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Quantified formulation of an interval model in mechanics

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Abstract. We consider summation of differently directed vectors (in geometric sense), the latter involved in linear equilibrium relations and defined by interval uncertainties. Such a summation is the basis of a variety of models in mechanics, for example. This work discusses a recent interval algebraic model of such a summation and presents the quantified formulation of this model. Quantified descriptions of the different interval models allow their better perception and comparison. Some restrictions are imposed on the solution of the algebraic interval model so that it has a meaningful interpretation. We prove some conditions under which the algebraic interval model and the traditional interval model have equivalent solution. Simple problems in mechanics illustrate the theoretical considerations. The quantified description of the algebraic interval model of truss structures is also presented.

Keywords: interval uncertainty, linear equilibrium, interval models

1 Introduction

For several decades the analysis and design of engineering systems consider and quantify the uncertainties that arise naturally in model parameters. Interval variables are widely used to represent non-probabilistic uncertainties. Methods based on interval analysis are applied not only to interval models but also to those based on fuzzy sets and hybrid models. According to [21] "interval analysis may be considered as the most widely adopted analytic tool among non-probabilistic analysts".

Most of the interval models considered so far in mechanics are based on classical interval analysis [1], [13]. Within this setting a lot of effort is put and a variety of special methods are proposed aiming at eliminating the interval dependency problem and obtaining sharp bounds for the unknowns. Aiming at an interval model which conforms to the physics of equilibrium equations, a novel model was proposed in [5], improved in [15] and further investigated in [16]–[19]. The new model (called algebraic) represents any vector (in geometric sense) model parameter (possessing magnitude and direction) by a directed interval (range + direction) and requires that all kinds of linear equilibrium equations be completely satisfied by the intervals for the unknowns. The model is embedded in an isomorphic algebraic extension of classical interval arithmetic which is known under a variety of names: Kaucher arithmetic [11], modal arithmetic [20], directed arithmetic [12], generalized (proper and improper) intervals, reflecting some of its aspects. Being a novel model, its results are compared to the results obtained by the traditional interval model based on classical interval arithmetic. However, these are two different models and they are applied for different purposes.

A goal of the present work is to further illuminate the algebraic interval model (and thus the scope of its application) by describing it via a logical expression involving the existential and universal quantifiers. The quantified description of the algebraic interval model allows its better perception and a comparison to the traditional interval model having a well-known quantified representation. The variety of solution sets related to the traditional linear interval models are described by corresponding quantified formulae and the well-known method of quantifier elimination [3] can be applied for finding quantifier free description of these solution sets, see e.g. [14]. Quantifier elimination is applied also to many applied mechanics and related problems, see, e.g., the works of Ioakimidis [7]–[10] and Charalampakis [2]. Here we also discuss this technique. The structure of the paper is as follows. Section 2 is devoted to the simplest problem of summation of differently directed vectors, the latter defined by interval uncertainty and involved in linear equilibrium relations. Such a summation is a background of a variety of models in mechanics, electrical engineering, etc., involving linear equilibrium relations, cf. [15]–[19]. In this section we first introduce some restrictions on the algebraic solution of the interval equation representing the model in the extended interval space. These restrictions aim at an interpretable solution (one with a unique direction), which completely satisfies the interval equilibrium equation. We prove necessary and sufficient conditions for these restrictions to be satisfied if the interval parameters have different uncertainty. Section 2.2 presents the equivalent quantified description of the algebraic interval model and its formal (algebraic) solution. Numerical examples illustrate the application of quantifier elimination for finding the unknown intervals. In Sect. 2.3 we prove some conditions under which the traditional and the algebraic interval models have equivalent solution. In Sect. 3 we discuss the quantified formulation of the more complicated models of statically indeterminate truss structures. The article ends by some conclusions.

2 Resultant of n collinear forces

Below we follow the notation used in [15]–[19] and apply some properties of Kaucher interval arithmetic presented also therein.

Consider a sum of n collinear forces $\overrightarrow{F}_i$. Each force has a fixed direction and magnitude F_i. The deterministic force equilibrium equation is $\overrightarrow{r} + \sum_i \overrightarrow{F}_i = 0$, where $\overrightarrow{r}$ is the unknown reaction force. The one dimensional problem is represented by the following magnitude equilibrium equation $s_r r + \sum_i s_i F_i = 0$, where all magnitudes r, F_i, $i = 1, \ldots, n$, are positive and the force directions are represented by the signs $s_r, s_i \in \{+, -\}$. Then the unknown reaction force is find as a solution $s_r r = -\sum_i s_i F_i$ of the above equation.

Let the forces F_i have uncertain magnitudes varying within intervals $\mathbf{F}_i \geq 0$,

$$\mathbf{F}_i = F_i + \delta_i F_i [-1, 1], \qquad 0 \leq \delta_i \leq 1.$$

Relation $0 \leq \delta_i \leq 1$ follows from the relations $\mathbf{F}_i \geq 0$, $\mathbf{F}_i \neq 0$.

The traditional interval equilibrium model replaces crisp force magnitudes by the corresponding intervals in the deterministic magnitude equilibrium equation,

$$s_r r + \sum_i s_i \mathbf{F}_i = 0 \tag{1}$$

and finds the unknown resultant force by $\mathbf{N} = \sum_i s_i \mathbf{F}_i$. Then, if $0 \notin \mathbf{N}$, $s_r = -\mathrm{sign}(\mathbf{N})$ and $\mathbf{r} = -s_r \mathbf{N}$. If there are forces $\overrightarrow{F}_i$ with different directions, it may happen that the direction of the resultant force $\overrightarrow{N}$ cannot be determined, e.g., for $\mathbf{F}_1 = [1, 3]$, $\mathbf{F}_2 = [2, 4]$, $s_1 = + = -s_2$, we have $\mathbf{N} = [-3, 1]$.

The demand [5] for canceling forces with equal interval magnitude and opposite direction is satisfied by the algebraic interval model proposed in [15]. According to the representation convention of this model the algebraic interval equilibrium is represented by interval equations in a generalized interval space $\mathbb{KR}$ (called Kaucher interval arithmetic [11]) which has group properties with respect to the operation addition and the operation multiplication of intervals that do not involve zero. Namely,

$$\sum_i s_i \left(\mathbf{F}_i\right)_{s_i} = [N], \quad \text{equivalently,} \quad \sum_i s_i \left(\mathbf{F}_i\right)_{s_i} - [N]_- \ = \ 0. \tag{2}$$

The subscript $-$ denotes the dual operator in $\mathbb{KR}$. Interval equations (2) have always a solution with respect to $[N] \in \mathbb{KR}$. Then, by the representation convention of the model, we find $s_r = -\mathrm{sign}([N])$ if $0 \notin \mathrm{pro}([N])$ and $\mathbf{r} = -s_r \mathrm{pro}([N])$.

Denote by s the sign operator in $\mathbb{KR} \backslash \mathcal{T}$, that is $s : \mathbb{KR} \backslash \mathcal{T} \to \{+, -\}$. In order to have meaningful interpretation of the interval resultant force (respectively the reaction force), it is required that

$$0 \notin \mathrm{interior}\left([N]_{\tau([N])}\right) = \mathrm{interior}\left(\mathrm{pro}([N])\right) \tag{3}$$

and

$$[N]_{s([N])} \in \mathbb{IR}. \tag{4}$$

Condition (3) implies that the direction of $\overrightarrow{N}$ (represented by its sign) is uniquely determined. It also implies that $s([N]) = s\left(\sum_i s_i F_i\right) = s_N$. Condition (4) implies, by the representation convention, that $s_N \mathbf{N} = [N]_{\tau([N])} = [N]_{s_N}$ completely satisfies the interval equilibrium equation

$$\sum_i s_i \left(\mathbf{F}_i\right)_{s_i} = s_N \mathbf{N}_{s_N}, \ \text{equivalently,} \ \sum_i s_i \left(\mathbf{F}_i\right)_{s_i} + (-s_N) \left(\mathbf{N}\right)_{-s_N} \ = \ 0. \tag{5}$$

Interval equilibrium equations (2) are applicable if the direction of the uncertain resultant force is not known, (cf, [16, end of Sect.4]). Since the direction of the deterministic resultant and the direction of the interval resultant forces should be the same, $s([N]) = s(\sum_i s_i F_i) = s_N$, and the first one is known before finding the interval resultant force, the true interval equilibrium equations are (5).

Below we investigate what are the necessary and sufficient conditions for the uncertainties in F_i so that the requirements (3) and (4) be satisfied.

2.1 Necessary and sufficient conditions

Lemma 1. *For $\lambda \in \mathbb{R}$, $\lambda[-1,1]_{s(\lambda)} = [-\lambda, \lambda]$.*

Proof. If $s(\lambda) = +$, then $\lambda[-1,1] = [-\lambda, \lambda]$. If $s(\lambda) = -$, then $\lambda[1,-1] = [-\lambda, \lambda]$. □

Proposition 1. *If $\delta_i = \delta$, $i = 1, \ldots, n$, then (5) holds true.*

Proof.

$$\begin{aligned} \sum_i s_i \left(\mathbf{F}_i\right)_{s_i} &= \sum_i s_i F_i + \delta \sum_i s_i F_i [-1,1]_{s_i} \\ &\overset{\text{distr.rel.}}{=} \sum_i s_i F_i + \delta \left(\sum_i s_i F_i \right) [-1,1]_{s(\sum s_i F_i)} \\ &\overset{\text{Lemma1}}{=} [\lambda - \delta\lambda, \lambda + \delta\lambda], \end{aligned}$$

where $\lambda = \sum_i s_i F_i$. Then, since $0 \le \delta \le 1$, the applied distributive relation is [18, Eqn (5)]. □

Theorem 1. *The relations*

$$\left| \sum_i s_i F_i \right| \ge \left| \sum_i s_i \delta_i F_i \right|, \tag{6}$$

$$\sum_i s_i F_i (1 - \delta_i) \ \le^{s(\sum_i s_i F_i)} \ \sum_i s_i F_i (1 + \delta_i) \tag{7}$$

are[1] *necessary and sufficient conditions for (3) and (4), respectively.*

Proof. As in the proof of Proposition 1,

$$\begin{aligned} \sum_i s_i \left(\mathbf{F}_i\right)_{s_i} &\overset{\text{distr.rel.}}{=} \sum_i s_i F_i + \left(\sum_i \delta_i s_i F_i \right) [-1,1]_{s(\sum_i \delta_i s_i F_i)} = \mu + \lambda[-1,1]_{s(\lambda)} \\ &= [\mu - \lambda, \mu + \lambda] = \begin{cases} [s\left(|\mu| - |\lambda|\right), s\left(|\mu| + |\lambda|\right)] & \text{if } s = s(\mu) = s(\lambda), \\ [s\left(|\mu| + |\lambda|\right), s\left(|\mu| - |\lambda|\right)] & \text{if } s = s(\mu) = -s(\lambda). \end{cases} \end{aligned}$$

[1] For $s \in \{+,-\}$ and $a, b \in \mathbb{R}$, we have $a \le^s b$ is equivalent to $\{a \le b \text{ if } s = +, a \ge b \text{ if } s = -\}$.

where $\mu = \sum_i s_i F_i$, $\lambda = \sum_i \delta_i s_i F_i$. It is obvious that the requirement (3) holds true if and only if relation (6) holds true. The requirement (4) is equivalent to

$$[\mu - \lambda, \mu + \lambda]_{s(\mu)} = \begin{cases} 0 \leq [\mu - \lambda, \mu + \lambda] \in \mathbb{IR} & \text{if } s(\mu) = + \\ \mathbb{IR} \ni [\mu + \lambda, \mu - \lambda] \leq 0 & \text{if } s(\mu) = -, \end{cases}$$

which is equivalent to (7). □

Proposition 2. *With the notation* $\mu = \sum_i s_i F_i$, $\lambda = \sum_i \delta_i s_i F_i$, *relation*

$$(s(\mu) = s(\lambda) \wedge \mu \geq \lambda) \vee (- = s(\mu) = -s(\lambda) \wedge -\mu \geq \lambda) \tag{8}$$

implies the relation (3) $\wedge$ *(4), which provides (5).*

Proof. It is obvious that $s(\mu) = s(\lambda) \wedge \mu \geq \lambda$ implies (3)$\wedge$(4).
If $- = s(\mu) = -s(\lambda) \wedge -\mu \geq \lambda$, then $[\mu - \lambda, \mu + \lambda]_- = [\mu + \lambda, \mu - \lambda] \in \mathbb{IR}$.
If $+ = s(\mu) = -s(\lambda) \wedge \mu \leq -\lambda$, then $0 \leq [\mu - \lambda, \mu + \lambda] \notin \mathbb{IR}$. □

2.2 Quantified formulation

The algebraic interval model of vector equilibrium can be formulated by means of the first order predicate calculus [4], via the existential and universal quantifiers. Denote $\mathbf{F}_{n+1} = \mathbf{N}$ and π', π'' be subsets of $\{1, \ldots, n, n+1\}$ such that $\pi' \cup \pi'' = \{1, \ldots, n+1\}$, $\pi' \cap \pi'' = \emptyset$ are defined by

$$\pi' := \{i \mid 1 \leq i \leq n+1, s_i = +\}, \quad \pi'' := \{i \mid 1 \leq i \leq n+1, s_i = -\}.$$

Following the semantic interpretation of proper and improper intervals, cf.[20], the interval equilibrium equations (5) are equivalent to the following logical expression (quantified formula) involving the existential and universal quantifiers

$$(\forall F_i \in \mathbf{F}_i, i \in \pi'')\,(\exists F_i \in \mathbf{F}_i, i \in \pi')\left(\sum_{i=1}^{n} s_i F_i = s_{n+1} F_{n+1}\right), \tag{9}$$

Note, that the equation in the third bracket above can be equivalently rearranged. This quantified formula, or its equivalents, can be used for finding the unknown interval for the resultant force by quantifier elimination [3]. In the examples that follow we use the software system *Mathematica*® [22], which supports quantifier elimination, in order to find the unknown interval for the resultant force. The *Mathematica*® function `Resolve[expr,dom]` attempts to resolve the logical expression `expr` into a form that eliminates the universal and existential quantifiers involved in `expr` over the domain `dom`. In the examples we work with the domain `Reals` and will skip this argument of the function.

Remark 1. Using a quantified formula for finding the unknown interval of the resultant force, it should be realized that when a universal quantifier `ForAll[x, ...]` is eliminated by the `Resolve` function, the result will contain no mention of the localized variable `x`.

From the quantified expression (9), and in view of Remark 1, the solution set for the interval resultant force magnitude is given by

if $s_N = +$,

$$\left\{ N \in \mathbb{R} \mid (\forall F_i \in \mathbf{F}_i, i \in \pi'')\, (\exists F_i \in \mathbf{F}_i, i \in \pi')\, \left(\sum_{i=1}^{n} s_i F_i = s_N N \right) \right\}, \quad (10)$$

if $s_N = -$,

$$\left\{ N \in \mathbb{R} \mid (\forall F_i \in \mathbf{F}_i, i \in \pi')\, (\exists F_i \in \mathbf{F}_i, i \in \pi'')\, \left(\sum_{i=1}^{n} s_i F_i = s_N N \right) \right\}. \quad (11)$$

Note, that the quantifiers in (10) and (11) differ, while the left-hand side of the equation in the third bracket is the same and the equation can be rearranged equivalently. This will be illustrated in the examples.

Example 1. Consider a box resting on a horizontal plane with three uncertain forces applied on the box as shown in Fig. 1. Assume that there is no friction force. Determine the resultant force applied on the box when $F_3 = 1200N$ and all forces are measured with 5% uncertainties. Since $\sum_{i=1}^{3} s_i F_i > 0$, the quantified

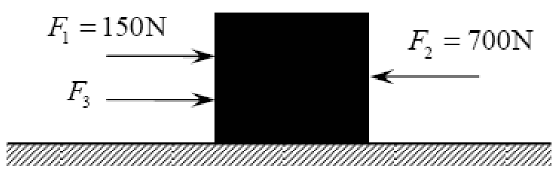

Fig. 1. Three forces acting on a box which rests on a horizontal plane, after [15]

formula (10) should be used for finding interval resultant force magnitude, that correspond to the interval equilibrium equations (5), and to those considered in [15, Example 4 c)]. In *Mathematica*® we have

```
In[1]:=Reduce[ForAll[{F2}, 665<=F2<=735,
        Exists[{F1,F3},425/10<=F1<=1575/10 && 1140<=F3<=1260
                            && F1-F2+F3==Nn]],{Nn},Reals]
Out[1]= 1235/2 <= Nn <= 1365/2
```

As mentioned above, the forces can be equivalently moved in the two sides of the equation constraint. Namely, the following evaluations (and other involving equivalent representation of the equation constraint) give the same result as `Out[1]`.

```
Reduce[ForAll[{F2}, 665<=F2<=735,
       Exists[{F1,F3}, 425/10<=F1<=1575/10 && 1140<=F3<=1260
```

```
                      && F1-F2+F3-Nn==0]],{Nn},Reals];

Reduce[ForAll[{F2}, 665<=F2<=735,
       Exists[{F1,F3}, 425/10<=F1<=1575/10 && 1140<=F3<=1260
                      && F1-Nn==F2-F3]],{Nn},Reals];
```

Example 2. Consider the problem of Example 1 for a) $F_3 = 300N$, b) $F_3 = 550N$, and all forces measured with 5% uncertainties.
In case a), $\sum_{i=1}^{3} s_i F_i < 0$, therefore, the quantified formula (11) should be used for finding interval resultant force magnitude,

```
In[2]:=Reduce[ForAll[{F1,F3}, 425/10<=F1<=1575/10 && 285<=F3<=315,
              Exists[{F2},665<=F2<=735&&-F1+F2-F3==Nn]],{Nn},Reals]
Out[2]= 475/2 <= Nn <= 525/2
```

Quantified expressions involving arbitrary equivalent representation of the equality constraint will give the same result.
In case b), $\sum_{i=1}^{3} s_i F_i = 0$, therefore any of the quantified formulae (10), (11) can be used for finding the interval resultant force magnitude.

In examples 1 and 2 all forces have the same level of uncertainty, which implies the validity of Proposition 1, therefore the existence of a proper solution $\mathbf{N}$. If the uncertainties in the forces differ, before applying quantifier elimination we can check the conditions of Proposition 2. As seen from the proof of Theorem 1, μ and λ specify both the condition (8) (resp. (6), (7)) and the interval solution sought.

Example 3. Consider the problem of Example 1 where the forces have the same directions and magnitudes as follows

a) $F_1 = 150\text{N}$, $\delta_1 = 3/75$, $F_2 = 700\text{N}$, $\delta_2 = 3/100$, $F_3 = 540\text{N}$, $\delta_3 = 5/100$;
b) $F_1 = 150\text{N}$, $\delta_1 = 2/100$, $F_2 = 700\text{N}$, $\delta_2 = 8/100$, $F_3 = 840\text{N}$, $\delta_3 = 5/100$;
c) $F_1 = 150\text{N}$, $\delta_1 = 2/100$, $F_2 = 700\text{N}$, $\delta_2 = 3/100$, $F_3 = 540\text{N}$, $\delta_3 = 5/100$.

In case a), $-10 = \sum_{i=1}^{3} s_i F_i$, $\sum_{i=1}^{3} s_i \delta_i F_i = 12$ and (6) does not hold. The latter means that $0 \in \mathbf{N}$.
In case b), $\mu = \sum_{i=1}^{3} s_i F_i = 290$, $\lambda = \sum_{i=1}^{3} s_i \delta_i F_i = -11$, therefore (6) holds true. However $[\mu - \lambda, \mu + \lambda]_{s(\mu)} = [301, 279] \notin \mathbb{IR}$.
In case c), $\mu = \sum_{i=1}^{3} s_i F_i = -10$, $\lambda = \sum_{i=1}^{3} s_i \delta_i F_i = 9$, therefore (6) holds true. However $[\mu - \lambda, \mu + \lambda]_{s(\mu)} = [-1, -19] \notin \mathbb{IR}$.
In any of the three cases, quantifier elimination applied to the corresponding quantified formula gives `False`, e.g., in case a)

```
In[3]:=Reduce[ForAll[{F1,F3}, 147<=F1<=153 && 513<=F3<=567,
              Exists[{F2},679<=F2<=721&&F1-F2+F3==-Nn]],{Nn},Reals]
Out[3]= False
```

In any of these three cases one can use the relation (8) in order to find bounds for the admissible uncertainties of the forces whose uncertainties can be controlled.

The algebraic interval model of linear equilibrium equations generalizes and applies transparently to summation of 2D and 3D vectors in geometric sense, see, e.g, [15, Example 3]. If each vector (geometric) is described by a single interval variable, then the requirements (8) (resp. (6), (7)) and the quantified formulae (10), (11)) also apply transparently to such 2D and 3D problems.

We end this section by the well-known quantified formulation of the equilibrium equations according to the traditional interval model (1), namely

$$(\exists F_i \in \mathbf{F}_i, i = 1, \ldots, n+1) \left(\sum_i s_i F_i = s_{n+1} F_{n+1} \right). \tag{12}$$

The difference between the two models is straightforward when comparing (9) and (12).

2.3 Equivalence of the interval models

Here we consider some special cases in which the traditional interval model and the algebraic interval model of linear equilibrium equations have the same solution.

Theorem 2. *The algebraic interval model of linear equilibrium equations and the classical one are equivalent in the following special cases*

(i) *all the uncertain forces have the same direction;*

(ii) *the uncertain forces $\overrightarrow{F}_i$ have directions s_i and all magnitudes depend affine linearly on the same interval parameter $p \in \mathbf{p} = \check{p} + \delta\check{p}[-1, 1] \geq 0$, such that $F_i = \alpha_i + \beta_i p$, $\alpha_i, \beta_i \in \mathbb{R}$, and some of the following relations holds true*

- *(ii.1)* $s\left(\sum_i s_i\alpha_i\right) = s\left(\sum_i s_i\beta_i\right)$,
- *(ii.2)* $+ = s\left(\sum_i s_i\alpha_i\right) \neq s\left(\sum_i s_i\beta_i\right)$ *and* $\sum_i s_i\alpha_i \leq -\left(\sum_i s_i\beta_i\right)\check{p}(1-\delta)$,
- *(ii.3)* $- = s\left(\sum_i s_i\alpha_i\right) \neq s\left(\sum_i s_i\beta_i\right)$ *and* $\sum_i s_i\alpha_i \leq -\left(\sum_i s_i\beta_i\right)\check{p}(1-\delta)$.

Proof. Let all the uncertain forces $\overrightarrow{F}_i$ have the same direction $s \in \{+, -\}$ and positive interval magnitude $\mathbf{F}_i$. According to the traditional interval model, the resultant force is $\mathbf{N} = \sum_i \mathbf{F}_i$ and satisfies the relation

$$\sum_i s\mathbf{F}_i = s\mathbf{N}. \tag{13}$$

According to the algebraic interval model $\sum_i s\left(\mathbf{F}_i\right)_s = s\left(\sum_i \mathbf{F}_i\right)_s$ and $\mathbf{N} = \sum_i \mathbf{F}_i$ satisfies the force equilibrium equation

$$\sum_i s\left(\mathbf{F}_i\right)_s = s\mathbf{N}_s. \tag{14}$$

If $s = +$, the equivalence is obvious. If $s = -$, we apply dual operator to both sides of (14) and obtain equivalently (13).

Case (ii). Consider the traditional interval model of summation. In order to reduce the overestimation due to the dependencies between the summands, we perform symbolic preprocessing of the expression and apply interval arithmetic at the simplified expression, where the interval variable $e = [-1, 1]$ appears only once. Namely,

$$\begin{aligned}
\sum_i s_i F_i &= \sum_i s_i \alpha_i + \sum_i s_i \beta_i p, \qquad p \in \mathbf{p} \\
&= \sum_i s_i \alpha_i + \left(\sum_i s_i \beta_i\right) \check{p}\,(1+\delta e)\,, \qquad e \in [-1,1] \\
&\in s\left[\left|\sum_i s_i\alpha_i + \left(\sum_i s_i\beta_i\right)\check{p}\,(1-\delta)\right|, \left|\sum_i s_i\alpha_i + \left(\sum_i s_i\beta_i\right)\check{p}\,(1+\delta)\right|\right] \\
&= s\mathbf{N},
\end{aligned}$$

where $s = s\left(\sum_i s_i\alpha_i\right)$ and some of the restrictions (ii.1)–(ii.3) applies. Note that $\mathbf{p} \geq 0$ implies $\check{p} \geq 0$.

According to the algebraic interval model, similarly to the proof of Theorem 1,

$$\begin{aligned}
[N] \quad &= \quad \sum_i s_i\alpha_i + \sum_i s_i\beta_i p_{s_i s(\beta_i)} \\
&\overset{\text{distr.rel}}{=} \sum_i s_i\alpha_i + \left(\sum_i s_i\beta_i\right)\check{p} + \left(\sum_i s_i\beta_i\right)\check{p}\delta\mathbf{e}_{s(\sum_i s_i\beta_i)} \\
&\overset{\text{Lemma1}}{=} \mu + \lambda\check{p} + [-\delta\check{p}\lambda, \delta\check{p}\lambda]\,,
\end{aligned}$$

where $\mu = \sum_i s_i\alpha_i$, $\lambda = \sum_i s_i\beta_i$. With any of the restrictions (ii.1)–(ii.3), the last interval $[N] \in \mathbb{KR}$ satisfies both (3), (4). Then, with the notation of s and $\mathbf{N}$, in the traditional model above, we have $[N]_s = -s\mathbf{N}$, which proves the assertion. □

Remark 2. Theorem 2 (i) follows straightforward by comparing the quantified expressions (9) and (12).

Theorem 2 gives theoretical justification of the numerical results discussed in [10], where the solution of the traditional interval model of mechanical problems with interval parameters depending on a single interval is compared to the solution of the corresponding interval algebraic model.

3 Statically indeterminate trusses

According to [18], the algebraic interval model of a truss structure is

$$\sum_{e=1}^{m} K_e\,(\mathbf{k}_e\mathbf{u})_{s(K_e)s(u)} \;=\; P\,(\mathbf{p}_f)_{s(P)}\,, \tag{15}$$

where $k_e = \frac{E_e A_e}{l_e}$ is the element stiffness, $K_e \in \mathbb{R}^{n\times n}$ is the element stiffness matrix, u is an n-vector of the unknown nodal displacements. A vector Pp_f collects the external loads p_f applied at the nodes with linear dependence. Arbitrary interval uncertainty can be considered in the external loads p_f and/or any of the parameters (Young modulus E_e, the area A_e, the length l_e, or an arbitrary combination of them) defining the element stiffness. In the deterministic case equation (15) combines the equilibrium equations $C^\top F = P.p_f$, the constitutive equations $F = D_k q$ and the compatibility equations $Cu = q$, where F and q are m-vectors of axial forces and deformations, respectively, D_k is a diagonal matrix of the internal stiffness $k = (k_{e_1}, \ldots, k_{e_m})^\top$, $C^\top \in \mathbb{R}^{n\times m}$ is the equilibrium matrix whose transpose is the so-called compatibility matrix. For each element e, K_e has rank one and thus $K_e = c_e c_e^\top$ and $C^\top = (c_{e_1}, \ldots, c_{e_m})$ collecting the vectors c_e.

Quantified formulation of the algebraic interval model (15) is applied to the equilibrium equations $C^\top \mathbf{F}_{s(C^\top)s(F)} = P(p_f)_{s(P)}$ as follows

$$\bigwedge_{i=1}^{n} \Big(\big(\forall F_j \in \mathbf{F}_j : C_{ij}^\top s(\mathbf{F}_j) < 0, \forall p_{f,k} \in \mathbf{p}_{f,k} : P_{ik} < 0\big) \big(\exists F_j \in \mathbf{F}_j : C_{ij}^\top s(\mathbf{F}_j) \geq 0, \exists p_{f,k} \in \mathbf{p}_{f,k} : P_{ik} \geq 0\big) \big(C^\top F = Pp_f\big)\Big), \quad (16)$$

where $\mathbf{F} = \Big((D_\mathbf{k})_{s(C\mathbf{u})} C\mathbf{u}\Big)_{s(C\mathbf{u})}$, cf. [19], and $\mathbf{u}$ is the interval vector for the displacements. The quantified expression (16) cannot be used for finding the unknown intervals for the axial forces or for the displacements because they may appear in different equations with different quantifiers. Intervals for the unknown displacements can be found by the methods proposed in [18] and obtaining the unknown axial forces is considered in [19].

Basing on the conditions for equilibrium, we qualify the algebraic interval model as a model which requires equilibrium between the uncertainties in the axial forces of the structure and the uncertainties in the external loads.

4 Conclusion

Basing on a strong theoretical background, we presented the quantified expressions of interval force equilibrium equations, where the position of each quantifier is precisely defined. The method of quantifier elimination is feasible only for very small problems with a few interval parameters, which makes it inefficient in comparison to numerical interval methods, cf. also [14], [10]. Despite of this, quantified formulation of a mechanical problem and its solution allows a better application and interpretation of the corresponding model. Since quantifier elimination is often applied as an alternative to interval computations, it should be mentioned that intervals for the unknowns in the algebraic interval model are also obtained without using interval arithmetic, cf. [18], [19]. The cases (Theorem 2) in which the traditional and the algebraic interval models have equivalent solution imply also corresponding equivalence to a methodology, proposed in [6], reducing the number of interval parameters in the traditional interval model to a single one.

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