SIAM J. MATRIX ANAL. APPL.
Vol. 36, No. 2, pp. 615–633

SOLVABILITY OF PARAMETRIC INTERVAL LINEAR SYSTEMS OF EQUATIONS AND INEQUALITIES*

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This work is dedicated to the memory of my father

Abstract. Considered are linear algebraic systems of equations and/or inequalities involving linear dependencies between interval-valued parameters. The goal is to characterize weak and strong solvability of such systems. Since any parametric interval equation is equivalent to a system of parametric interval inequalities, the focus is on the latter. We discuss an algorithmic procedure for characterizing the weak solutions, and thus weak solvability, of the considered systems. The set of strong solutions to a mixed system of parametric interval equations and inequalities is described explicitly. Applying the theory of weak solvability of parametric interval systems, we prove necessary and sufficient conditions for strong solvability of parametric interval inequalities.

Key words. interval linear systems, dependent data, weak solvability, strong solvability

AMS subject classification. 65G40

DOI. 10.1137/140966459

1. Introduction.

The problems of weak and strong solvability of systems of linear interval equations or inequalities, where the input data are independent intervals, have been studied for a long time; [20] contains a detailed overview. For some recent results and generalizations, see, for example, [10], [19], [21], [25], and the references therein.

The quest for a high level of detail and realism in modeling of practical problems requires considering linear algebraic systems of equations or/and inequalities involving dependencies between the interval model parameters; see, e.g., [2], [4], and the references therein. Thus far, mainly solutions of various problems related to parametric systems of interval equations have been considered. For example, [1], [7], [15], and [17] discuss how to obtain an explicit description of the set of all weak solutions (called the united parametric solution set) to a parametric system of interval equations; methods for outer interval estimation of the united parametric solution set are proposed by many authors; see the overviews presented in, e.g., [4], [11], and [14]. While strong solvability of interval equations or inequalities is studied mainly in the nonparametric case, a recent work of Hladík [11] proves necessary and sufficient conditions for strong solvability of interval inequalities involving some simple data dependencies. Strong solutions to parametric systems of interval equations can be considered as a special case of the more general class of parametric AE-solution sets [18]; the latter generalize the nonparametric AE-solution sets of interval equations [23]. Weak and strong solvability of interval equations and/or inequalities are closely related to the solution concepts of interval linear programming problems [8], [9], [12], [13]. Therefore, interval linear programming is a large application domain for the results on weak and strong solvability of parametric interval systems. In [8] the characterization of strong solvability for parametric interval linear systems is mentioned as an open problem.

*Received by the editors April 24, 2014; accepted for publication (in revised form) by A. Frommer February 5, 2015; published electronically May 26, 2015.
http://www.siam.org/journals/simax/36-2/96645.html

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In this work we consider linear algebraic systems of equations and/or inequalities which involve general linear dependencies between interval-valued parameters. The goal is to characterize weak and strong solvability as well as the strong solutions to such systems. Contrary to the nonparametric interval systems (see [10]), any parametric system of interval equations can be represented as an equivalent system of parametric interval inequalities. The same is true for any mixed system of parametric equations and inequalities. Therefore, the main focus of this work is parametric systems of interval inequalities. All the results for the latter are applicable to the other classes of parametric interval systems. In section 2 we discuss how to obtain an explicit description of the set of weak solutions to a parametric interval system. This problem was considered so far for parametric systems of interval equations [1], [7], [15], [17]. Interval systems of equations and inequalities involving some very specific dependencies are considered in [6]. Here, the so-called Fourier–Motzkin elimination of existentially quantified parameters is presented for parametric systems of interval inequalities. Explicit description of the set of strong solutions to a parametric interval system of equations and inequalities is derived in section 3. Systems of equations or inequalities are considered as special cases. A nonempty strong solution set presents a sufficient condition for strong solvability of the considered parametric interval system. In section 4 we prove necessary and sufficient conditions for strong solvability of parametric interval inequalities involving either general or some particular linear dependencies. The proofs in this section apply the theory of weak solvability of parametric interval systems. The paper ends with some conclusions.

Denote by $\mathbb{R}^{m\times n}$ and $\mathbb{R}^m := \mathbb{R}^{m\times 1}$ the set of real $m\times n$ matrices and the set of real vectors with m components, respectively. A real compact interval is $\mathbf{a} = [\underline{a}, \overline{a}] := \{a \in \mathbb{R} \mid \underline{a} \le a \le \overline{a}\}$. By $\mathbb{IR}^{m\times n}, \mathbb{IR}^m := \mathbb{IR}^{m\times 1}$ we denote the sets of interval $m \times n$ matrices and interval m-vectors, respectively. Consider a parametric system of linear algebraic equations

$$A(p)x = b(p) \tag{1.1}$$

and/or linear algebraic inequalities

$$C(p)y \le d(p), \tag{1.2}$$

where the parameters $p_1, \dots, p_K$ are considered as uncertain and varying within given intervals

$$p = (p_1, \dots, p_K)^\top \in (\mathbf{p}_1, \dots, \mathbf{p}_K)^\top = \mathbf{p}. \tag{1.3}$$

Although the dependencies between the parameters in (1.1) and/or (1.2) may be nonlinear in general, in this paper we consider parametric linear systems (1.1) and/or (1.2) involving *only* the linear structure of the uncertain (interval) parameters. That is,

$$\begin{aligned} A(p) &:= A_0 + \sum_{k=1}^{K} p_k A_k, \qquad & b(p) &:= b_0 + \sum_{k=1}^{K} p_k b_k, \\ C(p) &:= C_0 + \sum_{k=1}^{K} p_k C_k, \qquad & d(p) &:= d_0 + \sum_{k=1}^{K} p_k d_k, \end{aligned} \tag{1.4}$$

where $A_k \in \mathbb{R}^{m_1\times n_1}$, $C_k \in \mathbb{R}^{m_2\times n_2}$, $b_k \in \mathbb{R}^{m_1}$, $d_k \in \mathbb{R}^{m_2}$, $0 \le k \le K$, are given real matrices and vectors, respectively, and m_1, m_2, n_1, n_2 are positive integers.

For $\mathbf{a} = [\underline{a}, \overline{a}]$, define midpoint $\check{\mathbf{a}} := (\underline{a} + \overline{a})/2$, radius $\hat{\mathbf{a}} := (\overline{a} - \underline{a})/2$, and absolute value (magnitude) $|\mathbf{a}| := \max\{|\underline{a}|, |\overline{a}|\}$. These functions are applied to interval vectors and matrices componentwise. Without loss of generality and in order to have a unique representation (1.4), we assume that $\hat{\mathbf{p}}_k > 0$ for all $1 \leq k \leq K$. For a given index set $\Pi = \{\pi_1, \ldots, \pi_K\}$, denote $p_\Pi = (p_{\pi_1}, \ldots, p_{\pi_K})$. $\bigvee$, $\vee$ denote the logical "Or," $\bigwedge$, $\wedge$ denote the logical "And," and $\bigcup$, $\bigcap$ denote union and intersection, respectively. For a matrix $A_k \in \mathbb{R}^{m\times n}$, $A_{k,\bullet j}$ denotes the jth column of A_k and $A_{k,i\bullet}$ denotes the ith row of A_k. Denote by $\{\pm 1\}^m$ the set of all ± 1 vectors in $\mathbb{R}^m$, i.e., $\{\pm 1\}^m := \{x \in \mathbb{R}^m \mid |x| = (1, \ldots, 1)^\top\}$. For any vector v, (in)equalities like $v = 0$ are to be read $v = (0, \ldots, 0)^\top$.

Under a system of parametric interval linear equations

$$A(p)x = b(p), \quad p \in \mathbf{p}, \tag{1.5}$$

we understand the *family* of all systems of parametric linear equations (1.1) with data satisfying (1.3), (1.4). Similarly, a system of parametric interval linear inequalities will be denoted by

$$C(p)x \leq d(p), \quad p \in \mathbf{p}, \tag{1.6}$$

and a mixed system of parametric interval linear equations and inequalities will be denoted by

$$A(p)x = b(p), \quad C(p)y \leq d(p), \quad p \in \mathbf{p}. \tag{1.7}$$

In the system (1.7) or (1.1)–(1.2) the vector x may appear instead of y. All results presented in this paper are valid in this case, too. When an assertion is valid for any of the three systems (1.5)–(1.7), we will just call the system a parametric interval system without specifying its type.

The well-known and most studied nonparametric interval linear systems of equations $\mathbf{A}x = \mathbf{b}$ and/or inequalities $\mathbf{C}y \leq \mathbf{d}$ can be considered as special cases of the corresponding parametric interval systems. For example, $\mathbf{C}y \leq \mathbf{d}$, $\mathbf{C} \in \mathbb{IR}^{m\times n}$, $\mathbf{d} \in \mathbb{IR}^m$, can be considered as a special case of the corresponding parametric system involving $mn + m$ independent parameters $c_{ij} \in \mathbf{c}_{ij}$, $d_i \in \mathbf{d}_i$, $1 \leq i \leq m$, $1 \leq j \leq n$. For a parametric matrix $A(p)$, respectively, vector $b(p)$, depending on a number of parameters $p \in \mathbf{p}$, $A(\mathbf{p}), b(\mathbf{p})$ denote the *corresponding* interval nonparametric matrix, respectively, interval nonparametric vector,

$$A(\mathbf{p}) := A_0 + \sum_{k=1}^{K} \mathbf{p}_k A_k, \qquad b(\mathbf{p}) := b_0 + \sum_{k=1}^{K} \mathbf{p}_k b_k.$$

DEFINITION 1.1. *A parametric matrix $A(p) \in \mathbb{R}^{m\times n}$ is called a column-dependent parametric matrix (matrix involving only column dependencies) if for every parameter p_k, $k \in \{1, \ldots, K\}$, there exists $j \in \{1, \ldots, n\}$ such that $A_{k,\bullet j} \neq 0$ and $A_{k,\bullet i} = 0$ for all $i \neq j$.*

Any parametric vector $b(q)$, which is independent[1] of the parametric matrix $A(p)$, involves only column dependencies.

Following the notions introduced in [20], a system of real linear equations (inequalities or a mixed system) is called *solvable* if it has a solution and *feasible* if it has

[1] The vector and the matrix do not involve common parameters.

a nonnegative solution. A parametric interval system (1.5) (or (1.6) or (1.7)) is said to be *weakly* solvable (feasible) if there exists a numerical value of the parameter vector (1.3) such that the corresponding real system (1.1) (or (1.2) or (1.1)–(1.2), respectively) is solvable (feasible). A system (1.5) (or (1.6) or (1.7)) is said to be *strongly* solvable (feasible) if *each* real system (1.1) (or (1.2) or (1.1)–(1.2), respectively), obtained for a particular value of the parameter vector (1.3), is solvable (feasible). A real system (1.1) (or (1.2) or (1.1)–(1.2), respectively), obtained for a numerical vector $p \in \mathbf{p}$, is also called a *realization* of the parametric interval system.

2. Weak solvability of parametric interval systems. A weak solution to (1.7) is any vector $(x^\top, y^\top)^\top$, $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, satisfying (1.1)–(1.2) for some $p \in \mathbf{p}$. If $m_2 = 0$ or $m_1 = 0$, we obtain a weak solution of (1.5) or (1.6), respectively. A parametric interval system is weakly solvable if it has a weak solution.

The set of all weak solutions to (1.7), called a *united (or weak) parametric solution set*,[2] is

$$(2.1) \quad \Sigma_{uni}^{=,\leq}(A(p), C(p), b(p), d(p), \mathbf{p}) \\ := \left\{x \in \mathbb{R}^{n_1+n_2} \mid (\exists p \in \mathbf{p})(A(p)x = b(p), C(p)y \leq d(p))\right\}.$$

Similarly, we define the united parametric solution set $\Sigma_{uni}^{=} = \Sigma_{uni}^{=}(A(p), b(p), \mathbf{p})$ of (1.5) and the united parametric solution set $\Sigma_{uni}^{\leq} = \Sigma_{uni}^{\leq}(C(p), d(p), \mathbf{p})$ of (1.6). Thus, a parametric interval system is weakly solvable if its united parametric solution set is not empty.

The solution set $\Sigma_{uni}^{=}$ is characterized as follows, by a trivial set of inequalities:

$$\Sigma_{uni}^{=} = \{x \in \mathbb{R}^{n_1} \mid (\exists p_1 \in \mathbb{R}) \dots (\exists p_K \in \mathbb{R})((2.2)\text{–}(2.3) \text{ hold})\},$$

wherein

$$(2.2) \qquad A_0x - b_0 + \sum_{k=1}^{K} p_k(A_kx - b_k) \leq 0 \leq A_0x - b_0 + \sum_{k=1}^{K} p_k(A_kx - b_k),$$

$$(2.3) \qquad \underline{p}_k \leq p_k \leq \overline{p}_k, \quad k = 1, \dots, K.$$

Starting from such a description, Theorem 2.1 below shows how the parameters in this set can be eliminated successively in order to obtain a new description not involving p_k, $k = 1, \dots, K$. This parameter elimination process resembles the so-called Fourier–Motzkin elimination of variables (see, e.g., [24]) and will be called Fourier–Motzkin elimination of existentially quantified parameters.

THEOREM 2.1 (see Alefeld, Kreinovich, and Mayer [1]). *Let $f_{\lambda k}$, g_λ, $\lambda = 1, \dots, t$ (≥ 2), $k = 1, \dots, K$, be real-valued functions of $x = (x_1, \dots, x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there is a positive integer $t_1 < t$ such that $f_{\lambda 1}(x) \not\equiv 0$ for all $\lambda \in \{1, \dots, t\}$; $f_{\lambda 1}(x) \geq 0$ for all $x \in D$ and all $\lambda \in \{1, \dots, t\}$; and for each $x \in D$ there are an index $\beta^* = \beta^*(x) \in \{1, \dots, t_1\}$ with $f_{\beta^* 1}(x) > 0$ and an index $\gamma^* = \gamma^*(x) \in \{t_1 + 1, \dots, t\}$ with $f_{\gamma^* 1}(x) > 0$. For K parameters $p_1, \dots, p_K$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by*

$$S_1 := \{x \in D \mid (\exists p_1 \in \mathbb{R}) \dots (\exists p_K \in \mathbb{R})\,((2.4), (2.5) \text{ } hold)\},$$
$$S_2 := \{x \in D \mid (\exists p_2 \in \mathbb{R}) \dots (\exists p_K \in \mathbb{R})\,((2.6) \text{ } holds)\},$$

[2]Here we adopt the term *united parametric solution set*, which is mainly used in relation to systems of interval equations. Both "weak" and "united" will be used as synonyms.

where inequalities (2.4), (2.5), *and* (2.6), *respectively, are given by*

$$g_\beta(x) + \sum_{k=2}^{K} f_{\beta k}(x)p_k \le f_{\beta 1}(x)p_1, \quad \beta = 1, \dots, t_1, \tag{2.4}$$

$$f_{\gamma 1}(x)p_1 \le g_\gamma(x) + \sum_{k=2}^{K} f_{\gamma k}(x)p_k, \quad \gamma = t_1 + 1, \dots, t, \tag{2.5}$$

and for $\beta = 1, \dots, t_1$, $\gamma = t_1 + 1, \dots, t$

$$g_\beta(x)f_{\gamma 1}(x) + \sum_{k=2}^{K} f_{\beta k}(x)f_{\gamma 1}(x)p_k \le g_\gamma(x)f_{\beta 1}(x) + \sum_{k=2}^{K} f_{\gamma k}(x)f_{\beta 1}(x)p_k. \tag{2.6}$$

(Trivial inequalities such as $0 \le 0$ *can be omitted.) Then* $S_1 = S_2$.

If neither $f_{\lambda 1}(x) \ge 0$ nor $f_{\lambda 1}(x) \le 0$ holds uniformly on D, one can split D into several appropriate subdomains D_i with $\bigcup_i D_i = D$, for each of which the assumptions of Theorem 2.1 hold [1].

Remark 2.1 (see [1]). In the formulation of Theorem 2.1, $g_\lambda(x)$ denotes the expressions which are independent of the noneliminated parameters p_k. The assertion of the theorem remains true if the inequalities (2.4), (2.5) and the inequalities (2.6) are supplemented by inequalities which do not involve the currently eliminated parameter p_1, since these inequalities remain unchanged. Therefore, these inequalities are not involved in the description of the sets S_1 and S_2.

The following corollary simplifies the parameter elimination process.

COROLLARY 2.2 (see [1]). *With the notation and the assumptions of Theorem* 2.1 *let*

$$f_{\beta 1}(x) = h_{\beta\gamma}(x)\tilde{f}_{\beta 1}(x), \qquad f_{\gamma 1}(x) = h_{\beta\gamma}(x)\tilde{f}_{\gamma 1}(x),$$

where $h_{\beta\gamma}$, $\tilde{f}_{\beta 1}$, *and* $\tilde{f}_{\gamma 1}$ *are nonnegative functions defined on* D. *Then the assertion of Theorem* 2.1 *remains true if* $f_{\beta 1}(x)$, $f_{\gamma 1}(x)$ *are replaced in* (2.6) *by* $\tilde{f}_{\beta 1}(x)$ *and* $\tilde{f}_{\gamma 1}(x)$, *respectively.*

The application of Theorem 2.1, based on the endpoint inequalities (2.3), leads to a very large number of solution set characterizing inequalities. Therefore, in [17] it was proposed to employ the equivalent inequalities $\check{p}_k - \hat{p}_k \le p_k \le \check{p}_k + \hat{p}_k$, instead of the inequalities (2.3), and Theorem 2.1 was modified accordingly for the elimination of existentially quantified parameters in a system of linear algebraic equations. Here we do not recall Theorem 3.1 from [17]. Instead, in this section we consider only systems of parametric interval inequalities since any parametric interval system of equations (1.5) is equivalent to the system of parametric interval inequalities

$$\begin{aligned} A(p)x &\le \ b(p), \\ -A(p)x &\le \ -b(p), \quad p \in \mathbf{p}, \end{aligned} \tag{2.7}$$

and similarly any parametric interval system (1.7) is equivalent to a system of parametric interval inequalities. The representation (2.7) is equivalent to the representation (2.2)–(2.3). However, working with two-sided inequalities (2.2) has several advantages when characterizing weak solutions of parametric interval equations.

The solution set $\Sigma_{uni}^{\leq}\left(C(p), d(p), \mathbf{p}\right)$ is characterized by the following trivial set of inequalities:

$$\begin{aligned} &C_0 y - d_0 + \sum_{k=1}^{K} p_k (C_k y - d_k) \leq 0, \\ &\check{p}_k - \hat{p}_k \leq p_k \leq \check{p}_k + \hat{p}_k, \quad k = 1, \ldots, K. \end{aligned}$$

Starting from such a description, the theorem below shows how the parameters in this set can be eliminated successively in order to obtain a new description not involving p_k, $k = 1, \ldots, K$.

THEOREM 2.3. *Let $g_{\lambda k}(y)$, $\lambda = 1, \ldots, t (\geq 2)$, $k = 0, 1, \ldots, K$, be real-valued functions of $y = (y_1, \ldots, y_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that $g_{\lambda 1}(y) \not\equiv 0$ for all $\lambda \in \mathcal{T} = \{1, \ldots, t\}$, and define the sets $\mathcal{T}_1 := \{i \in \mathcal{T} \mid -g_{i1}(y) \geq 0\}$, $\mathcal{T}_2 := \{j \in \mathcal{T} \mid -g_{j1}(y) \leq 0\}$. For the parameters p_k, $k = 1, \ldots, K$, varying in $\mathbb{R}$ and for y varying in D define the sets S_1, S_2 by*

$$\begin{aligned} S_1 &:= \{y \in D \mid (\exists p_1 \in \mathbb{R}) \ldots (\exists p_K \in \mathbb{R})\, ((2.8), (2.9) \text{ hold})\}, \\ S_2 &:= \{y \in D \mid (\exists p_2 \in \mathbb{R}) \ldots (\exists p_K \in \mathbb{R})\, ((2.10), (2.11) \text{ hold})\}, \end{aligned}$$

where inequalities (2.8), (2.9) *and* (2.10), (2.11), *respectively, are given by*

$$g_{\lambda 0}(y) + \sum_{k=2}^{K} g_{\lambda k}(y) p_k \ \leq\ -g_{\lambda 1}(y) p_1, \quad \lambda = 1, \ldots, t, \tag{2.8}$$

$$\check{p}_1 - \hat{p}_1 \leq\ p_1\ \leq \check{p}_1 + \hat{p}_1, \tag{2.9}$$

$$g_{\lambda 0}(y) + g_{\lambda 1}(y)\check{p}_1 - |g_{\lambda 1}(y)|\hat{p}_1 + \sum_{k=2}^{K} g_{\lambda k}(y) p_k\ \leq 0, \quad \lambda = 1, \ldots, t, \tag{2.10}$$

$$\begin{aligned} \bigvee_{i \in \mathcal{T}_1 \neq \emptyset, j \in \mathcal{T}_2 \neq \emptyset} \Bigg(&- g_{i1}(y) \geq 0,\ -g_{j1}(y) \leq 0, \\ &g_{i0}(y) g_{j1}(y) - g_{j0}(y) g_{i1}(y)\ + \sum_{k=2}^{K} \left(g_{ik}(y) g_{j1}(y) - g_{i1}(y) g_{jk}(y)\right) p_k\ \leq 0 \Bigg). \end{aligned} \tag{2.11}$$

(Trivial inequalities which are true for any $y \in \mathbb{R}^n$ can be omitted.) Then $S_1 = S_2$.

Proof. For fixed $\mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{T}$, the inequalities (2.8)–(2.9) can be split into two subsets $\mathcal{I}, J$,

$$\begin{aligned} &g_{i0}(y) + \sum_{k=2}^{K} g_{ik}(y) p_k\ \leq\ -g_{i1}(y) p_1, \quad i \in \mathcal{T}_1, \\ &\check{p}_1 - \hat{p}_1\ \leq\ p_1, \end{aligned} \tag{$\mathcal{I}$}$$

$$\begin{aligned} &g_{j1}(y) p_1\ \leq\ -g_{j0}(y) - \sum_{k=2}^{K} g_{jk}(y) p_k, \quad j \in \mathcal{T}_2, \\ &p_1\ \leq\ \check{p}_1 + \hat{p}_1, \end{aligned} \tag{$\mathcal{J}$}$$

such that the set $\mathcal{I}$ corresponds to the index set β from Theorem 2.1 and the set $\mathcal{J}$ corresponds to the index set γ from Theorem 2.1. Obviously, in each set $\mathcal{I}, J$ there is at least one inequality where the coefficient function of p_1 is positive. Thus, applying Theorem 2.1, we obtain

$$\begin{aligned} &- g_{i1}(y) \geq 0 \ \wedge \ -g_{j1}(y) \leq 0 \ \wedge \ g_{i0}(y)g_{j1}(y) - g_{j0}(y)g_{i1}(y) \\ &\qquad + \sum_{k=2}^{K}(g_{ik}(y)g_{j1}(y) - g_{jk}(y)g_{i1}(y))p_k \leq 0, \quad i \in \mathcal{T}_1, j \in \mathcal{T}_2, \end{aligned} \tag{2.12}$$

$$g_{i0}(y) + g_{i1}(y)\check{p}_1 + g_{i1}(y)\hat{p}_1 + \sum_{k=2}^{K} g_{ik}(y)p_k \leq 0, \quad i \in \mathcal{T}_1, \tag{2.13}$$

$$g_{j0}(y) + g_{j1}(y)\check{p}_1 - g_{j1}(y)\hat{p}_1 + \sum_{k=2}^{K} g_{jk}(y)p_k \leq 0, \quad j \in \mathcal{T}_2, \tag{2.14}$$

$$-2\hat{p}_1 \leq 0. \tag{2.15}$$

The inequality (2.15) holds true for any y in the domain D. Inequalities (2.13) and (2.14) are equivalent to (2.10). The set of inequalities (2.12) is empty if either $\mathcal{T}_1$ or $\mathcal{T}_2$ is an empty set. □

Remark 2.1 applies to Theorem 2.3, too. The application of Theorem 2.3 provides a more compact representation of the inequalities characterizing a parametric united solution set. Corollary 2.2, with an obvious change in the notation, also applies with Theorem 2.3.

Theorem 2.3 provides a parametric generalization of Gerlach's theorem [5].

COROLLARY 2.4. *A parametric interval system* (1.6)*, where each parameter appears in only one inequality of the system, is weakly solvable if and only if the system* $C(\check{p})y - d(\check{p}) \leq \sum_{k=1}^{K} \hat{p}_k |C_k y - d_k|$ *is solvable.*

Gerlach's theorem [5] is a special case of Corollary 2.4.

Theorem 2.3, applied to a parametric interval system of equations (2.7), provides a parametric generalization of the Oettli–Prager theorem [16]. An explicit description of the united parametric solution set $\Sigma_{uni}^{=}$ is obtained in the special cases of systems with symmetric or skew-symmetric matrices [15], any two-dimensional united parametric solution set [17], and when every parameter is involved in only one equation of the system.

The inequalities (2.11) are called "cross-inequalities" [17]. The elimination of any parameter involved in only one inequality of (1.6) (or one equation of (1.5), or similarly one equation or inequality of (1.7)) does not generate cross-inequalities.

Remark 2.2. Neither of the parameter elimination theorems prescribes the order of parameter elimination. Nevertheless, we recommend ordering (if possible) the parameters p_k, $k = 1, \ldots, K$, with respect to their occurrence in the equations/inequalities of the initial system and following this order in the parameter elimination process, starting with the parameters appearing in a minimal number of equations/inequalities. This provides a minimal number of solution set characterizing inequalities and facilitates the proof of superfluous/redundant inequalities; cf. [17].

COROLLARY 2.5. *A parametric interval system* (1.7)*, where each parameter appears in only one equation or inequality of the system, is weakly solvable if and only* *if the system*

$$|A(\check{p})x - b(\check{p})| \le \sum_{k=1}^{K} \hat{p}_k |A_k y - b_k|,$$
$$C(\check{p})y - d(\check{p}) \le \sum_{k=1}^{K} \hat{p}_k |C_k y - d_k|$$

is solvable.

Example 2.1. Find the explicit description of the united (weak) solution set to the following parametric interval system of mixed equations and inequalities:

$$\begin{aligned} p_1x_1 - p_2x_2 &= 1 - p_2, \\ 2x_1 + 3x_2 &\le p_2, \\ 3x_1 + x_2 &\le 2p_2, \end{aligned}$$

where $p_1 \in [-\frac{3}{2} \pm \frac{1}{2}]$, $p_2 \in [\frac{3}{2} \pm \frac{1}{2}]$. The elimination of p_1 replaces the equation above with the two-sided inequality

$$x_1\check{p}_1 - |x_1|\hat{p}_1 - 1 \le -(1 - x_2)p_2 \le x_1\check{p}_1 + |x_1|\hat{p}_1 - 1.$$

Then, the elimination of p_2 from the latter inequalities together with the two initial inequalities generates the endpoint inequalities

$$|x_1\check{p}_1 - 1 + (1 - x_2)\check{p}_2| \le |x_1|\hat{p}_1 + |-1 + x_2|\hat{p}_2, \tag{2.16}$$

$$\begin{aligned} 2x_1 + 3x_2 - \check{p}_2 &\le \hat{p}_2, \\ 3x_1 + x_2 - 2\check{p}_2 &\le 2\hat{p}_2 \end{aligned} \tag{2.17}$$

and the following cross-inequalities:

$$\begin{pmatrix} -1 + x_2 \ge 0, \\ -2x_1(1 - x_2) - 3(1 - x_2)x_2 \le -1 - 3x_1/2 + |x_1|/2, \\ -3x_1(1 - x_2) - (1 - x_2)x_2 \le -2 - 3x_1 + |x_1| \end{pmatrix} \vee \begin{pmatrix} -1 + x_2 \le 0, \\ -2x_1(1 - x_2) - 3(1 - x_2)x_2 \ge -1 - 3x_1/2 - |x_1|/2, \\ -3x_1(1 - x_2) - (1 - x_2)x_2 \ge -2 - 3x_1 - |x_1| \end{pmatrix}. \tag{2.18}$$

The united solution set is presented in Figure 1.

3. Strong solutions of parametric equations and inequalities. A *strong solution* to (1.7) is any vector $(x^\top, y^\top)^\top$, $x \in \mathbb{R}^{n_1}$, $y \in \mathbb{R}^{n_2}$, such that it solves (1.1)–(1.2) for all $p \in \mathbf{p}$. If $m_2 = 0$ or $m_1 = 0$, we obtain a strong solution of (1.5) or (1.6), respectively. The set of all strong solutions to (1.7) is defined by

$$\begin{aligned} \Sigma_{str}^{=,\le} &= \Sigma_{str}^{=,\le}(A(p), C(p), b(p), d(p), \mathbf{p}) \\ &:= \{(x^\top, y^\top)^\top \in \mathbb{R}^{n_1+n_2} \mid (\forall p \in \mathbf{p})(A(p)x = b(p), C(p)y \le d(p))\} \end{aligned}$$

and is called its *strong parametric solution set.* The strong parametric solution set of (1.5) or (1.6) is defined similarly.

THEOREM 3.1. *A vector* $(x^\top, y^\top)^\top \in \mathbb{R}^{n_1+n_2}$ *is a strong solution of* (1.7)*, that is,* $(x^\top, y^\top)^\top \in \Sigma_{str}^{=,\le}$*, if and only if* $(x^\top, y^\top)^\top$ *satisfies the following system not involving*

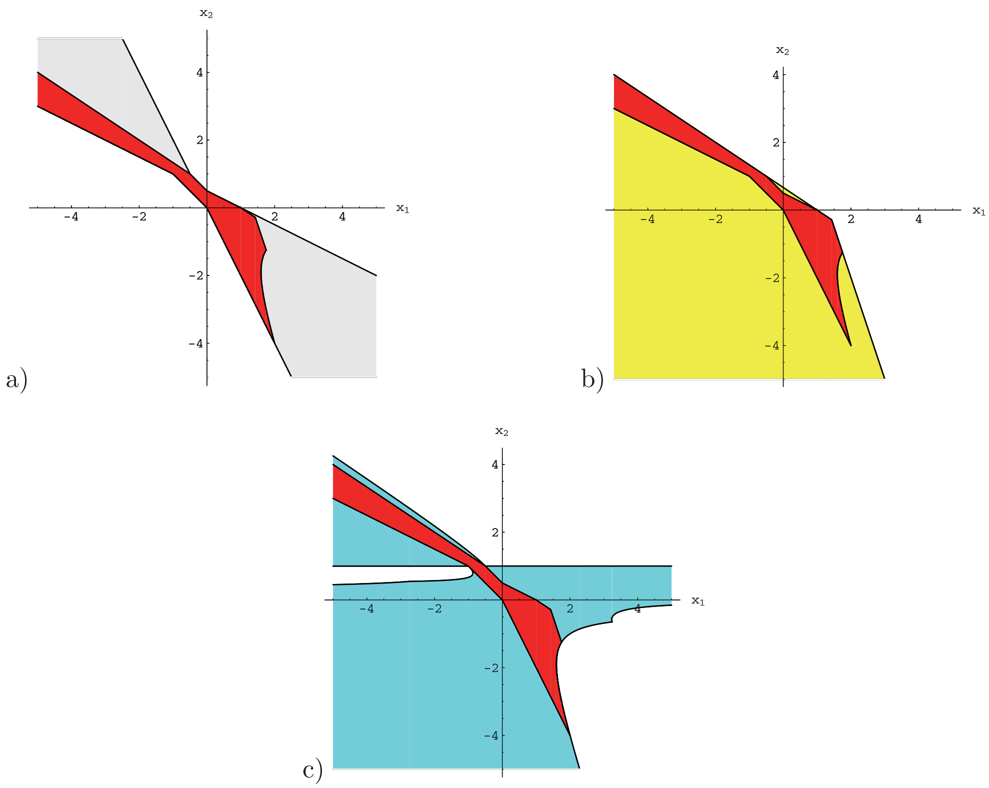

FIG. 1. *The united solution set (red/dark gray) to the parametric system from Example* 2.1 *together with* (a) *the solutions of inequalities* (2.16), (b) *the solutions of inequalities* (2.17), *and* (c) *the solutions of inequalities* (2.18).

the parameters p:

$$A_k x - b_k = 0, \quad k = 0, 1, \ldots, K, \tag{3.1}$$

$$C(\check{p})y - d(\check{p}) + \sum_{k=1}^{K} \hat{p}_k |C_k y - d_k| \leq 0. \tag{3.2}$$

Proof. The proof goes by successive elimination of the interval parameters $p_1 \in \mathbf{p}_1, \ldots, p_K \in \mathbf{p}_K$ from (1.7). Due to the distributivity of universal quantifiers over conjunctions, we apply the elimination of every interval parameter separately to each equation and each inequality of system (1.7). In the parameter elimination we also use the two relations[3]

$$\lambda\check{a} - |\lambda|\hat{a} \leq \lambda a \leq \lambda\check{a} + |\lambda|\hat{a}, \qquad \lambda \in \mathbb{R},\ a \in \mathbf{a} \in \mathbb{IR}, \tag{3.3}$$

$$(\forall t \in \mathbf{t})(v_1 \leq f(t) \leq v_2) \quad \Leftrightarrow \quad (v_1 \leq \min_{t \in \mathbf{t}} f(t)) \wedge (\max_{t \in \mathbf{t}} f(t) \leq v_2). \tag{3.4}$$

The system of equations is equivalent to $A(p)x - b(p) \leq 0 \leq A(p)x - b(p)$, which

[3] Note that (3.3) is equivalent to $\lambda[\check{a} - \hat{a}, \check{a} + \hat{a}] = [\lambda\check{a} - |\lambda|\hat{a}, \lambda\check{a} + |\lambda|\hat{a}]$, $\lambda \in \mathbb{R}$, $\mathbf{a} = [\check{a} - \hat{a}, \check{a} + \hat{a}]$.

is rearranged to

$$A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k) \leq \ -p_1(A_1x - b_1) \ \leq A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k).$$

The application of relation (3.4), with p_1 instead of t, to the latter two-sided inequality yields

$$(\forall p_1 \in \mathbf{p}_1)\left(A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k) \leq -p_1(A_1x - b_1)\right.$$
$$\left. \leq A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k)\right) \ \Leftrightarrow$$

$$\left(A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k) \leq \min_{p_1\in\mathbf{p}_1}(-p_1(A_1x - b_1))\right) \wedge \left(\max_{p_1\in\mathbf{p}_1}(-p_1(A_1x - b_1)) \leq A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k)\right). \tag{3.5}$$

The application of (3.4) to the components of $-p_1(A_1x - b_1)$ gives

$$\min_{p_1\in\mathbf{p}_1}(-p_1(A_1x - b_1)) = -\check{p}_1(A_1x - b_1) - \hat{p}_1|A_1x - b_1|,$$
$$\max_{p_1\in\mathbf{p}_1}(-p_1(A_1x - b_1)) = -\check{p}_1(A_1x - b_1) + \hat{p}_1|A_1x - b_1|.$$

Hence (3.5) is equivalent to

$$\left(A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k) \leq -\check{p}_1(A_1x - b_1) - \hat{p}_1|A_1x - b_1|\right)$$
$$\wedge \left(-\check{p}_1(A_1x - b_1) + \hat{p}_1|A_1x - b_1| \leq A_0x - b_0 + \sum_{k=2}^{K} p_k(A_kx - b_k)\right),$$

which is rearranged to the system of two-sided inequalities

$$A_0x - b_0 + \check{p}_1(A_1x - b_1) + \hat{p}_1|A_1x - b_1| + \sum_{k=3}^{K} p_k(A_kx - b_k) \leq -p_2(A_2x - b_2)$$
$$\leq A_0x - b_0 + \check{p}_1(A_1x - b_1) - \hat{p}_1|A_1x - b_1| + \sum_{k=3}^{K} p_k(A_kx - b_k)$$

for the elimination of $p_2 \in \mathbf{p2}$. Similarly, the elimination of the remaining interval parameters $p_2, \ldots, p_K$ finally yields

$$A(\check{p})x - b(\check{p}) + \sum_{k=1}^{K} \hat{p}_k|A_kx - b_k| \leq 0 \leq A(\check{p})x - b(\check{p}) - \sum_{k=1}^{K} \hat{p}_k|A_kx - b_k|,$$

which is equivalent to

$$|A(\check{p})x - b(\check{p})| \leq -\sum_{k=1}^{K} \hat{p}_k |A_k x - b_k|.$$

The latter inequality holds true if and only if its two sides are equal to zero, which gives (3.1).

The inequalities $C(p)y \leq d(p)$ are processed in the same way as the equalities:

$$(\forall p_1 \in \mathbf{p}_1) \left(C_0 y - d_0 + \sum_{k=2}^{K} p_k (C_k y - d_k) \leq -p_1(C_1 y - d_1) \right) \Leftrightarrow$$
$$C_0 y - d_0 + \sum_{k=2}^{K} p_k (C_k y - d_k) \leq \min_{p_1 \in \mathbf{p}_1} \left(-p_1 (C_1 y - d_1) \right).$$

The application of (3.3) implies

$$C_0 y - d_0 + \sum_{k=2}^{K} p_k (C_k y - d_k) \leq -\check{p}_1 (C_1 y - d_1) - \hat{p}_1 |C_1 y - d_1|,$$

which is rearranged to

$$C_0 y - d_0 + \check{p}_1 (C_1 y - d_1) + \hat{p}_1 |C_1 y - d_1| + \sum_{k=3}^{K} p_k (C_k y - d_k) \leq -p_2 (C_2 y - d_2).$$

The elimination of the remaining parameters $p_2, \ldots, p_K$ results in (3.2). ∎

Corollary 3.2. *$x \in \mathbb{R}^{n_1}$ is a strong solution of the system $A(p)x = b(p)$, $p \in \mathbf{p}$, if and only if x solves $A_k x - b_k = 0$ for all $k = 0, 1, \ldots, K$.*

Definition 3.3. *For a matrix $A \in \mathbb{R}^{m \times n}$, the set $\mathcal{N}(A) := \{x \in \mathbb{R}^n \mid Ax = 0\} \subseteq \mathbb{R}^n$ is called the* null space[4] *of A.*

Proposition 3.4. *$x \in \mathbb{R}^{n_1}$ is a strong solution of the system $A(p)x = b(q)$, $p \in \mathbf{p} \in \mathbb{IR}^{K_p}$, $q \in \mathbf{q} \in \mathbb{IR}^{K_q}$, if and only if*

$$x \in \bigcap_{k=1}^{K_p} \mathcal{N}(A_k),\ A_0 x = b_0,\ b_k = 0 \text{ for every } 1 \leq k \leq K_q. \tag{3.6}$$

Proof. According to the proof of Theorem 3.1, the following inequality describes the set of strong solutions to the considered system:

$$|A(\check{p})x - b(\check{q})| \leq -\sum_{k=1}^{K_p} \hat{p}_k |A_k x| - \sum_{k=1}^{K_q} \hat{q}_k |b_k|.$$

This inequality holds true if both of its sides are equal to the zero vector, which implies

$$\begin{aligned} A_k x &= 0, \quad 1 \leq k \leq K_p, \\ b_k &= 0, \quad 1 \leq k \leq K_q, \\ A_0 x &= b_0. \end{aligned}$$

[4]Some authors call the null space the *kernel* of the matrix.

The latter is equivalent to (3.6). ∎

Example 3.1. The parametric interval system

$$\begin{pmatrix} 2+p_1 & 3-3p_2 & 1+p_1 \\ 2-p_1+p_2 & 4 & 5-p_1+p_2 \\ 3+p_2 & 6+7p_2 & 1+p_2 \end{pmatrix} x = \begin{pmatrix} -t \\ 3t \\ -2t \end{pmatrix}, \quad p_1 \in \mathbf{p}_1,\, p_2 \in \mathbf{p}_2,\, t \in \mathbb{R},$$

has a nonempty strong solution set, which is

$$\{(-t, 0, t)^\top \mid t \in \mathbb{R}\}.$$

It is easy to verify that $(-t,0,t)^\top$ belongs to both $\mathcal{N}(A_1), \mathcal{N}(A_2)$ and $A_0 \cdot (-t,0,t)^\top = (-t, 3t, -2t)^\top$.

Corollary 3.5. *$y \in \mathbb{R}^{n_2}$ is a strong solution of the system $C(p)y \le d(p)$, $p \in \mathbf{p}$, if and only if y satisfies* (3.2).

Since the inequality (3.2) is equivalent to

$$C(\check{p})y - d(\check{p}) \le -\sum_{k=1}^{K} \hat{p}_k \left(\operatorname{diag}(u_k)(C_k y - d_k)\right) \tag{3.7}$$

for all vectors $u_k \in \{\pm 1\}^m$, $1 \le k \le K$, checking the existence of a strong solution to a parametric interval system $C(p)y \le d(p)$, $p \in \mathbf{p}$, reduces to checking the solvability of $K \cdot 2^m$ systems of real inequalities (3.7).

Proposition 3.6. *For a parametric matrix $\binom{A(p)}{C(p)}$ involving only column dependencies of the parameters, $\Sigma^{=,\le}_{str}(A(p), C(p), b(q), d(q), \mathbf{p}, \mathbf{q}) \equiv \Sigma^{=,\le}_{str}(A(\mathbf{p}), C(\mathbf{p}), b(\mathbf{q}), d(\mathbf{q}))$. The same is true for $A(p)$ involving only column dependencies and the strong solution sets $\Sigma^{=}_{str}(A(p), b(q), \mathbf{p}, \mathbf{q}) \equiv \Sigma^{=}_{str}(A(\mathbf{p}), b(\mathbf{q}))$, $\Sigma^{\le}_{str}(A(p), b(q), \mathbf{p}, \mathbf{q}) \equiv \Sigma^{\le}_{str}(A(\mathbf{p}), b(\mathbf{q}))$.*

Proof. The proof proceeds by finding explicit characterization of the strong solution sets mentioned as equivalent. Due to the distributivity of universal quantifiers over conjunctions, the elimination of the parameters in (1.7) is done independently for each equation and inequality in the parametric system (see the proof of Theorem 3.1). Since the parametric system involves only column dependencies, every parameter appears only once in each equation and inequality. Then, applying Theorem 3.1 to each equation and inequality of the parametric system and to the corresponding nonparametric system, the latter considered as a parametric system involving $(m_1+m_2)(n_1+n_2+1)$ interval parameters, we obtain equivalent representations. ∎

Clearly, if a parametric interval system has a nonempty strong solution set, it is strongly solvable.[5] However, the next example shows that a (parametric) interval system of equations may not have a strong solution while the system is strongly solvable.

Example 3.2. The system of parametric interval equations

$$\begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} x = \begin{pmatrix} p \\ 2p \end{pmatrix}, \quad p \in [1, 2],$$

does not have a strong solution, while the system is solvable for every $p \in \mathbf{p}$.

Hence, the conditions (3.1), (3.2) are only sufficient conditions for strong solvability of the respective parametric interval system. Therefore, strong solvability of parametric interval systems deserves a separate study.

[5]The definition is at the end of section 1.

4. Strong solvability of parametric interval systems. In this section we consider strong solvability of parametric interval inequalities $A(p)x \leq b(p)$, $p \in \mathbf{p} \in \mathbb{IR}^K$. The results are applicable to parametric interval systems of equations or mixed systems. In this section it will be assumed that $A(p) \in \mathbb{R}^{m\times n}$, $b(p) \in \mathbb{R}^m$, where m, n are arbitrary positive integers. The basic result concerning feasibility of real linear equations $Ax = b$ is proven by Farkas in [3]; for another proof see [20]. In this section we apply two versions of the Farkas theorem.

LEMMA 4.1. *A system $Ax \leq b$ has no solution if and only if the system $A^\top y = 0$, $b^\top y < 0$, $y \geq 0$ is solvable.*

LEMMA 4.2. *A system $Ax \leq b$, $x \geq 0$ has no solution if and only if the system $A^\top y \geq 0$, $b^\top y < 0$, $y \geq 0$ is solvable.*

THEOREM 4.3 (see [22]; see also Theorem 2.23 in [20]). *An interval system $\mathbf{A}x \leq \mathbf{b}$ is strongly solvable if and only if the system $\overline{A}x^1 - \underline{A}x^2 \leq \underline{b}$, $x^1 \geq 0$, $x^2 \geq 0$ is solvable.*

LEMMA 4.4. *For $\lambda_0, \lambda_i \in \mathbb{R}$, $\mathbf{a}_i \in \mathbb{IR}$, $i = 1, \ldots, K$,*

$$\lambda_0 + \sum_{k=1}^{K} \lambda_i \mathbf{a}_i \;=\; \left[\lambda_0 + \sum_{k=1}^{K} \lambda_i \check{a}_i - \sum_{k=1}^{K} |\lambda_i| \hat{a}_i,\ \lambda_0 + \sum_{k=1}^{K} \lambda_i \check{a}_i + \sum_{k=1}^{K} |\lambda_i| \hat{a}_i\right].$$

Proof. Apply the relation (3.3), and sum up the obtained inequalities. ∎

For $x \in \mathbb{R}$ we define its sign by $\operatorname{sign}(x) := \{1 \text{ if } x > 0, 0 \text{ if } x = 0, -1 \text{ if } x < 0\}$.

THEOREM 4.5. *Let $A(p)$, $p \in \mathbb{R}^{K_p}$, be a parametric matrix involving only column dependencies such that $\operatorname{sign}(a_{k,ij}) = \text{const}$ for all elements $a_{k,ij} \neq 0$ of A_k and every $1 \leq k \leq K_p$. Let $b(q)$, $q \in \mathbb{R}^{K_q}$, be such that $\operatorname{sign}(b_{k,i}) = \text{const}$ for $(b_{k,i}) \neq 0$, $1 \leq i \leq m$, and every $1 \leq k \leq K_q$. The parametric interval system $A(p)x \leq b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, is strongly solvable if and only if the corresponding nonparametric system $A(\mathbf{p})x \leq b(\mathbf{q})$ is strongly solvable.*

Proof. We prove the assertion by negation. Let $\tilde{p} \in \mathbf{p}$, $\tilde{q} \in \mathbf{q}$ be such that $A(\tilde{p})x \leq b(\tilde{q})$ is not solvable. By the Farkas lemma Lemma 4.1, there exists $y \in \mathbb{R}^m$, $y \geq 0$, such that

$$A^\top(\tilde{p})y = 0, \quad b^\top(\tilde{q})y \leq -1.$$

This means that y is a weak solution of the system

$$A^\top(p)y = 0, \quad b^\top(q)y \leq -1, \quad y \geq 0, \quad p \in \mathbf{p},\ q \in \mathbf{q}.$$

Since every parameter in $A^\top(p)$ is involved in only one equation of the above system and the latter has only one inequality $b^\top(q)y \leq -1$, by Corollary 2.5, y is a weak solution of the above system if and only if y satisfies the system

$$|A^\top(\check{p})y| \leq \sum_{k=1}^{K_p} \hat{p}_k |A_k^\top y|, \quad b^\top(\check{q})y - \sum_{k=1}^{K_q} \hat{q}_k |b_k^\top y| \leq -1, \quad y \geq 0. \tag{4.1}$$

Since $\operatorname{sign}(a_{k,ij}) = \text{const}$ for all elements $a_{k,ij} \neq 0$ of A_k, $|A_k^\top y| = |A_k^\top| y$ for $1 \leq k \leq K_p$, and similarly $|b_k^\top y| = |b_k^\top| y$ for $1 \leq k \leq K_q$. Hence, system (4.1) is equivalent to

$$\begin{aligned} &\left(A^\top(\check{p}) - \sum_{k=1}^{K_p} \hat{p}_k |A_k^\top|\right) y \leq 0 \leq \left(A^\top(\check{p}) + \sum_{k=1}^{K_p} \hat{p}_k |A_k^\top|\right) y, \\ &\left(b^\top(\check{q}) - \sum_{k=1}^{K_q} \hat{q}_k |b_k^\top|\right) y \leq -1, \quad y \geq 0. \end{aligned} \tag{4.2}$$

Due to Lemma 4.4 and due to the appearance of each parameter p_k in only one row of the above two-sided system of inequalities, (4.2) is equivalent to

$$\begin{aligned} &\underline{A^\top(\mathbf{p})}y \ \le 0 \le \ \overline{A^\top(\mathbf{p})}y, \\ &\underline{b^\top(\mathbf{q})}y \le -1, \quad y \ge 0. \end{aligned}$$

We use the Farkas lemma Lemma 4.2 to prove nonsolvability of the system

$$\overline{A(\mathbf{p})}x^1 - \underline{A(\mathbf{p})}x^2 \le \underline{b(\mathbf{q})}, \quad x^1 \ge 0,\ x^2 \ge 0.$$

According to Theorem 4.3, the latter means that the system $A(\mathbf{p}) \le b(\mathbf{q})$ is not strongly solvable. ◻

Theorem 4.5 generalizes Theorem 5 in [11]. Theorem 4.5 implies that checking strong solvability of a parametric interval system $A(p)x \le b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, involving only column dependencies as in Theorem 4.5, can be performed in polynomial time; see [20, section 2.15].

Corollary 4.6. *Let $A(p)$, $p \in \mathbb{R}^{K_p}$, be a parametric matrix involving only column dependencies, $q \in \mathbb{R}^{K_q}$, and the assumptions of Theorem* 4.5*. The parametric interval system $A(p)x \le b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, is strongly solvable if and only if the following system is solvable:*

$$\overline{A(\mathbf{p})}x^1 - \underline{A(\mathbf{p})}x^2 \le \underline{b(\mathbf{q})}.$$

Proof. The proof follows from Theorem 4.5 and [20, Theorem 2.23] (see also [22]). ◻

The next two theorems reveal the relations between the strong solvability and the strong solutions of parametric interval inequalities involving only column dependencies such as those in Theorem 4.5.

Theorem 4.7. *If a parametric interval system $A(p)x \le b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, involving only column-dependencies such as those in Theorem* 4.5*, is strongly solvable, then it has a strong solution.*

Proof. The proof follows from Theorem 4.5 and [20, Theorem 2.24] (see also [22]). ◻

The following theorem follows from [21, Theorem 4].

Theorem 4.8. *The following assertions are equivalent.*

(i) *x is a strong solution of $A(p)x \le b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, where the dependencies in $A(p)$, $b(q)$ are the same as those in Theorem* 4.5*.*

(ii) *x satisfies*

$$A(\check{p})x + \left(\sum_{k\in K_p} \hat{p}_k |A_k|\right)|x| \le \underline{b(\mathbf{q})}.$$

(iii) *x satisfies*

$$A(\check{p})x + \left(\sum_{k\in K_p} \hat{p}_k |A_k|\right)t \le \underline{b(\mathbf{q})}, \quad -t \le x \le t.$$

(iv) *$x^+ := \max\{x, 0\}$, $x^- := \max\{-x, 0\}$ (*max *is applied entrywise) satisfy*

$$\overline{A(\mathbf{p})}x^+ - \underline{A(\mathbf{p})}x^- \ \le \ \underline{b(\mathbf{q})}.$$

(v) $x = x^1 - x^2$, *where* x^1, x^2 *satisfy*

$$\overline{A(\mathbf{p})}x^1 - \underline{A(\mathbf{p})}x^2 \leq \underline{b(\mathbf{q})}, \quad x^1 \geq 0,\ x^2 \geq 0.$$

The next theorem considers general column dependencies in a parametric interval system.

THEOREM 4.9. *Let $A(p)$, $p \in \mathbb{R}^{K_p}$, be a parametric matrix involving only column dependencies. Let $K_{p,2}$ be the index set of the parameters involved in more than one row of $A(p)$, and for every $k \in K_{p,2}$ there exist $1 \leq j \leq n$, $i_1, i_2 \in \{1,\ldots,m\}$, $i_1 \neq i_2$, such that $\operatorname{sign}(a_{k,i_1 j}) \neq \operatorname{sign}(a_{k,i_2 j})$, $a_{k,i_1 j} \neq 0$, $a_{k,i_2 j} \neq 0$. Let $K_{p,1} = \{1,\ldots,K_p\} \setminus K_{p,2}$ be the index set of the remaining parameters involved in $A(p)$. Let $K_{q,1} \subseteq \{1,\ldots,K_q\}$ be the index set of the parameters q such that $\operatorname{sign}(b_{k,i}) = \operatorname{sign}(b_{k,j})$ for $i \neq j$ and $b_{k,i} \neq 0$, $b_{k,j} \neq 0$. Let $K_{q,2}$ be the index set of the remaining parameters q. The parametric interval system $A(p)x \leq b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, is strongly solvable if and only if the system*

$$\overline{C}x^1 - \underline{C}x^2 \ \leq\ \underline{d}, \quad x^1 \geq 0,\ x^2 \geq 0, \tag{4.3}$$

wherein

$$\begin{aligned} \underline{C} &:= A_0 + \sum_{k\in K_{p,1}} \underline{\mathbf{p}_k A_k} + \sum_{k\in K_{p,2}} \left(\check{p}_k A_k - \hat{p}_k \delta_k A_k\right), \\ \overline{C} &:= A_0 + \sum_{k\in K_{p,1}} \overline{\mathbf{p}_k A_k} + \sum_{k\in K_{p,2}} \left(\check{p}_k A_k + \hat{p}_k \delta_k A_k\right), \end{aligned} \tag{4.4}$$

$$\underline{d} := b_0 + \sum_{k\in K_{q,1}} \underline{\mathbf{q}_k b_k} + \sum_{k\in K_{q,2}} \left(\check{q}_k b_k - \hat{q}_k, \delta_k b_k\right), \tag{4.5}$$

is solvable for each $\delta_k \in \{\pm 1\}$, $k \in K_{p,2} \bigcup K_{q,2}$.

Proof. The proof proceeds similarly to that of Theorem 4.5 up to the system (4.1). Let K_j be the index sets of the parameters that appear in the jth column of $A(p)$, respectively. The first inequality of system (4.1) is equivalent to

$$\begin{aligned} &\{A^\top(\check{p})\}_j y - \sum_{k\in K_{p,1}\bigcap K_j} \hat{p}_k(|(A_{k,\bullet j})^\top|y) - \sum_{k\in K_{p,2}\bigcap K_j} \hat{p}_k\delta_k((A_{k,\bullet j})^\top y) \leq 0, \\ &0 \leq \{A^\top(\check{p})\}_j y + \sum_{k\in K_{p,1}\bigcap K_j} \hat{p}_k(|(A_{k,\bullet j})^\top|y) + \sum_{k\in K_{p,2}\bigcap K_j} \hat{p}_k\delta_k((A_{k,\bullet j})^\top y), \quad j = 1,\ldots,n, \end{aligned}$$

for some $\delta_k \in \{\pm 1\}$, $k \in K_{p,2}$.

The second inequality in (4.1) is equivalent to

$$b^\top(\check{q})y - \sum_{k\in K_{q,1}} \hat{q}_k(|b_k^\top|y) - \sum_{k\in K_{q,2}} \hat{q}_k\delta_k(b_k^\top y) \ \leq -1$$

for some $\delta_k \in \{\pm 1\}$, $k \in K_{q,2}$. Then, applying Lemma 4.4, we obtain that the system (4.1) is equivalent to the system

$$\begin{aligned} &\underline{C}^\top y \ \leq 0 \leq\ \overline{C}^\top y, \\ &\underline{d}^\top y \leq -1, \quad y \geq 0, \end{aligned}$$

wherein $\underline{C}$, $\overline{C}$, and $\underline{d}$ are defined by (4.4) and (4.5), respectively, for some $\delta_k \in \{\pm 1\}$, $k \in K_{p,2} \bigcup K_{q,2}$. By the Farkas lemma Lemma 4.2, we obtain that there exist $\delta_k \in$

$\{\pm 1\}$, $k \in K_{p,2} \bigcup K_{q,2}$, such that system (4.3) is not strongly solvable. The negation of this assertion yields the assertion of the theorem. $\square$

Theorem 4.9 generalizes Theorem 6 in [11]. Theorem 4.5 presents a special case of Theorem 4.9. Checking strong solvability of a parametric interval system $A(p)x \leq b(q)$, $p \in \mathbf{p}$, $q \in \mathbf{q}$, by Theorem 4.9 (if applicable) requires solving 2^t systems (4.3), where $t = \mathrm{Card}(K_{p,2} \bigcup K_{q,2})$.

Example 4.1. Check strong solvability of the parametric interval system

$$\begin{array}{rcl} 2x_1 + x_2 & \leq & q_1, \\ x_1 - x_2 & \leq & -2q_1, \end{array} \qquad q_1 \in [1, 2]. \tag{4.6}$$

This system involves column dependencies of the only parameter q_1. Below we check strong solvability of (4.6) in three ways, (a)–(c).

(a) By Theorem 3.1, the set of strong solutions to (4.6) is described by the inequalities

$$\begin{aligned} 2x_1 + x_2 &\leq 1, \\ x_1 - x_2 &\leq -4, \end{aligned} \tag{4.7}$$

whose solution set (found in *Mathematica*) is $(x_1 < -1,\ 4 + x_1 \leq x_2 \leq 1 - 2x_1) \vee (x_1 = -1, x_2 = 3)$. Since the set of strong solutions to (4.6) is not empty, the parametric interval system (4.6) is strongly solvable.

(b) By Proposition 3.6, the set of strong solutions to (4.6) is equivalent to the set of strong solutions to the nonparametric interval system

$$\begin{aligned} 2x_1 + x_2 &\leq \mathbf{q}_1, \\ x_1 - x_2 &\leq -2\mathbf{q}_1. \end{aligned}$$

If the latter nonparametric interval system has a strong solution, then the parametric system (4.6) has the same strong solution, and therefore it is strongly solvable.

(c) By Theorem 4.9, the system (4.6) is strongly solvable if and only if each of the two systems

$$\begin{aligned} 2x_1 + x_2 - 2y_1 - y_2 &\leq 1, \\ x_1 - x_2 - y_1 + y_2 &\leq -2, \\ x_1 \geq 0,\ x_2 \geq 0,\ y_1 \geq 0,\ y_2 &\geq 0, \end{aligned}$$

and

$$\begin{aligned} 2x_1 + x_2 - 2y_1 - y_2 &\leq 2, \\ x_1 - x_2 - y_1 + y_2 &\leq -4, \\ x_1 \geq 0,\ x_2 \geq 0,\ y_1 \geq 0,\ y_2 &\geq 0, \end{aligned}$$

is solvable. Since both real systems are solvable (checked by linear programming [21, section 4] or by suitable tools for inequalities in some computer algebra environments), the parametric system (4.6) is strongly solvable.

The theorem below may be used for proving strong solvability of linear parametric inequalities in the most general case.

THEOREM 4.10. *The parametric system $A(p)x \leq b(p)$, $p \in \mathbf{p}$, is strongly solvable if and only if the real system that explicitly describes the united parametric solution* *set of the system*

$$\begin{aligned} &A^\top(p)y = 0,\\ &b^\top(p)y \le -1,\\ &y \ge 0, \quad p \in \mathbf{p}, \end{aligned}$$

is not solvable.

Proof. We prove the assertion by negation. Let $\tilde{p} \in \mathbf{p}$ be such that the system $A(\tilde{p})x \le b(\tilde{p})$ is not solvable. By the Farkas lemma Lemma 4.1, there exists $y \ge 0$ such that

$$A^\top(\tilde{p})y = 0, \quad b^\top(\tilde{p})y \le -1.$$

This means that $y \in \mathbb{R}^m$ is a weak solution of the mixed system

$$A^\top(p)y = 0, \quad b^\top(p)y \le -1, \quad y \ge 0, \quad p \in \mathbf{p}.$$

Successively eliminating the parameters $p_1, \dots, p_K$ in the latter system by applying Theorem 2.3, we obtain a new system of inequalities which do not involve the parameters p, and the vector $y \in \mathbb{R}^m$ satisfies this system. ∎

Theorem 4.10 generalizes Theorem 8 in [11].

It is obvious that the application of Theorem 4.10 requires much more effort than the application of Proposition 3.6 in the cases not covered by Theorem 4.9.

Example 4.2. We apply Theorem 4.10 to prove strong solvability of the parametric interval system from Example 4.1. By Theorem 4.10 we have to prove that the parametric interval system

$$\begin{aligned} 2y_1 + y_2 &= 0,\\ y_1 - y_2 &= 0,\\ q_1(y_1 - 2y_2) &\le -1,\\ y_1 \ge 0,\ y_2 \ge 0, \quad q_1 &\in [1, 2], \end{aligned} \tag{4.8}$$

is not solvable. The elimination of the parameter q_1 by Theorem 2.3 yields the real system

$$\begin{aligned} &2y_1 + y_2 = 0,\\ &y_1 - y_2 = 0,\\ &1 \le -3(y_1 - 2y_2)/2 + |y_1 - 2y_2|/2,\\ &y_1 \ge 0,\ y_2 \ge 0. \end{aligned}$$

The solution of the two equations is $y_1 = 0$, $y_2 = 0$ and does not satisfy the inequality combining y_1 and y_2. Therefore, system (4.8) is not solvable, which, by Theorem 4.10, implies solvability of the parametric interval system from Example 4.1.

Proposition 4.11. *The system $A(p)x = b(q)$, $p \in \mathbf{p} \in \mathbb{IR}^{K_p}$, $q \in \mathbf{q} \in \mathbb{IR}^{K_q}$, where $A(p)$ involves only column dependencies, is strongly solvable if and only if*

$$\overline{C}x^1 - \underline{C}x^2 \le \underline{d}, \quad x^1 \ge 0, \quad x^2 \ge 0,$$

where

$$\overline{C} = \begin{pmatrix} A_0 \\ -A_0 \end{pmatrix} + \sum_{k=1}^{K_p} \left(\check{p}_k \begin{pmatrix} A_k \\ -A_k \end{pmatrix} + \hat{p}_k \delta_k \begin{pmatrix} A_k \\ -A_k \end{pmatrix} \right),$$

$$\underline{C} = \begin{pmatrix} A_0 \\ -A_0 \end{pmatrix} + \sum_{k=1}^{K_p} \left(\check{p}_k \begin{pmatrix} A_k \\ -A_k \end{pmatrix} - \hat{p}_k \delta_k \begin{pmatrix} A_k \\ -A_k \end{pmatrix} \right),$$

$$\underline{d} = \begin{pmatrix} b_0 \\ -b_0 \end{pmatrix} + \sum_{k=1}^{K_q} \left(\check{q}_k \begin{pmatrix} b_k \\ -b_k \end{pmatrix} - \hat{q}_k \delta_k \begin{pmatrix} b_k \\ -b_k \end{pmatrix} \right),$$

is solvable for each $\delta_k \in \{\pm 1\}$*,* $k \in \{1, \ldots, K_p\} \bigcup \{1, \ldots, K_q\}$*.*

Proof. The proof follows from (2.7) and Theorem 4.9. ∎

Example 4.3. Strong solvability of the parametric system of interval equations from Example 3.2 can be proven either by applying Proposition 4.11 or by applying (2.7) and Theorem 4.10.

5. Conclusion. The decision problems related to weak or strong solvability of interval equations and/or inequalities are put into a general framework of arbitrary linear dependencies between interval-valued parameters. We presented some general approaches and conditions for proving weak or strong solvability of parametric interval systems of equations and/or inequalities. Since practical problems involve dependent rather than independent interval data, the above results will help in obtaining more realistic results. An example illustrating the difference between parametric and non-parametric solvability is given in [11]. The examples of the present paper illustrate the application of the methodology discussed here.

Since weak and strong solvability of interval equations and/or inequalities are closely related to the solution concepts of interval linear programming problems, the results presented here are applicable to linear programming problems involving linear parameter dependencies. For example, in [8], a general algorithm to determine the optimal value range in linear programming is proposed. This algorithm requires proving weak and strong solvability of parametric interval linear systems, the latter mentioned as an open problem. Thus, the results presented in this paper allow for a further parametric generalization of the approach from [8] and for an expansion of its applicability.

Acknowledgment. The author thanks the reviewers for their comments, which helped improve the paper.

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