, but in a way that does not lead to prestressing (e.g., by slightly moving appropriate supports when necessary).

In the worst-case analysis, interval enclosures for the unknown reactions and moments are found by bounding the united parametric solution set of the system (13) defined by

$$
\Sigma^p := \{x\in\mathbb{R}^8 \mid \exists q\in\mathbf{q}, \exists l_{12}\in\mathbf{l}_{12}, \exists l_{23}\in\mathbf{l}_{23}, \exists l_{24}\in\mathbf{l}_{24}, A(l_{12},l_{23},l_{24})x=b(q,l_{12},l_{24})\}.
$$

We consider 0.5 % uncertainty in the interval parameters, that is

$$l_{12} \in [0.995, 1.005], l_{24} \in [0.995, 1.005], l_{23} \in [0.74625, 0.75375], q \in [9.95, 10.05].$$

Following Popova (2006) it is proven (by guaranteed parametric solvers) that the parametric solution of the system (13) is monotonic with respect to each interval parameter. The type of monotonicity is presented in Table 1 and corrects that reported in Popova (2006). For example, the lower bound of $\mathbf{R}_3^y$ is attained at the upper bound of $\mathbf{l}_{12}$ and the lower bounds of the remaining interval parameters; the upper bound of $\mathbf{R}_3^y$ is attained at the lower bound of $\mathbf{l}_{12}$ and the upper bounds of the remaining interval parameters. Then, the sharpest intervals enclosing the exact ranges for the magnitude of the unknown reactions and momenta together with their directions (signs in front of the intervals) are presented in Table 2. For this example, in exact arithmetic, the bounds obtained by the combinatorial approach, which gives the interval hull of all 16 point solutions to the linear systems at all possible endpoints of the parameter intervals, gives the same result as that presented in Table 2. A Monte Carlo computation involving the 16 vertex solutions also gives the same result. However, for systems involving a large number of interval parameters, the combinatorial approach is not feasible. If the interval hull of the solution set Σ^p is not attained at particular endpoints of the parameters, both the combinatorial approach and the computed Monte Carlo bounds will underestimate the true ranges of the solution components. The power of self-verified parametric interval solvers is in proving mathematical statements (e.g., Table 1) by fast floating-point computations which are feasible even for large number of interval parameters, see the example in Section 5 of Popova (2017b).

Table 1 Monotonic dependence of the solution components of the system (13) with respect to interval parameters l_{12}, l_{23}, l_{24}, q having 1% uncertainties in lengths and 15% uncertainty in the load

	$\mathbf{M}_1$	$\mathbf{M}_{21}$	$\mathbf{M}_{24}$	$\mathbf{R}_1^y$	$\mathbf{R}_3^y$	$\mathbf{R}_4^y$	$\mathbf{R}_1^x$	$\mathbf{R}_3^x$
l_{12}	−1	1	1	1	−1	1	−1	1
l_{23}	1	−1	1	−1	1	1	1	−1
l_{24}	1	−1	−1	−1	1	1	−1	1
q	1	−1	−1	−1	1	1	−1	1

Basing on classical interval arithmetic, the interval model of the equilibrium equations (7)–(11) just replaces the deterministic parameters by interval ones. With the exact values from Table 2 the evaluated values for all equations are presented in Table 3. These values deviate from the desired interval $[0,0]$ and thus confirm the claim in Elishakoff et al. (2015), Mazandarani (2015) that interval equilibrium equations based on classical interval arithmetic may not correspond to the problem physics.

Table 2   Interval bounds for the reactions and the momenta in the interval planar frame system (13) obtained by Table 1. Rows $u_{0.5\%}$ and $u_{1\%}$ are for 0.5%, respectively 1%, uncertainties in all interval parameters; $u_{1+15\%}$ denotes the enclosures obtained for 1% uncertainty in lengths and 15%% uncertainty in the load

	$\mathbf{M}_1$	$\mathbf{M}_{21}$	$\mathbf{M}_{24}$	$\mathbf{R}_1^x$
$u_{.5\%}$	[.24479, .25530]	−[.48958, .51059]	−[.98309, 1.0171]	−[.64953, .68421]
$u_{1\%}$	[.23966, .26068]	−[.47934, .52135]	−[.96639, 1.0344]	−[.63279, .70215]
$u_{1+15\%}$	[.20577, .29681]	−[.41155, .59362]	−[.82973, 1.1778]	−[.54330, .79948]
	$\mathbf{R}_1^y$	$\mathbf{R}_3^y$	$\mathbf{R}_4^y$	$\mathbf{R}_3^x$
$u_{.5\%}$	− [.73072, .76973]	[6.6698, 6.8309]	[3.9601, 4.0401]	[.64953, .68421]
$u_{1\%}$	− [.71189, .78992]	[6.5905, 6.9126]	[3.9204, 4.0804]	[.63279, .70215]
$u_{1+15\%}$	− [.61121, .89941]	[5.6585, 7.8708]	[3.366, 4.646]	[.54330, .79948]

Table 3   Evaluating the interval equations corresponding to (7)–(11) with the exact bounds for the unknowns from Table 2 (value$_1$ for 0.5% uncertainty in all interval variables; value$_2$ for 1% uncertainty in lengths and 15% uncertainty in the load)

equation	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*
value$_1$	[−.035, .035]	[−.24, .24]	[−.48, .48]	[−.04, .04]	[−.16, .16]
value$_2$	[−.26, .26]	[−3.49, 3.49]	[−5.23, 5.23]	[−.29, .292]	[−1.71, 1.71]

The results in Table 3 may lead to a wrong conclusion that the interval enclosures of the unknowns, presented in Table 2, are not sharp enough. Therefore we consider the correct model of interval equilibrium equations corresponding to the deterministic equations (7)–(11). According to the considerations in Popova (2017a) and Section 3, the deterministic equations (7)–(11) are transformed into the following interval equations

$$\mathbf{R}_1^x + \mathbf{R}_3^x = 0, \tag{14}$$

$$\mathbf{R}_1^y + \mathbf{R}_3^y + \mathbf{R}_4^y - \mathrm{Dual}(\mathbf{ql}_{24}) = 0, \tag{15}$$

$$-\mathrm{Dual}(\mathbf{M}_1) + \mathbf{R}_4^y(l_{12} + l_{24}) + \mathbf{R}_3^y l_{12} + \mathbf{R}_3^x l_{23} - \mathrm{Dual}\left(\mathbf{ql}_{24}\left(\mathbf{l}_{12} + \frac{1}{2}\mathbf{l}_{24}\right)\right) = 0, \tag{16}$$

$$-\mathrm{Dual}(\mathbf{R}_1^y \mathbf{l}_{12}) - \mathrm{Dual}(\mathbf{M}_1) + \mathbf{M}_{21} = 0, \tag{17}$$

$$\mathbf{R}_4^y \mathbf{l}_{24} - \mathrm{Dual}\left(\frac{1}{2}\mathbf{ql}_{24}^2\right) - \mathrm{Dual}(\mathbf{M}_{24}) = 0. \tag{18}$$

Following the deterministic model, the exact interval values from Table 2 are replaced in the equations (14)–(18) according to the conversion convention of Section 3. The latter means that the negative intervals $\mathbf{M}_{21}, \mathbf{M}_{24}, \mathbf{R}_1^y, \mathbf{R}_1^x$ from Table 2 are replaced in equations (14)–(18) as $\mathrm{Dual}(\mathbf{M}_{21}), \mathrm{Dual}(\mathbf{M}_{24}), \mathrm{Dual}(\mathbf{R}_1^y), \mathrm{Dual}(\mathbf{R}_1^x)$. Then, the evaluation in Kaucher arithmetic of the left-hand sides of equations (14)–(18) gives the

intervals presented in Table 4. These intervals show that the interval enclosures of the unknowns, presented in Table 2 and obtained by the classical interval approach, are quite good.

Table 4   Evaluating the interval equilibrium equations (14)–(18) with the unknowns from Table 2 (value$_1$ for 0.5% uncertainty in all interval variables; value$_2$ for 1% uncertainty in lengths and 15% uncertainty in the load)

equation	(14)	(15)	(16)	(17)	(18)
Value$_1$	[0, 0]	[−.0019, .0019]	[−.027, .029]	[0, 0]	[−.0037, .0038]
Value$_2$	[0, 0]	[−.003122, .004332]	[−.0453, .0658]	[0, 0]	[−.0063, .0087]

Example 2 shows that the new interval model of equilibrium equations, which corresponds exactly to the physics of static equilibrium, can be used for verifying the quality of the obtained interval enclosures of the unknown vector quantities.

Why there are nonzero intervals in Table 4 will be discussed in the next example.

Example 3 In this example we consider the same planar frame from Example 2 and use the new interval model of equilibrium equations (14)–(18) to find interval bounds for the unknown momenta and reactions.

Since the equations (14)–(18) involve more unknowns than equations we find first interval enclosures for the momenta M_1, M_{21}, M_{24} as exact (or the sharpest possible) bounds for the solution set of the parametric interval system defined by equation (12). This means that we use the interval enclosures of the momenta M_1, M_{21}, M_{24} from Table 2. Then, we replace the obtained intervals $\mathbf{M}_1$, $\mathrm{Dual}(\mathbf{M}_{21})$ and $\mathrm{Dual}(\mathbf{M}_{24})$ in the equilibrium equations (14)–(18) and by Kaucher interval arithmetic find proper algebraic solutions for $\mathbf{R}_1^y, \mathbf{R}_2^y, \mathbf{R}_3^y$ and $\mathbf{R}_1^x, \mathbf{R}_3^x$ as follows.

From equation (18) by applying properties (3) we obtain

$$[R_4^y] = (\mathbf{q}\mathbf{l}_{24}^2 / 2 + \mathrm{Dual}(\mathbf{M}_{24})) / \mathrm{Dual}(\mathbf{l}_{24}) \in \begin{cases} [3.9243, 4.0764] & \text{for } u_{1\%}, \\ [3.3693, 4.6414] & \text{for } u_{1+15\%}. \end{cases} \quad (19)$$

$[R_4^y]$ is a proper interval which implies $\mathbf{R}_4^y = [R_4^y]$. With the obtained values for $\mathbf{M}_{24}$ and $\mathbf{R}_4^y$ and in exact (rational) arithmetic the equilibrium equation (18) is completely satisfied for any of the three levels of uncertainty in Table 2. The newly obtained intervals $\mathbf{R}_4^y$ are slightly narrower than those in Table 2 because the expression (19) involves subtraction and division of differently directed intervals.

From equation (17), by applying properties (3) and (2), we obtain

$$[R_1^y] = (-\mathrm{Dual}(\mathbf{M}_1) + \mathrm{Dual}(\mathbf{M}_{21})) / \mathrm{Dual}(\mathbf{l}_{12})$$

$$= \mathrm{Dual}((-\mathbf{M}_1 + \mathbf{M}_{21}) / \mathbf{l}_{12}) \in \begin{cases} [-.71189, -.78992] & \text{for } u_{1\%}, \\ [-.61121, -.89941] & \text{for } u_{1+15\%}. \end{cases} \quad (20)$$

$[R_1^y]$ is an improper interval which implies $\mathbf{R}_1^y = \mathrm{pro}([R_1^y])$. The newly obtained $\mathbf{R}_1^y$ is the same (in exact arithmetic) as that in Table 2 because $[R_1^y]$ is negative and the

expression inside the Dual in (20) is in classical interval arithmetic. Since $\mathbf{R}_1^y$ is negative interval, it should be substituted in equilibrium equations as $\mathrm{Dual}(\mathbf{R}_1^y)$.

From equation (15) by applying properties (3) we obtain

$$[R_3^y] = -\mathrm{Dual}(\mathrm{Dual}(\mathbf{R}_1^y)) - \mathrm{Dual}(\mathbf{R}_4^y) + \mathbf{ql}_{24} \in \begin{cases} [6.5885, 6.9146] & \text{for } u_{1\%}, \\ [5.6568, 7.8731] & \text{for } u_{1+15\%}. \end{cases}$$

$[R_3^y]$ is a proper interval which implies $\mathbf{R}_3^y = [R_3^y]$. With the newly obtained intervals for $\mathbf{R}_1^y$, $\mathbf{R}_3^y$, $\mathbf{R}_4^y$ and in exact arithmetic, the equilibrium equation (15) is completely satisfied for any of the three levels of uncertainty in Table 2.

From equation (16) by applying properties (3) we obtain

$$[R_3^x] = \Big(\mathbf{M}_1 - \mathrm{Dual}(\mathbf{R}_4^y(\mathbf{l}_{12} + \mathbf{l}_{24})) - \mathrm{Dual}(\mathrm{Dual}(\mathbf{R}_3^y \mathbf{l}_{12}))$$
$$+ \mathbf{ql}_{24}\Big(\mathbf{l}_{12} + \frac{1}{2}\mathbf{l}_{24})\Big)\Big)/\mathrm{Dual}(\mathbf{l}_{23}) \in \begin{cases} [.67514, .65645] & \text{for } u_{1\%}, \\ [.57966, 74744] & \text{for } u_{1+15\%}. \end{cases} \tag{21}$$

For $u_{1\%}$, $[R_3^x]$ is an improper interval and we have to take its proper projection $\mathbf{R}_3^x = \mathrm{pro}([R_3^x]) = [.65645, .67514]$. In this case the expression of the interval equilibrium equation (16) is evaluated to $[-.01389, .01417]$. With denominator $\mathbf{l}_{23}$ in (21) case $u_{1\%}$, we obtain a proper interval

$$[\tilde{R}_3^x] = \Big(\mathbf{M}_1 - \mathrm{Dual}(\mathbf{R}_4^y(\mathbf{l}_{12} + \mathbf{l}_{24})) - \mathrm{Dual}(\mathrm{Dual}(\mathbf{R}_3^y \mathbf{l}_{12}))$$
$$+ \mathbf{ql}_{24}\Big(\mathbf{l}_{12} + \frac{1}{2}\mathbf{l}_{24}\Big)\Big)/\mathbf{l}_{23} \in [.66177, .66971]. \tag{22}$$

Since $\tilde{\mathbf{R}}_3^x = [\tilde{R}_3^x]$ is narrower than $\mathbf{R}_3^x$ and the evaluation of the interval equilibrium equation (16) with $\tilde{\mathbf{R}}_3^x$ is also sharper $[-.00993, .0101]$, we take $\mathbf{R}_3^x = \tilde{\mathbf{R}}_3^x$ in case of 1% uncertainties in all parameters. The choice of $\mathbf{R}_3^x$ between (21) and (22) is because $\mathbf{R}_3^x$ is a compound in the expression $\mathbf{R}_3^x \mathbf{l}_{23}$ of the equilibrium equation (16). In case of $u_{1+15\%}$, we have $\mathbf{R}_3^x = [R_3^x]$ and the expression of the interval equilibrium equation (16) is evaluated to 0.

From equation (14) we obtain

$$[R_1^x] = -\mathrm{Dual}(\mathbf{R}_3^x)$$
$$\mathbf{R}_1^x = \mathrm{pro}([R_1^x]) \in \begin{cases} -[.65645, .67514] & \text{for } u_{1\%}, \\ -[.57966, .74744] & \text{for } u_{1+15\%}. \end{cases}$$

Being negative interval, $\mathbf{R}_1^x$ is substituted in interval equilibrium equations by $\mathrm{Dual}(\mathbf{R}_1^x)$. Interval equilibrium equation (14) is satisfied completely by both the newly obtained intervals $\mathbf{R}_1^x, \mathbf{R}_3^x$ and by those in Table 2 because the equation involves only two opposite vectors.

Summarising the results for Example 3, where the unknown reactions are obtained by an algebraic approach applied to interval equilibrium equations (14)–(18), we have:

(i) Newly obtained interval enclosures for $\mathbf{R}_1^x$, $\mathbf{R}_3^x$, $\mathbf{R}_4^y$ are sharper than those in Table 2. The enclosure $\mathbf{R}_3^y$ from Table 2 underestimates the newly obtained enclosure $\mathbf{R}_3^y$. This is because interval forces in the interval equilibrium model are like connected vessels — expanding some interval enclosures shrinks the enclosure of others (and vice versa), so that the equilibrium equations are always satisfied. This property is particularly important because not always we can obtain the sharpest interval enclosures of the unknowns involved in the additional relations.

(ii) In case of 1% uncertain lengths and 15% uncertain load, the newly obtained enclosures of the reactions completely satisfy all interval equilibrium equations (14)–(18). In case of 1% uncertainty in all interval parameters, only the interval equilibrium equation (16) is not completely satisfied.

The algebraic approach should be applied carefully. It is not universally applicable to all models based on equilibrium equations. For example, the equations modelling the displacements at the nodes of a statically indeterminate truss structure with applied loads are based on force equilibrium equations. However, similarly to equation (12), the force equilibrium equations are linked to the material properties of the beams. Therefore, similarly to equation (12), the equations for the displacements in a statically indeterminate truss structure may be solved by the classical interval methods. The new interval equilibrium equations model summation of vector quantities with opposite directions and interval magnitude. This implies the applicability of the interval algebraic approach to those models or those parts of models which involve unknown vector summands.

5 Conclusion

In this article we considered and compared two different interval models of linear equilibrium equations:

1 The deterministic equilibrium equation(s) modelled by classical interval arithmetic. This model provides only a rough enclosure of the equilibrium even by the exact ranges of the unknowns.

2 A new interval model of linear equilibrium equations is presented in Section 3. It is based on the properties of an algebraic completion of classical interval arithmetic and aims at a complete (or a better) satisfaction of the equilibrium principle.

Both models (1) and (2), and the interval enclosures they provide, are mathematically correct. However, the two models have different properties, some of them discussed in the present article. For a logical interpretation of models (1) and (2) see, e.g., Sainz et al. (2014) and Elishakoff et al. (2015).

The new interval model of linear equilibrium equations in mechanics contributes to the development of more realistic and accurate interval models that conform to the problem physics. An attractive property of this model is its straightforward and transparent application to the deterministic models. By a simple representation

convention one can easily transform a deterministic formulation into a unique interval arithmetic formulation in the interval space $(\mathbb{KR}, +, \times, \subseteq)$ and the computations are done in the same rich algebraic space. Since the ranges of the unknown physical quantities are proper intervals, the obtained results must be always interpreted in the original physical setting of the problem. The latter means that the results in $\mathbb{KR}$ must be projected on the interval space $\mathbb{IR}$.

The main and the widest application of the interval algebraic approach to linear equilibrium equations is for computing the resultant of differently directed uncertain vector quantities. As demonstrated by several examples, the new interval equilibrium equations allow accounting for the dependencies between interval parameters from the very beginning of the modelling process, and thus providing realistic and sharp interval enclosures of some model parameters to be used in the subsequent computations, cf. Example 1. In more complicated examples of statically indeterminate structures involving equilibrium equations, the algebraic approach in Kaucher arithmetic and the classical interval arithmetic approach should be combined in a hybrid approach, as it is done in Example 3.

The interval model of linear equilibrium equations in terms of proper and improper intervals is the true interval model of these equations because it conforms to the problem physics. Independently of the way interval enclosures of the unknown quantities are obtained from an interval model involving equilibrium equations, the genuine interval model of these equations can be used for verifying the quality of the obtained interval enclosures of the unknowns.

The comparison of the two interval models of linear equilibrium equations will continue in some other papers.

Acknowledgement

References

Beer, F.P., Johnston, E.R., Mazurek, D.F., Cornwell, P.J. and Eisenberg, E.R. (2010) *Vector Mechanics for Engineers: Statics and Dynamics*, 9th ed., McGraw-Hill.

Dimitrova, N., Markov, S.M. and Popova, E.D. (1992) 'Extended interval arithmetics: new results and applications', in Atanassova, L. and Herzberger, J. (Eds): *Computer Arithmetic and Enclosure Methods*, Elsevier Sci. Publishers B.V., pp.225–232.

Elishakoff, I., Gabriele, S. and Wang, Y. (2015) 'Generalized Galileo Galilei problem in interval setting for functionally related loads', *Archive of Applied Mechanics*, Vol. 86, No. 7, pp.1203–1217.

Kaucher, E. (1980) 'Interval analysis in the extended interval space IR', *Computing Suppl.*, Vol. 2, pp.33–49.

Kulpa, Z., Pownuk, A. and Skalna, I. (1998) 'Analysis of linear mechanical structures with uncertainties by means of interval methods', *CAMES*, Vol. 5, pp.443–477.

Markov, S.M. (1999) 'An iterative method for algebraic solution to interval equations', *Applied Numerical Mathematics*, Vol. 30, pp.225–239.

Markov, S.M., Popova, E.D. and Ullrich, C.P. (1996) 'On the solution of linear algebraic equations involving interval coefficients', in Margenov, S. and Vassilevski, P. (Eds): *Iterative Methods in Linear Algebra, II*, IMACS Series in Computational and Applied Mathematics, Vol. 3, pp.216–225.

Mazandarani, M. (2015) *IEEE standard 1788-2015 vs. multidimensional RDM interval arithmetic*, Posting to IEEE P1788 working group. Available online at: http://grouper.ieee.org/groups/1788/email/msg08439.html

Moens, D. and Hanss, M. (2011) 'Non-probabilistic finite element analysis for parametric uncertainty treatment in applied mechanics: Recent advances', *Finite Elements in Analysis and Design*, Vol. 47, pp.4–16.

Moore, R.E. (1966) *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Moore, R.E., Kearfott, R.B. and Cloud, M.J. (2009) *Introduction to Interval Analysis*, Society for Industrial and Applied Mathematics.

Muhanna, R.L., Rama Rao, M.V. and Mullen, R.L. (2013) 'Advances in interval finite element modelling of structures', *Life Cycle Reliability and Safety Engineering*, Vol. 2, No. 3, pp.15–22.

Nadezhin, D.Y. and Zhilin, S.I. (2014) 'JInterval library: principles, development, and perspectives', *Reliable Computing*, Vol. 19, pp.229–247.

Popova, E.D. (1998) 'Algebraic solutions to a class of interval equations', *Journal of Universal Computer Science*, Vol. 4, No. 1, pp.48–67.

Popova, E.D. (2001) 'Multiplication distributivity of proper and improper intervals', *Reliable Computing*, Vol. 7, No. 2, pp.129–140.

Popova, E.D. (2006) 'Computer-assisted proofs in solving linear parametric problems', *Proceedings of SCAN'06*, IEEE Computer Society Press, p.35. DOI: 10.1109/SCAN.2006.12

Popova, E.D. (2016) 'Interval Model of Equilibrium Equations in Mechanics', in Freitag, S., Muhanna, R.L. and Mullen, R.L. (Eds): *Proceedings of REC'2016*, Ruhr University Bochum, pp.241–255.

Popova, E.D. (2017a) 'Improved solution to the generalized Galilei's problem with interval loads', *Archive of Applied Mechanics*, Vol. 87, No. 1, pp.115–127.

Popova, E.D. (2017b) 'Enclosing the solution set of parametric interval matrix equation $A(p)X = B(p)$ ', *Numerical Algorithms*, Vol. 78, No. 2, pp.423–447. doi:10.1007/s11075-017-0382-1.

Popova, E.D. and Ullrich, C.P. (1996) *Directed interval arithmetic in Mathematica. Implementation and applications*, Technical Report 96-3, Universität Basel, Switzerland. Available online at: http://www.math.bas.bg/epopova/papers/tr96-3.pdf

Ratschek, H. and Sauer, W. (1982) 'Linear interval equations', *Computing*, Vol. 26, pp.105–115.

Sainz, M.A., Armengol, J., Calm, R., Herrero, P., Jorba, L. and Vehi, J. (2014) *Modal Interval Analysis: New Tools for Numerical Information*, Lecture Notes in Mathematics 2091, Springer p.318.

Shary, S.P. (2002) 'A new technique in systems analysis under interval uncertainty and ambiguity', *Reliable Computing*, Vol. 8, No. 5, pp.321–418.