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Enclosing the Solution Set of Parametric Interval Matrix Equation $A(p)X = B(p)$

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Abstract Consider the parametric matrix equation $A(p)X = B(p)$, where the elements of the matrices $A(p)$ and $B(p)$ depend linearly on a number of uncertain parameters varying within given intervals. We prove that the united parametric solution sets of the matrix equation and that of the corresponding linear system with multiple right-hand sides, although different as sets, have the same interval hull. A generalization of the parametric Krawczyk iteration with low computational complexity for the matrix equation is presented. Some details improving the implementation and the application of this method are discussed. An interval method, designed by A. Neumaier and A. Pownuk for enclosing the united solution set of parametric linear systems with particular dependency structure, is generalized for arbitrary linear dependencies between the parameters and for systems with multiple right-hand sides. A new, more powerful, sufficient condition for regularity of a parametric interval matrix is proven. An important application of the linear systems with multiple right-hand sides is presented as a key methodology for feasibility in computing the interval hull of a class of united parametric solution sets that appear in practical problems.

Keywords linear matrix equations · interval parameters · solution enclosure

Mathematics Subject Classification (2000) 65G40 · 15A24 · 65F10

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1 Introduction

Consider the parametric matrix equation

$$\begin{aligned} A(p)X &= B(p), \qquad p = (p_1, \ldots, p_K)^\top \in (\mathbf{p}_1, \ldots, \mathbf{p}_K)^\top = \mathbf{p}, \\ A(p) &:= A_0 + \sum_{k=1}^{K} p_k A_k, \quad B(p) := B_0 + \sum_{k=1}^{K} p_k B_k, \end{aligned} \tag{1}$$

where $A_i \in \mathbb{R}^{m\times m}$, $B_i \in \mathbb{R}^{m\times n}$, $i = 0, \ldots, K$; $\mathbb{R}^{m\times n}$ denotes the set of real $m \times n$ matrices and $\mathbb{R}^m := \mathbb{R}^{m\times 1}$ denotes the set of real vectors with m components. A real compact interval is

$$\mathbf{a} = [a^-, a^+] := \{a \in \mathbb{R} \mid a^+ \leq a \leq a^+; a^-, a^+ \in \mathbb{R}\}.$$

By $\mathbb{IR}^m, \mathbb{IR}^{m\times n}$ we denote the sets of interval m-vectors and interval $m \times n$ matrices, respectively. Denote by $B_{\bullet k}(p)$ the k-th column of the matrix $B(p)$. For a given index set $\mathit{\Pi} = \{\pi_1, \ldots, \pi_k\}$, $p_{\mathit{\Pi}}$ denotes $(p_{\pi_1}, \ldots, p_{\pi_k})$.

Equation (1) can be considered in two different ways, [1], [2], as a matrix equation or as a parametric linear system with multiple right-hand sides. The parametric united solution set to equation (1) considered as a parametric linear system with multiple right-hand sides is

$$\begin{aligned} \overline{\Xi}^p_{uni} = \overline{\Xi}(A(p), B(p), \mathbf{p}) := \big\{ X = (X_{\bullet 1} | \ldots | X_{\bullet n}) \in \mathbb{R}^{m\times n} | \\ (\forall i = 1, \ldots, n)(\exists p \in \mathbf{p})\, (A(p)X_{\bullet i} = B_{\bullet i}(p)) \big\}. \end{aligned} \tag{2}$$

The parametric united solution set to the matrix equation (1) is

$$\Xi^p_{uni} = \Xi(A(p), B(p), \mathbf{p}) := \big\{ X \in \mathbb{R}^{m\times n} | (\exists p \in \mathbf{p})\, (A(p)X = B(p)) \big\}. \tag{3}$$

It was shown in [1] that $\Xi^p_{uni} \subseteq \overline{\Xi}^p_{uni}$ and the goal is to find interval outer estimate of Ξ^p_{uni} with a considerably reduced computational complexity. Proposed are modified versions of for the parametric Krawczyk iteration, the parametric Bauer-Skeel method, the Hansen-Bliek-Rohn method, Gaussian elimination and the parametric Gauss-Seidel method.

In this work we prove that, although the two parametric solution sets Ξ^p_{uni} and $\overline{\Xi}^p_{uni}$ are different as sets, if they are nonempty and bounded they have the same exact interval hull, $\square\, \Xi^p_{uni} = \square\, \overline{\Xi}^p_{uni}$, cf. section 2. For a nonempty and bounded set $\Xi \subset \mathbb{R}^{m\times n}$,

$$\square \Xi := \bigcap \{\mathbf{X} \in \mathbb{IR}^{m\times n} \mid \mathbf{X} \supseteq \Xi\}.$$

The above result is a background for the reduced computational complexity of the methods providing interval enclosure of Ξ^p_{uni}. In section 3 we propose an improved version of the parametric Krawczyk iteration that has an expanded application domain and the same reduced computational complexity compared to the method from [1].

Most of the interval methods for solving parametric interval linear systems with a single right-hand side $A(p)x = b(p)$, $p \in \mathbf{p}$, e.g., [3,5,12,24], and the methods with reduced computational complexity proposed in [1], require strong regularity of the parametric matrix, the latter defined in [11]. Since strong regularity is only a sufficient condition for regularity of a parametric matrix in a given interval box, these methods sometimes fail especially when the parameter intervals are large. In [10] Neumaier and Pownuk proposed an iterative method, known as the best known so far interval method for solving parametric linear systems in a special form

$$(A_0 + LDR)x \;=\; b_0 + Fq, \tag{4}$$

with interval parameters p isolated in a diagonal matrix D and in the right-hand side vector q. This method does not require strong regularity of $A(p)$ on $\mathbf{p}$. In [10] it is demonstrated on linear systems with over 5000 variables and over 10000 parameters. In general, not every parametric system can be represented in the form (4). In [17] some constructive sufficient conditions for a parametric matrix $A(p)$ to be representable in the form $A_0 + LDR$ are proven and the method is generalized to parametric systems involving dependencies in both the matrix and the right hand side. Then a natural question arises: Is it possible any parametric system involving arbitrary linear dependencies be represented in an equivalent form similar to (4) and the generalized method of Neumaier and Pownuk be applicable. The answer is positive and presented in Section 4, where a constructive theorem shows how any parametric linear system in general form can be transformed into a form similar to (4). Based on the latter, a more powerful sufficient condition for regularity of a parametric matrix in a box is proven. These justify the expanded scope of applicability of the generalized method of Neumaier and Pownuk, which is demonstrated by numerous numerical examples. In Section 5 we first present an algorithm (with reduced computational complexity) implementing the method of Neumaier and Pownuk to parametric matrix equations. Thereafter in the same section, an important application of the linear systems with multiple right-hand sides is presented as a key methodology for feasibility in computing the interval hull of a large class of united parametric solution sets. The paper ends with some concluding remarks.

The following notation will be used. For $\mathbf{a} = [a^-, a^+] \in \mathbb{IR}$, define midpoint $\mathrm{mid}(\mathbf{a}) = \check{a} := (a^- + a^+)/2$, radius $\mathrm{rad}(\mathbf{a}) = \hat{a} := (a^+ - a^-)/2$, absolute value (magnitude) $|\mathbf{a}| := \max\{|a^-|, |a^+|\}$. Define sign of an interval by

$$\mathrm{sign}(\mathbf{a}) := \{1 \text{ if } a^- > 0,\; -1 \text{ if } a^+ < 0,\; 0 \text{ otherwise}\}.$$

The end point functionals $(\cdot)^-$, $(\cdot)^+$ and the functionals, defined above, are applied to interval vectors and matrices componentwise.

Denote $\{\pm 1\}^m := \{u \in \mathbb{R}^m \mid |u| = (1, ..., 1)^\top\}$. For $\mathbf{a} \in \mathbb{IR}^n$ and $u \in \{\pm 1\}^n$, a^u is defined by $a_i^u := \{a_i^- \text{ if } u_i = -1;\; a_i^+ \text{ if } u_i = 1\}$, $i = 1, \ldots, n$.

$\varrho(A)$ denotes the spectral radius of a square matrix A and I is the identity matrix of appropriate dimension.

Given an interval vector $\mathbf{p} \in \mathbb{IR}^K$ and $K+1$ numerical matrices $A_k \in \mathbb{R}^{n\times n}$, $k = 0, \ldots, K$, the set of matrices

$$\{A(p), \mathbf{p}\} := \left\{ A(p) = A_0 + \sum_{k=1}^{K} p_k A_k \mid p \in \mathbf{p} \right\}$$

is called a (square) parametric interval matrix with linear parameter dependencies defined by the numerical matrices A_k, $k = 0, \ldots, K$. For every parametric interval matrix $\{A(p), \mathbf{p}\}$ involving only linear dependencies, the corresponding nonparametric interval matrix is

$$A(\mathbf{p}) := \Box\{A(p), \mathbf{p}\} = A_0 + \sum_{k=1}^{K} \mathbf{p}_k A_k,$$

where the expression at the most right side is evaluated in interval arithmetic.

Definition 1 A square parametric interval matrix $\{A(p), \mathbf{p}\}$ is called *regular* if $A(p)$ is regular for every $p \in \mathbf{p}$. $\{A(p), \mathbf{p}\}$ is said singular otherwise, i.e., if $A(p)$ is singular for some $p \in \mathbf{p}$.

For $\mathbf{a}, \mathbf{b} \in \mathbb{IR}$, $\mathbf{a} \subseteq \mathbf{b}$, the percentage by which $\mathbf{b}$ overestimates $\mathbf{a}$ is defined by

$$100(1 - \omega(\mathbf{a})/\omega(\mathbf{b})), \qquad \omega(\mathbf{a}) := a^+ - a^-.$$

2 Interval Hulls of two United Solution Sets

Consider the parametric matrix equation (1) and the two parametric solution sets (2), (3).

Definition 2 Kernel (or null space) of a matrix $A(p) \in \mathbb{R}^{m\times m}$ is

$$\ker(A(p)) := \{x \in \mathbb{R}^m | A(p)x = 0\}.$$

Let the kernel of the parametric matrix $A(p)$ (considered as a symbolic matrix) be the trivial zero vector, $\ker(A(p)) = \{0\}$. Then the matrix $X(p) = A^{-1}(p)B(p)$ has explicit representation and

$$\left(A^{-1}(p)B_{\bullet 1}(p) | \ldots | A^{-1}(p)B_{\bullet n}(p)\right) = A^{-1}(p)B(p).$$

If $A(p)$ is regular for every $p \in \mathbf{p}$, then the sets Ξ^p_{uni} and $\overline{\Xi}^p_{uni}$ are bounded. Since the interval hull provides componentwise bounds for bounded sets, we have

$$\{\Box\, \Xi^p\}_{ij} = [\min_{p\in\mathbf{p}} \{\Xi^p\}_{ij}, \max_{p\in\mathbf{p}} \{\Xi^p\}_{ij}] = \{\Box\, \overline{\Xi}^p\}_{ij}, \qquad i, j = 1, \ldots, m.$$

Thus, $\Box\, \Xi^p_{uni} = \Box\, \overline{\Xi}^p_{uni}$ and we proved the following theorem.

Theorem 1 *For nonempty and bounded Ξ^p_{uni}, $\overline{\Xi}^p_{uni}$, we have $\Box\, \Xi^p_{uni} = \Box\, \overline{\Xi}^p_{uni}$.*

The following example, required by one of the reviewers, illustrates the proof of Theorem 1.

Example 1 For the parametric matrix $A(a) = \begin{pmatrix} a, & -2a \\ 3+a/2, & -a \end{pmatrix}$ we have $\ker(A(a)) = \{0\}$. Therefore, $A^{-1}(a) = \begin{pmatrix} -1/6, & 1/3 \\ (6+a)/12a, & 1/6 \end{pmatrix}$ and, for any right hand side $B(a)$, the solution $X(a) = A^{-1}(a)B(a)$ has explicit representation. Since $\det(A(a)) = 6a$, for any $a \in \mathbf{a} \in \mathbb{IR}$, $0 \notin \mathbf{a}$, the parametric matrix is nonsingular and both solution sets Ξ^p_{uni}, $\overline{\Xi}^p_{uni}$ are nonempty. The latter implies that Theorem 1 holds true.

Theorem 1 shows that we cannot obtain sharper outer interval bounds for Ξ^p_{uni} of the matrix equation (1) than the outer interval bounds for $\overline{\Xi}^p_{uni}$ of the corresponding system with multiple right-hand sides. This makes the task of designing methods for interval bounding of Ξ^p_{uni} more or less trivial since the problem reduces to finding outer interval estimate of the solution set to a parametric system with multiple right-hand sides. The latter is easily achievable with considerably less computational effort (inversion of one $m \times m$ matrix) instead of solving an augmented parametric system which requires inversion of an $mn \times mn$ matrix. In fact, looking at the modified numerical methods proposed in [1] one can see that they all provide outer estimates of the united solution set to the corresponding parametric system with multiple right-hand sides. By Theorem 1 the tedious proof of the Krawczyk iteration in [1, Section 4.1] could be avoided. Theorem 1 allows designing a generalization of the method of Neumaier and Pownuk for parametric matrix equations.

3 Parametric Krawczyk Iteration for $A(p)X = B(p)$

Definition 3 **([11])** A square parametric matrix $\{A(p), \mathbf{p}\}$ is strongly regular if $A(\check{p})$ is regular and some of the following matrices is regular

$$\mathbf{B} := \square\{A^{-1}(\check{p})A(p) \mid p \in \mathbf{p}\}, \qquad \mathbf{B}' := \square\{A(p)A^{-1}(\check{p}) \mid p \in \mathbf{p}\}. \tag{5}$$

Theorem 2 **([11])** *Let* $\check{A} = A(\check{p})$ *be regular and* $\mathbf{C} = \mathbf{B}$ *or* $\mathbf{C} = \mathbf{B}'$ *be defined by (5). If* $\operatorname{rad}(\mathbf{C}) < \operatorname{rad}\left(\check{A}^{-1}A(\mathbf{p})\right)$[1] *and* $\operatorname{rad}(\mathbf{C}) + \operatorname{rad}\left(\check{A}^{-1}A(\mathbf{p})\right)$ *is irreducible, then*

$$\varrho(\operatorname{rad}(\mathbf{C})) < \varrho(\operatorname{rad}(\check{A}^{-1}A(\mathbf{p}))).$$

The parametric Krawczyk iteration for matrix equations, [1, Theorem 5.3 and Algorithm 1] are derived by applying [23, Theorem 10.11] to an augmented parametric system and subsequent rearrangements based on the properties of the Kronecker product. Since the iteration matrix $I - RA(\mathbf{p})$ in [23, Theorem 10.11] uses the nonparametric matrix $A(\mathbf{p})$, the same do [1, Theorem 5.3 and Algorithm 1]. It follows from Theorem 2 above, cf. [11], that the parametric

[1] For $A, B \in \mathbb{R}^{m\times n}$, $A < B$ if $A \leq B$ and $A \neq B$.

Krawczyk iteration using a sharper iteration matrix $I - \square\{RA(p)|p \in \mathbf{p}\}$ [12, Theorem 2.3] has a larger scope of applicability than [23, Theorem 10.11] and, respectively, than [1, Theorem 5.3, Algorithm 1]. Historically, [23, Theorem 10.11] generalizes the method from [4].

The following theorem follows straightforward from the corresponding theorem [12, Theorem 2.3] for parametric linear systems and Theorem 1.

Theorem 3 *Consider (1). Let $R \in \mathbb{R}^{m\times m}$, $\mathbf{Y} \in \mathbb{IR}^{m\times n}$, $\tilde{X} \in \mathbb{R}^{m\times n}$ be given and define $\mathbf{Z} \in \mathbb{IR}^{m\times n}$, $\mathbf{C} \in \mathbb{IR}^{m\times m}$ by*

$$\mathbf{Z} := R(B_0 - A_0\tilde{X}) + \sum_{k=1}^{K} \mathbf{p}_k(R(B_k - A_k\tilde{X})),$$

$$\mathbf{C} := I - \square\{RA(p)|p \in \mathbf{p}\} = I - RA_0 - \sum_{k=1}^{K} \mathbf{p}_k(RA_k).$$

Define $\mathbf{V} \in \mathbb{IR}^{m\times n}$ by the following Gauss-Seidel iteration

$$\begin{aligned} \mathbf{V}_{i\bullet} &= \mathbf{Z}_{i\bullet} + (\mathbf{C}\cdot\mathbf{U})_{i\bullet}, \quad 1 \le i \le m, \\ \mathbf{U} &= \left(\mathbf{V}_{1\bullet}^{\top}|\ldots|\mathbf{V}_{i-1\bullet}^{\top}|\mathbf{Y}_{i\bullet}^{\top}|\ldots|\mathbf{Y}_{m\bullet}^{\top}\right)^{\top}. \end{aligned} \tag{6}$$

If $\mathbf{V} \subsetneqq \mathbf{Y}$[2], then R and every matrix $A(p)$ with $p \in \mathbf{p}$ are regular, and for every $p \in \mathbf{p}$ the unique solution $X(p) = A^{-1}(p)B(p)$ of (1) considered as a parametric system with multiple right-hand sides satisfies $X(p) \in \tilde{X} + \mathbf{V}$. Respectively, for every $p \in \mathbf{p}$ the unique solution $X(p) = A^{-1}(p)B(p)$ of the parametric matrix equation (1) satisfies $X(p) \in \tilde{X} + \mathbf{V}$.

With $\mathbf{D} := \mathbf{C}\cdot\mathbf{V} \in \mathbb{IR}^{m\times n}$, the following inner estimate of the interval hull $\square\,\Xi_{uni}^{p}$ satisfies

$$\left[\tilde{X} + \inf(\mathbf{Z}) + \sup(\mathbf{D}), \tilde{X} + \sup(\mathbf{Z}) + \inf(\mathbf{D})\right] \subseteq \square\,\Xi_{uni}^{p} = \square\,\tilde{\Xi}_{uni}^{p},$$

where inf, sup *are applied componentwise.*

Algorithm 1 describes the implementation of Theorem 3. In order to force $\mathbf{X} \subsetneqq \mathbf{Y}$, the concept of ε-inflation, cf. [22], is applied on line 6 of the algorithm. ε-inflation is applied to interval vectors and matrices componentwise.

Algorithm 1 *Parametric Krawczyk iteration for systems with multiple right-hand sides and for the matrix equation (1).*

Input: *matrices $A_k \in \mathbb{R}^{m\times m}$, $B_k \in \mathbb{R}^{m\times n}$, $k = 0,\ldots,K$,*
interval vector $\mathbf{p} = (\mathbf{p}_0,\ldots,\mathbf{p}_K)^{\top}$, where $\mathbf{p}_0 = [1,1]$,
integer number maxItr *and $\varepsilon \in \mathbb{R}_+$ (usually $\varepsilon = 0.1$).*

Output: *matrices $\mathbf{X}$, (optionally) $\mathbf{H}$, such that $\mathbf{H} \subseteq \square\,\Xi_{uni}^{p} \subseteq \mathbf{X}$, or the message "Strong regularity of the parametric matrix cannot be proven.".*

1. $R \approx A^{-1}(\check{p})$; $\tilde{X} \approx A^{-1}(\check{p})B(\check{p})$;

[2] $\mathbf{V},\mathbf{Y} \in \mathbb{IR}^{m\times n}$, $\mathbf{V} \subsetneqq \mathbf{Y} \Leftrightarrow \mathbf{V} \subseteq \mathbf{Y}$ and $(\mathbf{V}_{ij} \neq \mathbf{Y}_{ij},$ for every $1 \le i \le m, 1 \le j \le n)$.

2. $\mathbf{Z} = \sum_{k=0}^{K} \mathbf{p}_k(R(B_k - A_k\tilde{X}))$;
3. $\mathbf{C} = I - \sum_{k=0}^{K} \mathbf{p}_k(RA_k)$;
4. $q = 0$; $\mathbf{X} = \mathbf{Z}$;
5. **repeat**
6. $\mathbf{Y} = \mathbf{X} = [1-\varepsilon, 1+\varepsilon]\mathbf{X}$;
7. **for** $i = 1$ *to* m **do** $\mathbf{X}_{i\bullet} = \mathbf{Z}_{i\bullet} + \mathbf{C}_{i\bullet} \cdot \mathbf{X}$;
8. $q = q + 1$;
9. *IsVerified*$= \mathbf{X} \subsetneqq \mathbf{Y}$;
10. **until** *IsVerified* $=$ *false* **and** $q \leq$maxItr;
11. **If** (**not** *IsVerified*) **then** *Message "The parametric matrix may not be strongly regular."; Exit;*
12. **else**
13. $\mathbf{X} = \tilde{X} + \mathbf{X}$;
14. $\mathbf{D} = \mathbf{C} \cdot \mathbf{X}$;
15. $\mathbf{H} = \left[\tilde{X} + \inf(\mathbf{Z}) + \sup(\mathbf{D}),\ \tilde{X} + \sup(\mathbf{Z}) + \inf(\mathbf{D})\right]$.
16. **Return**: $\mathbf{X}$ *and (optionally)* $\mathbf{H}$.

Theorem 3 and Algorithm 1 have a larger scope of applicability than [1, Theorem 5.3 and Algorithm 1] due to the sharper iteration matrix $\mathbf{C}$, see the examples in [12], [18]. Beside, compared to [1, Theorem 5.3 and Algorithm 1], Theorem 3 and Algorithm 1 involve a Gauss-Seidel iteration on line 8 of the algorithm, which provides a slightly narrower interval enclosure $\mathbf{X}$, and an inner estimate $\mathbf{H}$ of the interval hull $\square\, \Xi_{uni}^{p}$. The latter allows estimating the quality of the computed outer estimate $\mathbf{X}$, cf. [18], [23]. How to implement correctly rounded floating-point computations at line 15 of Algorithm 1 is described in [18], [23]. The computational complexity of Algorithm 1 is $O(c(K+1, Itr)m^2 \max\{m, n\})$, where $c(K+1, Itr)$ is a constant depending on the number of the parameters and the iterations.

The quality of the solution set enclosure can be slightly improved by running a subsequent refinement iteration, [21]. Thus, Algorithm 1 can be expanded (between lines 12 and 13) by the following refinement iteration

12.1 q=0;
12.2 **repeat**
12.3 $\mathbf{Y} = \mathbf{X}$;
12.4 $\mathbf{X} = \mathbf{Y} \bigcap (\mathbf{Z} + \mathbf{C}\mathbf{Y})$;
12.5 q=q+1;
12.6 **until** $\max_{i,j}$`dist`$(\mathbf{X}_{ij}, \mathbf{Y}_{ij}) > \delta$ **and** $q \leq$maxRefItr;

wherein $\delta \in \mathbb{R}_+$ and `maxRefItr` are appropriate prescribed constants; `dist`$(\cdot, \cdot)$ denotes the distance between two intervals, cf. [9]. The stopping criterion at line 12.6 can be another one.

Two constants are involved in Algorithm 1 as input values: `maxItr` and ε. These constants are important for verifying that the parametric matrix $A(p)$ is strongly regular on $\mathbf{p}$. Increasing the value of ε usually reduces the number of the iterations necessary for the convergence of the method but slightly enlarges the obtained outer interval estimate of the solution set, [22].

Example 2 We apply Algorithm 1, expanded by a refinement iteration, to the parametric matrix equation from [1, Example 5.2]

$$\begin{pmatrix} 2p_1 & p_2 \\ -p_2 & 2p_1 \end{pmatrix} X = \begin{pmatrix} 0, & p_3 \\ p_3, & -\frac{1}{2} \end{pmatrix}, \qquad \begin{matrix} p_1 \in [1,2], \\ p_2 \in [-\frac{6}{5}, 2], \end{matrix} \quad p_3 \in [-3,3].$$

For this matrix equation [1, Algorithm 1] exits by the message "Method cannot obtain outer estimation". This message is incorrect since the method (Theorem 3) is capable to find an outer estimate of the solution set but in the implementation ([1, Algorithm 1]) the combination of *maxItr* and ε cannot provide convergence of the method. Namely, [1, Algorithm 1] employs an absolute ε-inflation, which increases the number of iterations necessary for the convergence.

Algorithm 1 above converges in 15 initial iterations and after sufficiently many refinement iterations (with $\delta = 10^{-10}$) yields an interval matrix enclosing the interval hull of the parametric matrix equation. The obtained guaranteed interval enclosure is compared to the interval enclosure, obtained by the parametric Bauer-Skeel method from [1] executed in exact (rational) arithmetic. The two enclosures differ in the 10th digit after the decimal point for each interval end point.

Some researchers, e.g., [25,1], compare the quality of the solution enclosures obtained (for systems with a single right-hand side) by the parametric Krawczyck iteration and by the parametric Bauer-Skeel method, [24,3], without discussing the most important difference between these two methods. While the parametric Krawczyck iteration provides *guaranteed* enclosure of the solution set by floating point computations, the interval enclosure of the Bauer-Skeel method is guaranteed only if the data and the computations are in exact (rational) arithmetic. The next example illustrates this.

Example 3 Consider the following parametric interval linear system

$$\begin{pmatrix} p_1 & p_1 \\ p_1 & p_1 + 0.01 \end{pmatrix} x = \begin{pmatrix} p_2 \\ p_2 + 0.01 \end{pmatrix}, \qquad \begin{matrix} p_1 \in [0.9, 1.1], \\ p_2 \in [1.9, 2.1]. \end{matrix}$$

The parametric Bauer-Skeel method executed in exact arithmetic gives the vector $([2/3, 4/3],\ [1,1])^\top$. It encloses the parametric solution set, which is the interval vector $\left([\frac{8}{11}, \frac{4}{3}],\ [1,1]\right)^\top$. The parametric Bauer-Skeel method executed in floating-point arithmetic gives the following vector (where the floating points have 16 digits mantissas), which does not enclose the parametric solution set,

$$([0.6...66939, 1.3...3363],\ [0.9...9715, 0.9...9717])^\top .$$

The parametric Krawczyck iteration after one initial iteration and 15 refinement iterations (with $\delta = 10^{-16}$) gives

$$([0.6...61077, 1.3...38497],\ [0.9...95374, 1.0...04625])^\top ,$$

where the floating points have also 16 digits mantissas.

4 Generalizing the Method of Neumaier and Pownuk

4.1 The method applicable to a class of parametric systems

Consider a parametric linear algebraic system in the form

$$\begin{aligned} A(p)x &= b(p,q), \quad p \in \mathbf{p},\ q \in \mathbf{q}, \\ A(p) &:= A_0 + \sum_{k=1}^{K} p_k A_k, \qquad b(p,q) := b_0 + \sum_{k=1}^{K} p_k b_k + \sum_{k=1}^{Q} q_k b_k. \end{aligned} \tag{7}$$

Definition 4 A parametric solution set is called *linear*, in other words its shape is linear, if the boundary of the solution set consists of parts of hyperplanes.

Theorem 4 ([17]) *Consider the parametric system (7). Denote by $g_k(x)$ the polynomial greatest common divisor (GCD) of the elements of $A_k x$, $k = 1,\dots,K$. Let $g_k(x)$ be a nonconstant polynomial for every $k = 1,\dots,K$. Define*

$$L := (l_1|\dots|l_K) \in \mathbb{R}^{n\times K}, \ \text{ where } \ l_k := A_k x / g_k(x) \in \mathbb{R}^n$$
$$R := (r_1|\dots|r_K)^\top \in \mathbb{R}^{K\times n}, \ \text{ where } \ r_k := \left(\frac{\partial g_k(x)}{\partial x_1},\dots,\frac{\partial g_k(x)}{\partial x_n}\right)^\top \in \mathbb{R}^n.$$

If there exists $t_k \in \mathbb{R}$ such that $t_k l_k = b_k := \partial b(p,q)/\partial p_k$ for every $k = 1,\dots,K$, then

(i) $\Sigma_{uni}(A(p), b(p,q), \mathbf{p}, \mathbf{q})$ has linear shape and
(ii) the parametric system (7) is equivalently represented as

$$(A_0 + LDR)x = b_0 + LDt + Fq, \quad p \in \mathbf{p},\ q \in \mathbf{q}, \tag{8}$$

where $F := (b_1|\dots|b_Q) \in \mathbb{R}^{n\times Q}$, $t = (t_1,\dots,t_K)^\top$ and $D = Diag(p)$.

Theorem 4 shows when[3] and how a linear system in general parametric form (7) can be represented in an equivalent form (8), where the parameters that appear in the matrix $A(p)$ are isolated with a single entry in a diagonal matrix. Therefore, Theorem 4 is called *conversion theorem*. The representation (8) allows a more efficient handling of some parameter dependencies by a method of Neumaier and Pownuk, [10,17].

Theorem 5 ([17, Theorem 4]) *For the parametric interval linear system (8), let $p_0 \in \mathbf{p}$ be such that $A_0 + L\mathrm{Diag}(p_0)R$ is invertible and put $D_0 = \mathrm{Diag}(p_0)$, $C := (A_0 + LD_0R)^{-1}$.*
(i) The solution $x = x(p,q)$ of (8) is related to $y = Rx(p,q)$ by the equations

$$x = Cb_0 + CFq + CL(D_0 t + h), \tag{9}$$
$$y = RCb_0 + RCFq + RCL(D_0 t + h), \tag{10}$$

[3] if there is a particular structure of the parameter dependencies defined by the sufficient conditions of the theorem

where

$$h = (D_0 - D)(y - t), \quad D = \mathrm{Diag}(p). \tag{11}$$

(ii) If there are vectors $w \geq 0$, $w' > 0$ *and* w'' *such that*

$$\begin{aligned} w' &\leq w - |D_0 - D|\,|RCL|w, \\ w'' &\geq |D_0 - D||RCb_0 + RCFq + RCLD_0t - t|, \end{aligned} \tag{12}$$

then

$$h \in \mathbf{h} := [-\alpha w, \alpha w], \quad \alpha = \max_i \frac{w_i''}{w_i'}. \tag{13}$$

Since (12) implies that $w' > 0$ if $w > 0$ and D_0 is close enough to D, in the implementation of Theorem 5 we take p_0 as the midpoint of $\mathbf{p}$, and w as the vector with all entries one. Then (12) is satisfied with

$$\begin{aligned} w' &:= w - |\mathrm{Diag}(p_0 - \mathbf{p})||RCL|w \\ w'' &:= |\mathrm{Diag}(p_0 - \mathbf{p})||RCb_0 + (RCF)\mathbf{q} + RCLD_0t - t|. \end{aligned}$$

If $w' > 0$, the enclosure (13) is valid. If this is not true, due to Theorem 7, we may compute the largest eigenvalue ϱ (= the spectral radius) of the matrix

$$M := |\mathrm{Diag}(p_0 - \mathbf{p})||RCL|. \tag{14}$$

If $\varrho < 1$, any $w > 0$, which is close to an associated eigenvector, makes $w' > 0$.

With the initial interval enclosure $\mathbf{h}$ from (13) and using interval arithmetic in (9)–(11), we get enclosures $\mathbf{x}$ for x, $\mathbf{y}$ for y and a generally improved enclosure for h. The enclosures can be further improved by iterating and intersecting with the previously computed enclosures. Thus we iterate

$$\begin{aligned} \mathbf{y} &= (RCb_0 + (RCF)\mathbf{q} + (RCL)(D_0t + \mathbf{h})) \cap \mathbf{y}, \\ \mathbf{h} &= (\mathrm{Diag}(p_0 - \mathbf{p})\mathbf{y} - \mathrm{Diag}(p_0 - \mathbf{p})t)) \cap \mathbf{h} \end{aligned}$$

until some stopping test holds, and then get the enclosure

$$\mathbf{x} := (Cb_0) + (CF)\mathbf{q} + (CL)(D_0t + \mathbf{h})$$

of all x satisfying (8). The stopping criterion could be as that one used in [10]. For the sake of comparing the parametric Krawczyck iteration and the method of Neumaier and Pownuk we applied the stopping criterion at line 12.6 of the refinement iteration for the parametric Krawczyck algorithm.

In what follows we generalize the conversion Theorem 4 for arbitrary parameter dependencies in the matrix $A(p)$ and the vector $b(p, q)$ in (7), which will allow the application of Theorem 5 to any parametric linear system.

4.2 Generalization to arbitrary parametric interval linear systems

Usually, in practical problems, the representation of a general parametric system (7) into the form (8) is guided by the physical properties of the problem modeled by the system. For example, the method of Neumaier and Pownuk in [10] is motivated by finite element models (FEM) of truss structures in structural mechanics. A special procedure, called Element-By-Element technique for linear interval equations resulting from static FE problems in solid and structural mechanics, is designed by Muhanna and Mullen [8] in order to obtain a representation in the form (8), which is different from the conventional FEM formulation and allows a better handling of the interval dependencies. Below we propose a methodology for obtaining a representation in the form (8) which is easy to obtain and the methodology is exact in exact arithmetic.

Assume that for every parameter p_k, $1 \le k \le K$, there is a decomposition of its coefficient matrix

$$A_k = \sum_{i=1}^{\alpha_k} A_{k_i} + A_k^*, \tag{15}$$

where $\alpha_k \ge 0$, A_{k_i} has more than one nonzero row and $A_{k_i}x$ satisfies the conditions of Theorem 4 for every i, $1 \le i \le \alpha_k$. In other words, each A_{k_i} involves linearly dependent nonzero rows and, if $A_k^* \neq 0$, it does not involve linearly dependent nonzero rows.

Theorem 6 *Let $\mathcal{K}_1$ be an index set of the parameters for which $\alpha_k > 0$ in the decomposition (15), then a parameter index k is involved α_k times in $\mathcal{K}_1$. Let $\mathcal{K}_2$ be an index set of the parameters for which $A_k^* \neq 0$ in (15). Let for $k \in \mathcal{K}_2$, the parameter p_k appear in n_k equations, and the index set of these equations be E_k. Let $k_1 = Card(\mathcal{K}_1)$, $k_2 = Card(\mathcal{K}_2)$. Define a* block*-diagonal matrix:*

$$D = \begin{pmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_{k_1+k_2} \end{pmatrix} \in \mathbb{R}^{s\times s}, \quad s = k_1 + \sum_{k\in\mathcal{K}_2} n_k$$

and block *matrices $L = (l_1|\dots|l_{k_1+k_2}) \in \mathbb{R}^{n\times s}$, $R = (r_1|\dots|r_{k_1+k_2})^\top \in \mathbb{R}^{s\times n}$, $t = (t_1^\top, \dots, t_{k_1+k_2}^\top)^\top$ in the following way.*

For $k \in \mathcal{K}_1$, $D_k = p_k$ and l_k, r_k, t_k are defined as in Theorem 4.

For $k \in \mathcal{K}_2$, $D_k = \mathrm{Diag}((p_k, \dots, p_k)) \in \mathbb{R}^{n_k\times n_k}$, $l_k = (l_{i_1}|\dots|l_{i_{n_k}})$, $r_k = (r_{i_1}|\dots|r_{i_{n_k}})$, $l_k, r_k \in \mathbb{R}^{n\times n_k}$, $t_k^\top = (t_{i_1}, \dots, t_{i_{n_k}})$, are such that

$$l_{i_j} = \begin{cases} 1 & \text{in its } i_j\text{-th element} \\ 0 & \text{elsewhere,} \end{cases} \qquad r_{i_j} = \left(A_{k,i_j\bullet}\right)^\top, \quad i_j \in E_k.$$

Vector t is obtained as a solution of the equation $LDt = b(p)$. Then $A(p) = A_0 + LDR$ and $b(p) = b_0 + LDt + Fq$.

Proof The proof follows from the construction. □

Theorem 6 defines the dependencies for each parameter p_k, $k \in \mathcal{K}_2$, separately in each equation of the system which contains p_k. In the equivalent LDR representation, obtained by the decomposition (15) and Theorem 6, the order of the parameters in the diagonal of D, and respectively the order of the columns in L and the rows in R can be changed without any change in the properties of the representation. Similarly, the sign of the GCD $g_k(x)$ in Theorem 4, with obvious changes in the corresponding column of L and row of R, can be changed.

The maximal size of the LDR representation, obtained by the decomposition (15) and Theorem 6, depends of the length of the diagonal of D, which is $s = \sum_{k=1}^{K} \operatorname{rank}\left((A_k|b_k)\right)$.

The LDR representation of a parametric matrix/system is not unique. The decomposition (15) and the conversion Theorems 4 and 6 specify a fixed equivalent representation of the parametric system in a way which preserves the rank of the matrices $(A_k|b_k)$, $1 \le k \le K$, and aims at accounting for both row and column dependencies of the parameters when estimating regularity of the parametric interval matrix (Theorem 7). Aiming at the best estimate of a parametric solution set, some modifications in the equivalent LDR representation are possible, as demonstrated by Example 6. If a parametric matrix involves more row- than column-dependencies of the parameters, then an equivalent representation based on

$$\left(A_k^\top\right)^\top = \left(L_k D_k R_k\right)^\top = L_k^\top D_k R_k^\top, \quad 1 \le k \le K \tag{16}$$

may have better regularity properties. This is illustrated by Example 8 below.

The next theorem gives a sufficient condition for the applicability of the method from Theorem 5 to any parametric interval linear system

$$A(p)x = b(p,q), \qquad p \in \mathbf{p} \in \mathbb{IR}^K,\ q \in \mathbf{q},$$

which has the equivalent representation

$$\begin{aligned}\left(A_0 + L\operatorname{Diag}(g(p))R\right)x = b_0 + L\operatorname{Diag}(g(p))t + Fq,\\ g(p) \in g(\mathbf{p}) \in \mathbb{IR}^s, \quad q \in \mathbf{q}\end{aligned} \tag{17}$$

with suitable numerical matrices L, R, numerical vector t and a parameter vector $g(p)$, which is obtained from p by involving some parameters p_k, $1 \le k \le K$, more than once. The vector $g(p) \in \mathbb{R}^s$ can be represented as

$$g_i(p) = \begin{cases} (p_j)^\top & \text{for } j \in \mathcal{K}_1 \subset \mathcal{K} = \{1,\dots,K\}, \\ \left(p_{j,1},\dots,p_{j,n_j}\right)^\top & \text{for } j \in \mathcal{K}_2 \subseteq \mathcal{K},\ \mathcal{K}_2 \ne \emptyset, \end{cases} \tag{18}$$

wherein $p_{j,l} \in \mathbf{p}_j$, $l = 1,\dots,n_j$, $j \in \mathcal{K}_2$, the index sets $\mathcal{K}_1$, $\mathcal{K}_2$ correspond to the equivalent representation from Theorem 6.

Theorem 7 *Let $A_0 + L\operatorname{Diag}(g(p))R$, $g(p) \in g(\mathbf{p})$, be an equivalent representation of $\{A(p), \mathbf{p}\}$ with particular numerical matrices L, R and $g(p) \in \mathbb{R}^s$ represented by (18). If $\varrho\left(\left|(RC_0^{-1}L)\operatorname{Diag}(g(p_0 - \mathbf{p}))\right|\right) < 1$ for any nonsingular $C_0 = A_0 + L\operatorname{Diag}(g(p_0))R$, $p_0 \in \mathbf{p}$, then $\{A(p), \mathbf{p}\}$ is regular.*

Proof Denote $\mathbf{M} := (RC_0^{-1}L)\mathrm{Diag}(g(p_0-\mathbf{p}))$. By [9, Corollary 3.2.3], $\varrho\,(|\mathbf{M}|) < 1$ is equivalent to $|\mathbf{M}|u < u$ for some vector $u > 0$. The latter is equivalent to $||\mathbf{M}||_u < 1$ for some vector $u > 0$. Since the last relation is equivalent to $||I-(I-\mathbf{M})\,||_u < 1$ for some $u > 0$, then by [9, Proposition 3.7.2] we have that $I - \mathbf{M}$ is an H-matrix and therefore a regular interval matrix, which implies for every $\tilde{p} \in \mathbf{p}$

$$\left(I - RC_0^{-1}L\mathrm{Diag}(g(p_0 - \tilde{p}))\right) y = 0 \;\Leftrightarrow\; y = 0.$$

Then the equation $x - C_0^{-1}L\mathrm{Diag}(g(p_0 - \tilde{p}))y = 0$ has the only solution $x = 0$, which implies regularity of $A(p)$ for each $p \in \mathbf{p}$. □

Theorem 7 shows that each multiply occurring parameter in the diagonal matrix D of the representation (17) can be considered as a distinct (different) parameter at each occurrence and can be renamed. The renamed parameters corresponding to one multiply occurring parameter vary within the same interval bounds.

The condition from Theorem 7 is only a sufficient condition for regularity of a parametric matrix $\{A(p), \mathbf{p}\}$. In general, $\{A(p), \mathbf{p}\}$ can be regular and Theorem 7 be not satisfied, cf. Example 4.

Example 4 Consider the parametric interval matrix

$$\begin{pmatrix} 3p & 1 \\ -1 & 3p-1 \end{pmatrix}, \quad p \in [0,1]. \tag{19}$$

The united parametric solution set $\Sigma_{uni}(A(p), 0, \mathbf{p})$ has explicit characterization by the following system of inequalities, obtained by [15, Theorem 4.1],

$$\begin{aligned} |3x_1/2 + x_2| &\le 3|x_1|/2, \\ |-x_1 + x_2/2| &\le 3|x_2|/2, \\ \left|3x_1^2 + 3x_1x_2 + 3x_2^2\right| &\le 0. \end{aligned}$$

The latter system of inequalities has the only real solution $x_1 = x_2 = 0$. Therefore, the parametric interval matrix (19) is regular. By Theorem 6, the parametric matrix (19) has the equivalent representation $A_0 + L\mathrm{Diag}((p,p))R$, wherein

$$A_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then, for $p_0 = \check{p}$, $A(\check{p})$ is nonsingular and

$$\left|\left(RA^{-1}(\check{p})L\right)\begin{pmatrix} \frac{1}{2} - [0,1] & 0 \\ 0 & \frac{1}{2} - [0,1] \end{pmatrix}\right| = \begin{pmatrix} \frac{3}{7} & \frac{6}{7} \\ \frac{6}{7} & \frac{9}{7} \end{pmatrix}.$$

The spectral radius (= the maximal eigenvalue) of the last numerical matrix is $(6 + 3\sqrt{5})/7 \approx 1.815$, and thus Theorem 7 is not satisfied.

4.3 Applicability shown by numerical examples

Theorem 6 together with the decomposition (15) generalize the first conversion Theorem 4 for parameters that do not satisfy the latter theorem and for parameters that satisfy Theorem 4 in part of the equations of the system. Here we illustrate the application of the conversion Theorem 6 and its features. The expanded scope of applicability of the generalized method of Neumaier and Pownuk (Theorem 5), due to the new sufficient condition (Theorem 7) for regularity of a parametric interval matrix, and its advantages over other enclosure methods, are also illustrated by numerical examples. Theorem 7 provides a more powerful sufficient condition for regularity of a parametric interval matrix than the conditions for strong regularity. This means that a parametric matrix may be regular by Theorem 7 but is not strongly regular, illustrated by Example 5.

Example 5 Consider the parametric interval linear system

$$\begin{pmatrix} 1+p_1 & 2+p_1 & 1+p_2 \\ -0.5-p_2 & 1+p_2 & 1+p_1 \\ 1+p_2 & -1-p_2 & 1+p_1 \end{pmatrix} x = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}, \quad \begin{array}{l} p_1 \in [-0.5, 0.5], \\ p_2 \in [-0.5, 0.5]. \end{array}$$

The parametric interval matrix is not strongly regular. For the coefficient matrices A_1, A_2 we have $A_1 = A_1' + A_1^*$, where $A^*_{1,11} = A^*_{1,12} = 1$, $A^*_{1,ij} = 0$ for $ij \notin \{11, 12\}$, and $A_1'x$ satisfies Theorem 4 with $g_1'(x) = x_3$. The matrix $A_2 = A_2' + A_2^*$, where $A^*_{2,13} = 1$, $A^*_{2,ij} = 0$ for $ij \neq 13$, and $A_2'x$ satisfies Theorem 4 with $g_2'(x) = -x_1 + x_2$. The conversion Theorem 6 gives an equivalent representation with $\mathrm{Diag}((p_1, p_1, p_2, p_2))$, $t = 0 \in \mathbb{R}^4$ and the matrices

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

Then Theorem 7 is satisfied and the method from Theorem 5 (with $\delta = 10^{-3}$) yields the following enclosure of the parametric solution set

$$([-24.001, 24.001],\ [-16.001, 16.001],\ [-12.001, 24.001])^\top.$$

Visualization of the parametric united solution sets, or their projections, and computing the corresponding interval hulls are done by the methods from [14]. The outwardly rounded interval hull of the parametric solution set is

$$([-12, 1.60770], [-10, .803848], [3.74999, 18])^\top.$$

While the conversion Theorem 4 applies to a class of parametric linear systems having united solution set with linear boundary, Theorem 6, respectively the method from Theorem 5, are applicable also to parametric solution sets that do not have linear boundary, see Examples 5, 6, 7, 8.

The conversion Theorem 6 has a specific feature that affects the application of Theorem 5. This feature is illustrated in Examples 6 and 8.

Example 6 Consider the parametric interval linear system

$$\begin{pmatrix} \frac{1}{2}-p_2 & p_2 & p_1 \\ p_1 & -p_2 & p_3 \\ p_1 & p_3 & 1 \end{pmatrix} x = \begin{pmatrix} p_2 \\ 2p_2 \\ 3p_2 \end{pmatrix}, \quad \begin{array}{l} p_1 \in [\frac{3}{4}, \frac{5}{4}], \\ p_2 \in [\frac{1}{2}, \frac{3}{2}], \\ p_3 \in [\frac{1}{2}, \frac{3}{2}]. \end{array}$$

For the matrices A_1, A_2 we have $A_1 = A_1' + A_1^*$, where $A^*_{1,13} = 1$, $A^*_{1,ij} = 0$ for $ij \neq 13$ and A_1' satisfies Theorem 4 with $g_1'(x) = x_1$. Matrix $A_2 = A_2' + A_2^*$, where $A^*_{2,11} = -1$, and $A_2'x$ satisfies Theorem 4 with $g_2'(x) = x_2$. A_3x does not satisfy Theorem 4. The conversion Theorem 6 yields an equivalent representation defined by $\mathrm{Diag}((p_1, p_1, p_2, p_2, p_2, p_3, p_3))$, $t = (0, 0, -2, 3, 3, 0, 0)^\top$ and

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The fifth element of the diagonal vector, the fifth column of L and the fifth row of R are introduced to account for the presence of p_2 in the third equation. If we remove these in the diagonal matrix, in L and in R, Theorem 7 will be satisfied, that is the parametric interval matrix is regular. However, we need the above representation for the application of Theorem 5. The latter finds that the computed w' is not positive. Due to R involving a zero row, the subsequent computation of the maximal eigenvalue of $|RCL||D_0 - \mathbf{D}|$ and its corresponding eigenvector gives $\varrho < 1$ and eigenvector having zero at its fifth component. Nevertheless, since this zero component affects only the right-hand side of the system, we continue the computations with w equal to this eigenvector and exclude the fifth component of w' (and w'') in the computation of α in (13). Thus, we obtain (with $\delta = 10^{-5}$) the following interval enclosure of the parametric united solution set of the system

$$([-81.334, 84.001], [-38.667, 39.667], [-73.001, 75.334])^\top.$$

The projection of the parametric solution set on the x_1, x_2 coordinate space is presented in Figure 1. The interval hull of the parametric solution set is

$$([0.59999, 1.70691], [-0.20271, 0.859375], [0.24999, 3.81082])^\top.$$

Since the parametric interval linear system is also strongly regular, the parametric Krawczyck method with $\varepsilon = 0.1$ converges in 17 initial iterations and with sufficiently many refinement iterations results in an interval vector, which has the same quality as the solution enclosure obtained by the parametric Bauer-Skeel method executed in rational arithmetic

$$([-30.0001, 32.6667], [-14.7778, 15.7778], [-28.2223, 30.5556])^\top.$$

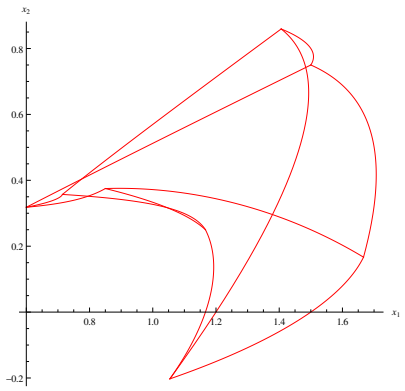

Fig. 1 Projection of the parametric united solution set to the system from Example 6 on the x_1, x_2 coordinate space.

We see above that the method from Theorem 5 yields a worse solution enclosure than the parametric Krawczyck iteration. Note that if the fifth column of L is replaced by $(1,2,3)^\top$ and with $t = (0,0,0,0,1,0,0)^\top$, the method from Theorem 5 yields the solution enclosure

$$([-37.33393, 40.00060], [-18.00029, 19.00029], [-33.66721, 36.00054])^\top .$$

Remark 1 Example 5 in [17] reports that the single step parametric Bauer-Skeel method [24,3] yields a better enclosure of the considered solution set than the method of Neumaier and Pownuk [10]. This is not true and resulted from a bug. Both methods yield interval vectors of similar quality for the considered example. Example 1 from [17] can be used as another illustration of the problems, for which the method from Theorem 5 yields a worse enclosure than the parametric Krawczyck iteration or the parametric single step method.

If the parametric matrix involves a parameter, which satisfies the conversion Theorem 4 at least partly, the application of the method from Theorem 5 may be superior to the parametric Krawczyck method (respectively the parametric Bauer-Skeel method) even when the parametric matrix $\{A(p), \mathbf{p}\}$ is strongly regular, cf. Example 7.

Example 7 Consider the parametric interval linear system

$$\begin{pmatrix} \frac{1}{2} - p_2 & p_2 & 2p_1 \\ p_2 & -p_2 & p_3 \\ 2p_1 & p_3 & -5 + 3p_2 \end{pmatrix} x = \begin{pmatrix} 3p_1 \\ 2p_3 \\ 2p_2 \end{pmatrix}, \quad \begin{array}{l} p_1 \in [\frac{3}{4}, 1\frac{1}{4}], \\ p_2 \in [\frac{1}{2}, \frac{3}{2}], \\ p_3 \in [\frac{1}{2}, \frac{3}{2}]. \end{array}$$

The parametric interval matrix is strongly regular due to regularity of $\mathbf{B} = \square\{A^{-1}(\check{p})A(p) \mid p \in \mathbf{p}\}$. The parametric Krawczyck method with $\varepsilon = 0.1$ converges in 13 initial iterations and with sufficiently many refinement iterations results in an interval vector, which has the same quality as the solution enclosure obtained by the parametric Bauer-Skeel method in rational arithmetic

and then outwardly rounded to

$$([-11.1367, 14.7156], [-17.3927, 19.7085], [-1.65595, 4.39279])^\top . \qquad (20)$$

In the application of Theorem 6, $A_2 = A_2' + A_2^*$, where $A^*_{2,33} = 3$, $A^*_{2,ij} = 0$ for $ij \neq 33$, and A_2' satisfies Theorem 4 with $g_2'(x) = x_1 - x_2$. The equivalent representation is defined by $\mathrm{Diag}((p_1, p_1, p_2, p_2, p_3, p_3))$, $t = (3, 0, 0, 0, 2, 2)^\top$ and the matrices

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & 0 & 2 \\ 2 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Theorem 7 is satisfied and the method from Theorem 5 (with $\delta = 10^{-6}$) yields the following interval enclosure of the parametric united solution set

$$([-1.37087, 4.94982], [-2.44828, 4.76407], [0.5634128, 2.173430])^\top . \qquad (21)$$

The enclosure (20) overestimates the enclosure (21) by $(81\%, 85\%, 79\%)^\top$. The projection of the parametric solution set on the x_1, x_2 coordinate space is presented in Figure 2. The interval hull of the parametric solution set is

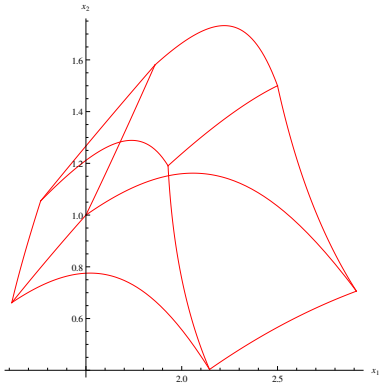

Fig. 2 Projection of the parametric united solution set to the system from Example 7 on the x_1, x_2 coordinate space.

$$([1.11290, 2.91177], [0.403225, 1.73237], [0.999999, 1.54839])^\top .$$

A parametric interval matrix $\{A(p), \mathbf{p}\}$ may be strongly regular due to regularity of $\mathbf{B}' = \square\{A(p)A^{-1}(\check{p}) \mid p \in \mathbf{p}\}$ despite of $\mathbf{B} = \square\{A^{-1}(\check{p})A(p) \mid p \in \mathbf{p}\}$ being not regular. In this case, all existing enclosure methods for the united parametric solution set of $A(p)x = b(p)$, $p \in \mathbf{p}$, which require or check

regularity of $\mathbf{B}$, fail. A recent paper on strong regularity [26] also considers only half of Definition 3, the one restricted to $\mathbf{B}$. The next example shows that Theorem 5 with appropriate LDR decomposition may provide a solution enclosure in the considered case.

Example 8 Consider the parametric interval linear system

$$\begin{pmatrix} \frac{1}{2} - p_2 & p_1 & p_1 \\ p_2 & -p_2 & p_3 \\ p_1 & p_3 & 1 \end{pmatrix} x = \begin{pmatrix} p_2 \\ 2p_2 \\ 3p_2 \end{pmatrix}, \quad \begin{array}{l} p_1 \in [\frac{3}{4}, \frac{5}{4}], \\ p_2 \in [\frac{1}{2}, \frac{3}{2}], \\ p_3 \in [\frac{1}{2}, \frac{3}{2}]. \end{array}$$

For this system $\mathbf{B} = \square\{A^{-1}(\check{p})A(p) \mid p \in \mathbf{p}\}$ is not regular while the matrix $\mathbf{B}' = \square\{A(p)A^{-1}(\check{p}) \mid p \in \mathbf{p}\}$ is regular. By Definition 3 the parametric matrix is strongly regular and therefore regular. However, all enclosure methods based on regularity of $\mathbf{B}$ fail on this system. The decomposition (15) and conversion Theorem 6 imply a matrix M (in (14)) whose spectral radius is greater than one and the method from Theorem 5 fails. In order to account for the row dependencies in both p_1, p_2, we consider an equivalent representation based on (16). Thus, $A^\top(p) - A_0^\top$ has an equivalent representation with $\mathrm{Diag}((p_1, p_1, p_2, p_2, p_2, p_3, p_3))$, $t = (0, 0, 2, -1, 3, 0, 0)^\top$ and the matrices

$$L = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

In this representation, the fifth element in the diagonal matrix, the fifth column of L and the fifth row of R are introduced in order to account for the dependencies between p_2 in the matrix and the third element of right-hand side vector. Removing the fifth columns and rows, we obtain a representation of the parametric matrix $A(p) = A_0 + R^\top D L^\top$ for which Theorem 7 is satisfied. The method from Theorem 5, applied by accounting for the zero row in $L^\top$ as in Example 6, yields (with $\delta = 10^{-5}$) the following interval enclosure of the parametric united solution set

$$([-41.11159, 43.77826], [-43.11161, 44.11161], [-51.88949, 54.22282])^\top. \quad (22)$$

Although this enclosure is quite rough, since the smallest outer enclosure of the parametric united solution set is

$$([0.69999, 1.7157], [-0.4501, 1.0938], [0.3818, 3.3244])^\top,$$

(22) is the only available solution enclosure.

5 Application of Parametric Linear Systems with Multiple Right-Hand Sides

In this section we first present an algorithm (Algorithm 2) implementing the method from Theorem 5, [10], for parametric linear systems with multiple right-hand sides independent of the parametric matrix. For simplicity of the presentation, special cases discussed in examples 6, 8, when the numerical matrix R involves some zero rows, are not reflected by this algorithm.

Algorithm 2 *Iteration for parametric interval systems*

$$(A_0 + L\mathrm{Diag}(p)R)\,x = B_0 + Fq, \qquad p \in \mathbf{p},\ q \in \mathbf{q},$$

with $A_0 \in \mathbb{R}^{m\times m}$, $L \in \mathbb{R}^{m\times s}$, $R \in \mathbb{R}^{s\times m}$ *and* $\mathrm{Diag}(p) \in \mathbb{R}^{s\times s}$; *matrices* $B_0 = (b_{0,1}|\ldots|b_{0,n}) \in \mathbb{R}^{m\times n}$, *and* $F = (F_1|\ldots|F_n) \in \mathbb{R}^{m\times n\mu}$ *correspond to* λ *right-hand sides* $b_{0,i} + F_i q$, $i = 1,\ldots,m$, $q \in \mathbb{R}^{\mu}$.

Input: *matrices* A_0, $L, R, \mathrm{Diag}(p)$, B_0, F; *interval vectors* $\mathbf{p}$, $\mathbf{q}$; *integer number* maxItr *and* $\varepsilon \in \mathbb{R}_+$.

Output: *interval matrix* $\mathbf{X}$, *such that* $\square\,\Xi^p_{uni} \subseteq \mathbf{X}$, *or the error message "Regularity of the parametric matrix cannot be proven.".*

1. $C \approx (A_0 + L\mathrm{Diag}(\check{p})R)^{-1}$; $w = (1,\ldots,1)^\top \in \mathbb{R}^s$; $w' = w - \mathrm{Diag}(\hat{p})|RCL|w$;
2. **if** $w' \not> 0$ **then**
3. compute the largest eigenvalue ϱ and its corresponding eigenvector w for the matrix $\mathrm{Diag}(\hat{p})|RCL|$;
4. **if** $\varrho < 1$ and $w > 0$ **then** $w' = w - \mathrm{Diag}(\hat{p})|RCL|w$;
5. **else return:** *"Regularity of the parametric matrix cannot be proven.";*
6. $W'' = \mathrm{Diag}(\hat{p})\left|RCB_0 + \left((RCF)^\top \mathbf{q}\right)^\top\right|$;
7. $\mathbf{H} = (\mathbf{h}_1|\ldots|\mathbf{h}_n)$, $\mathbf{h}_j = [-\alpha_j, \alpha_j]w$, $\alpha_j = \max_i\{W''_{ij}/w'_i\}$, $j = 1,\ldots,n$;
8. $\mathbf{Y} = RCB_0 + \left((RCF)^\top \mathbf{q}\right)^\top + (RCL)\,\mathbf{H}$;
9. $l = 0$;
10. **repeat**
11. $\mathbf{H}^{\mathrm{old}} = \mathbf{H}$; $\mathbf{Y}^{\mathrm{old}} = \mathbf{Y}$;
12. $\mathbf{H} = (\mathrm{Diag}(\hat{p})\mathbf{Y}) \cap \mathbf{H}$;
13. $\mathbf{Y} = \left(RCB_0 + \left((RCF)^\top \mathbf{q}\right)^\top + (RCL)\,\mathbf{H}\right) \cap \mathbf{Y}$;
14. $l = l + 1$;
15. $\nu = \max_j\left\{\sum_{i=1}^n \mathrm{width}(\mathbf{H}^{\mathrm{old}}_{ij}) - \sum_{i=1}^n \mathrm{width}(\mathbf{H}_{ij})\right\}$;
16. **until** $\nu > \varepsilon$ **and** $l \leq$maxItr;
17. **return** $\mathbf{X} = CB_0 + \left((CF)^\top \mathbf{q}\right)^\top + (CL)\,\mathbf{H}$.

Assuming $w' > 0$, the computational complexity of Algorithm 2 is $O(m^3 + Itr(ms\max\{m,s\} + s^2 n))$, where $O(ms\max\{m,s\} + s^2 n)$ is the complexity of computing $(RCL)\,\mathbf{H}$.

In enclosing a $\Sigma_{\mathrm{uni}}(A(p), b(p), \mathbf{p})$ the ultimate goal is that the enclosure be as close to the minimal one, $\square\Sigma^p_{\mathrm{uni}}$, as possible. Also, every new interval

enclosure method tests its quality by comparing the enclosing interval vector to $\square\Sigma^p_{\text{uni}}$. In general computing $\square\Sigma^p_{\text{uni}}$ is an NP-hard problem [20]. A large class of parametric linear systems possess the property that the componentwise infimum and supremum of their Σ^p_{uni} are attained at particular end points of $\mathbf{p}$. We call this property *end point* property of the united parametric solution set. Examples of such problems are all parametric systems where (some of) the parameters satisfy Theorem 4; for practical domain-specific problems see [10, 17, 19] and the references given therein. The end point property is possessed by many solution sets even if the solution set has nonlinear boundary (it is not monotone with respect to all the parameters). The most common approach for computing $\square\Sigma^p_{\text{uni}}$ for solution sets that possess the discussed end point property is the combinatorial approach. By the combinatorial approach

$$\square\Sigma^p \;=\; [\min_{u\in\{\pm1\}^K} A^{-1}(p^u)b(p^u),\;\; \max_{u\in\{\pm1\}^K} A^{-1}(p^u)b(p^u)],$$

wherein K is the dimension of the parameter vector p. However, even for modest number of parameters involved in the system, the combinatorial approach requires solving a prohibitively big number of point linear systems, see, e.g., [19] and Example 9. A very efficient technique, which proves the end point property of a parametric solution set w.r.t a given parameter and simultaneously finds which parameter end point contributes to the inf/sup of $\square\Sigma^p_i$, $i = 1, \ldots, n$, is proposed in [13] and further studied in [6]. This technique does computer-assisted proof of the global and/or local monotonicity of the solution components with respect to each interval parameter. The computer-assisted proof is based on guaranteed interval enclosures for the partial derivatives of the parametric solution with respect to each interval parameter. Details about the methodology can be found in [13]. Here we make a significant contribution to the computational efficiency of this methodology by considering the partial derivatives with respect to all parameters simultaneously in a parametric linear system with multiple right-hand sides. Algorithm 3 below presents the computer-assisted proof of the end point property for the lower bound of a solution component, i.e., $\inf \Sigma^p_i$. The algorithm proves global and/or local monotonicity of a solution component with respect to given interval parameters and the type of the monotonicity determines which parameter end points yield the corresponding exact bound of the solution component. The same algorithm, modified accordingly, can be used for $\sup \Sigma^p_i$.

Algorithm 3 *Proving the end point property of* $\inf \Sigma^p_i$ *to a parametric interval linear system* $A(p)x = b(p)$, $p \in \mathbf{p} \in \mathbb{IR}^K$.

1. $L_+ = L_- = \emptyset$; $L_0 = \{1, \ldots, K\}$; $m = 0 \in \mathbb{R}^K$;
2. **repeat**
3. *find* $\mathbf{x}^*$ *by a self-verified solver of the parametric linear system*

$$A(p^-_{L_+}, p^+_{L_-}, p_{L_0})x(p_{L_0}) = b(p^-_{L_+}, p^+_{L_-}, p_{L_0}), \qquad p_{L_0} \in \mathbf{p}_{L_0}.$$

4. *find* $\mathbf{X}$ *by a self-verified solver of the parametric linear system with multiple right-hand sides*

$$A(p^-_{L_+}, p^+_{L_-}, p_{L_0})X(p_{L_0}) = B(p^-_{L_+}, p^+_{L_-}, p_{L_0}, x^*), \quad p_{L_0} \in \mathbf{p}_{L_0}, x^* \in \mathbf{x}^*$$

wherein, for $\nu \in L_0$,

$$X_{\bullet\nu}(p_{L_0}) = \frac{\partial x(p^-_{L_+}, p^+_{L_-}, p_{L_0})}{\partial p_\nu}, \qquad B_{\bullet\nu}(p^-_{L_+}, p^+_{L_-}, p_{L_0}, x^*) = b_\nu + A_\nu x^*.$$

5. $s = \operatorname{sign}(\mathbf{X}_{i\bullet})$;
6. **for** $\nu \in L_0$
7. **if** $s_\nu = 1$ **then** $L_+ = L_+ \cup \{\nu\}$; $L_0 = L_0 \setminus \{\nu\}$; $m_\nu = s_\nu$;
8. **if** $s_\nu = -1$ **then** $L_- = L_- \cup \{\nu\}$; $L_0 = L_0 \setminus \{\nu\}$; $m_\nu = s_\nu$;
9. **until** $L_0 \neq \emptyset$ *and* L_+, L_- *are updated;*
10. **return** m.

A key requirement of Algorithm 3 is the usage of self-verified solvers for parametric interval linear systems with single (step 3) and multiple (step 4) right-hand sides providing *guaranteed* solution enclosures in floating-point arithmetic. We recommend the parametric Krawczyk iteration (Section 3). The method of Neumaier and Pownuk can be used if the parametric matrix is not strongly regular and Theorem 7 is satisfied. The latter usually happens for large parameter intervals and in this case the overestimation of the method is much bigger than the round-off errors. Example 3 shows why methods, which do not provide guaranteed enclosure in floating-point arithmetic, should not be used in computer-assisted proofs.

Solving parametric linear systems with multiple right-hand sides improves essentially the computational efficiency of the above methodology by saving the inversion of as many matrices as the number of the parameters whose monotonicity is to be proven. Various minor improvements in the implementation and the application of the discussed methodology are also possible. In the example below we demonstrate that computer-assisted proofs in floating-point arithmetic by solving parametric linear systems with multiple right-hand sides makes obtaining the interval hull of a parametric solution set, possessing the end point property and depending on many interval parameters, feasible.

Example 9 Consider a finite element model of a one-bay 20-floor truss cantilever presented in Fig. 3, after [7]. There are 42 nodes and 101 elements, resulting in a parametric linear system with 81 variables and 101 uncertain parameters. The bay is L; every floor is $0.75L$. 20 horizontal loads are applied at the left nodes. The boundary conditions are determined by the supports: at A the support is a pin, at B the support is roller. A complete description of this example is available in [7].

We assume 5% uncertainty in the modulus of elasticity E_k of each element, that is $E_k \in [0.975, 1.025]$, $k = 1, \ldots, 101$. The goal is to find enclosures for the normalized displacements U_x, U_y at corner $\mathbf{D}$.

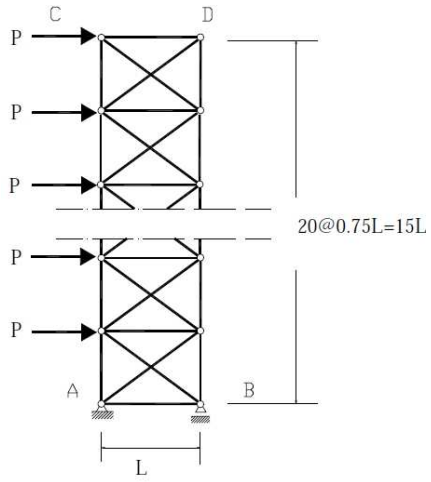

Fig. 3 One-bay 20-floor truss cantilever after [7].

solution component U_x						
iteration	1	2	3	4	5	6
mon. parameters	21	29	24	14	10	3

solution component U_y					
iteration	1	2	3	4	5
mon. parameters	20	29	28	19	5

Table 1 Number of the parameters whose global/local monotonicity properties are proven at each iteration of the Algorithm 3 applied to both lower and the upper endpoints of the solution components U_x and U_y of the system from Example 9.

The method of Neumaier and Pownuk is designed especially for such parametric systems and an outer solution enclosure is reported in [10]. For the considered uncertainty intervals the parametric Krawczyck iteration is also applicable. By Theorem 4 the parametric solution set of the considered system has linear boundary, which means that the exact interval hull of the solution set is attained at particular end points of the parameter intervals, [17]. Since the number of the interval parameters is 101, the combinatorial approach requires solving $2^{101} \approx 2.5353 * 10^{30}$ linear systems which is prohibitively large. Therefore, we run Algorithm 3 separately for the lower and the upper endpoints of the solution components U_x and U_y, in order to find which interval end points generate the exact lower and upper bounds of these solution components, respectively. Table 1 presents the number of the parameters whose global/local monotonicity properties are proven at each iteration of the Algorithm 3. At the first iteration the algorithm proves global monotonicity, while in the subsequent iterations the monotonicity is local.

For both the lower and the upper bounds of the solution component U_x, all parameters have the same monotonicity type presented by arrows below ($\downarrow$

means monotone decreasing, ↑ means monotone increasing).

$$m = (\ \downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,$$
$$\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,$$
$$\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\downarrow,\uparrow)^{\top}.$$

With the above end point properties, the sharpest interval enclosure of U_x is

$$[17764.344224458626, 18675.352625309442].$$

Global/local monotonicity properties of the solution component U_y w.r.t the parameters and to both the lower and the upper bounds are as follows

$$m = (\ \uparrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,[\downarrow,\uparrow],\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,$$
$$\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,$$
$$\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow,\uparrow,\uparrow,\uparrow,\downarrow,\downarrow)^{\top}.$$

Note that U_y is not monotone w.r.t E_{11} but has the end point property: both lower and upper bounds of U_y are attained at the upper bound of $\mathbf{E}_{11}$. With the above end point properties we obtain the sharpest interval enclosure of U_y

$$[-828.0756224626116, -786.3032528207145].$$

6 Conclusion

Proving that the exact interval hull of the parametric united solution set to the matrix equation $A(p)X = B(p)$, $p \in \mathbf{p}$, is the same as the exact interval hull of the parametric solution set to the corresponding parametric system with multiple right-hand sides, we justify the considerably improved computational complexity of the enclosure methods generalized to the matrix equation.

We discussed some details improving the implementation and the application of the parametric Krawczyck iteration with reduced computational complexity for parametric interval systems with multiple right-hand sides. The applicability of the method of Neumaier and Pownuk, known as the best one so far, is greatly expanded to parametric interval linear systems with arbitrary linear dependencies. Numerous numerical examples in Section 4.3 demonstrate that the presented new sufficient condition for regularity of a parametric interval matrix, applied to the generalized method of Neumaier and Pownuk, expands the scope of applicability of the latter and improves the quality of the solution enclosure for many parametric interval linear systems.

Although computing the exact interval hull of a parametric united solution set is NP-hard in general, we demonstrated in Section 5 that for a very large class of parametric united solution sets, which possess the end point property, a methodology based on computer-assisted proofs in floating point arithmetic and parametric linear systems with multiple right-hand sides makes computing the interval hull feasible even for parametric systems involving a big number

of interval parameters. This improved methodology allows computing sharp enclosures of the parametric united solution sets even when the end point property is valid only for some of the interval parameters.

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