Modelling a cavity expansion problem under uncertainties

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Considered is a cylindrical cavity expansion boundary value problem which is a theoretical basic for interpretation of the results from pressuremeter test in geomechanics. The uncertainties in model parameters are represented by tolerance intervals. Interval Finite Element technique is applied to bound the solution for the displacements in the considered linear elastic cavity expansion BVP with interval uncertainty in the Elastic modulus. Different interval methods for sharp enclosure of the displacements are compared. The analytic and discrete solutions are presented together with the interval band enclosing the displacements' range.

***Key words:** cavity expansion theory, finite element method, interval arithmetic.*

INTRODUCTION

The cylindrical cavity expansion theory is a simple theory that has many applications in geotechnical engineering. In particular, it is widely used to analyse the results from the pressuremeter test in geomechanics. The pressuremeter test in soil mechanics is a promissing method for characterizing the mechanical properties of soil and rock. This test simulates the expansion of a cylindrical cavity and because the model has well defined boundary conditions, it is more amenable to rigorous theoretical analysis than most other in-situ tests (Fig. 1). The applied pressure at the cavity wall is prescribed and controlled; as a result we can obtain the relative growth in cavity radius. Thus the pressuremeter test gives an in situ stress-strain curve of the soil material. Many authors solve the problem for cavity expansion as 2D axisymmetric or 2D plane strain problem using finite element approximations of different order. The material parameters considered in soil mechanical models are only vaguely determined. From engineering point of view it is interesting to estimate the range of the response of the material under some uncertainties in the material properties, applied pressure or geometry. There are different approaches for dealing with the uncertainty in mechanical models. Here we represent the uncertainties in model parameters by intervals and apply different interval methods to bound the range of values for the system response.

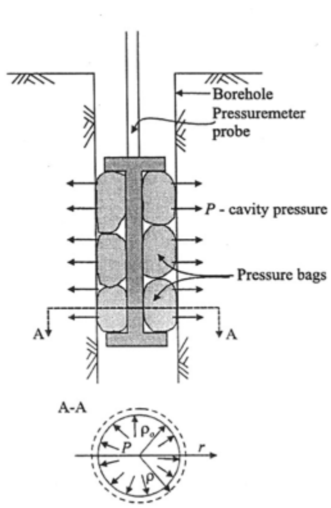

Fig. 1: Schematic presentation of pressuremeter test.

FORMULATION OF THE PROBLEM

The expansion of the pressuremeter probe is theoretically considered to be the expansion of an infinitely long cylinder in an infinite mass of soil. The position of the cylindrical probe is over 2-3 meter in depth. These assumptions allow considering a plane strain boundary value problem. We consider a plain strain formulation of a boundary value problem in a circle with a finite radius R. The cavity is modelled by a circle with prescribed radius r_0. The ratio between the cavity radius and the outer radius is $R/r_0 = 50$.

Real pressuremeters have length to diameter ratios between 3 and 10. The assumption for plane strain formulation is more reasonable when this ratio is higher. To simplify the problem, we use the following additional assumptions: All displacements are small. This means that we can use the infinitesimal strain tensor to characterize the deformation. We do not need to distinguish between stress measures, and we do not need to distinguish between the deformed and the undeformed configuration of the solid when writing the equilibrium equations and the boundary conditions. These assumptions are reasonable for rock materials and for most of geomaterials.

The material is an isotropic linear elastic medium with Young's modulus E_p, considered to be uncertain and varying within interval $E_p \in [E-\delta, E+\delta]$, where δ is the level of uncertainty.

We solve the boundary value problem for the displacement field u_i, the strain field ε_{ij} and the stress field σ_{ij} satisfying the following equations: displacement – strain relation: $\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$; stress – strain relation: $\sigma_{ij} = \frac{E_p}{1+\nu}\left\{\varepsilon_{ij} + \frac{\nu}{1-2\nu}\varepsilon_{kk}\delta_{ij}\right\}$; equilibrium equation: $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$. Stress boundary condition $\sigma_{ij} n_i = t_j$ on parts of the boundary where pressure is prescribed. Displacement boundary condition is $u_i = d_i$, on parts of the boundary, where the displacements are prescribed.

The analysis is performed by finite element approximation with three nodes linear finite elements. The plain strain formulation is applied. Finite element calculations are implemented by first developing the element stiffness matrices in parametric form (with Young's modulus as a parameter) using a standard non-interval formulation. The MATLAB Symbolic Tool Box is used to generate in symbolic form the element stiffness matrices and the element load vectors. After the assemblage of elements stiffness matrices a final linear system of parametric interval equations $K(E_p)U = F$ is obtained. To find a guaranteed interval enclosure for the displacement vector U we apply and compare three methods for solving parametric interval linear systems: Rump's parametric fixed-point iteration method [5], a generalisation of Rump's method [4], and a hybrid method based on the monotonicity properties of the parametric solution [3]. An application of the hybrid method to another linear elastic model can be found in [1].

NUMERICAL EXAMPLE AND RESULTS

We illustrate the application of the considered interval methods for solving parametric interval linear systems to the model of plane strain deformation on a domain described as a quarter of ring with inner radius $r_0 = 0.1[m]$ and outer radius $R = 5\,[m]$, see Fig.2. The prescribed pressure is $p = 0.01\,[MPa]$ on the cavity wall. Fixed displacements are prescribed $u_x = 0$ on the line $y = 0$, $u_y = 0$ on the line $x = 0$ and $u_x = u_y = 0$ on the outer contour $\sqrt{x^2 + y^2} = R$. The finite element discretization model is shown on Fig. 2. The crisp value for the uncertain parameter is $E_p = 0.15\,[GPa]$, the Poison's ratio is 0.3, and the tolerance δ is taken to be 10% of the crisp value.

The deterministic solutions for the displacement obtained by MSC.MARC [6] and by MATLAB are shown on Fig. 3. The analytical solution [2] for the same problem is compared with the numerical ones. The numerical solutions are in a good agreement with the analytical solution of the same problem and show that the number of finite elements approximates well the gradient of the displacement (u_x) along the line $y = 0$.

The parametric Rump's method [5] is unable to find an enclosure for the solution to the system. This is because the stiffness matrix $K(E_p)$ involves the so-called column dependencies of the parameter. The contraction matrix of the method is overestimated because the column dependencies are not accounted for during its computation.

The deficiency of the parametric Rump's method is resolved by a recent generalization of the parametric fixed-point method [4]. Fig. 4 shows the interval response of the displacement for the same sequence of nodes, that is used on Fig. 3, along the line $y = 0$.

Interval fixed-point iteration methods usually overestimate the exact interval hull of the solution set. To get a very sharp and guaranteed enclosure of the exact solution hull we exploit the monotonicity properties of the parametric solution (see [1], [3]). However the guaranteed proof of the monotonicity properties of the parametric solution is a quite heavy and inefficient procedure especially for large practical problems. That is why many authors apply the monotonicity approach in a non-rigorous way.

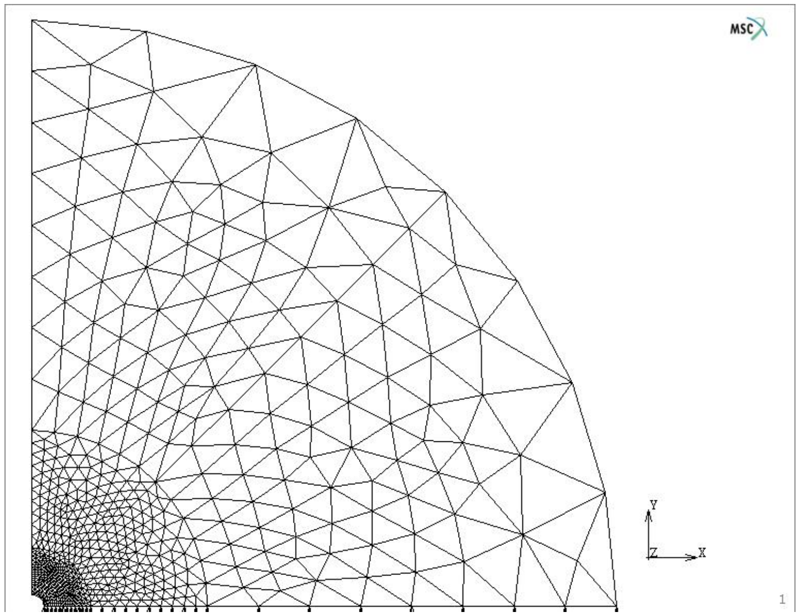

Fig 2: Finite element disretization; 611 nodes, 1160 linear triangle finite elements.

All numerical interval computations are done in the environment of CAS *Mathematica* by a corresponding package for solving interval linear systems [3]. The interval solution, obtained by the generalized parametric fixed-point iteration, overestimates the exact interval hull of the solution set, obtained by the monotonicity approach, with about 9%. This shows a very good sharpness of the iteration method despite of the big dimension of the system. However, the considered parametric system involves only one uncertain parameter.

CONCLUSIONS

A methodology for including material parameter uncertainty into cylindrical cavity expansion problem is presented here. Three interval methods for bounding the system response are applied to a numerical example and compared. The generalized fixed-point parametric iteration is computationally much more efficient than the hybrid interval monotonicity approach and still possesses a good sharpness of the solution enclosure. Interval approach provides limit values for an exponentially growing combination of scalar solutions in a single analysis. It can be concluded that the application of interval methods has a good ability and prospective for handling uncertainties in material parameters. These methods are also applicable to other mechanical systems as well as to handle uncertainties in geometry, applied pressure, etc.

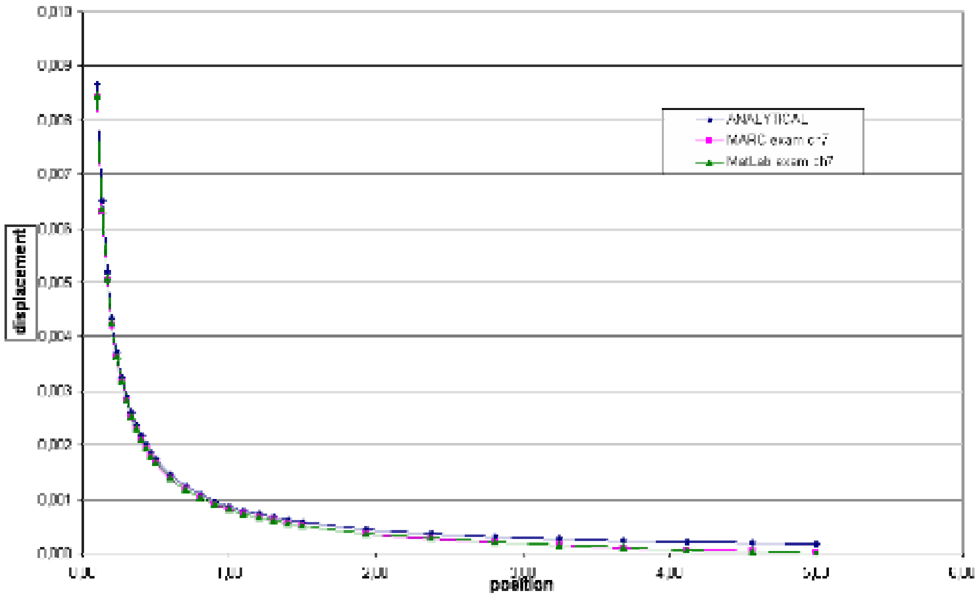

Fig. 3: Compression of numerical and analytical solutions. Displacements u_x along the line $y=0$.

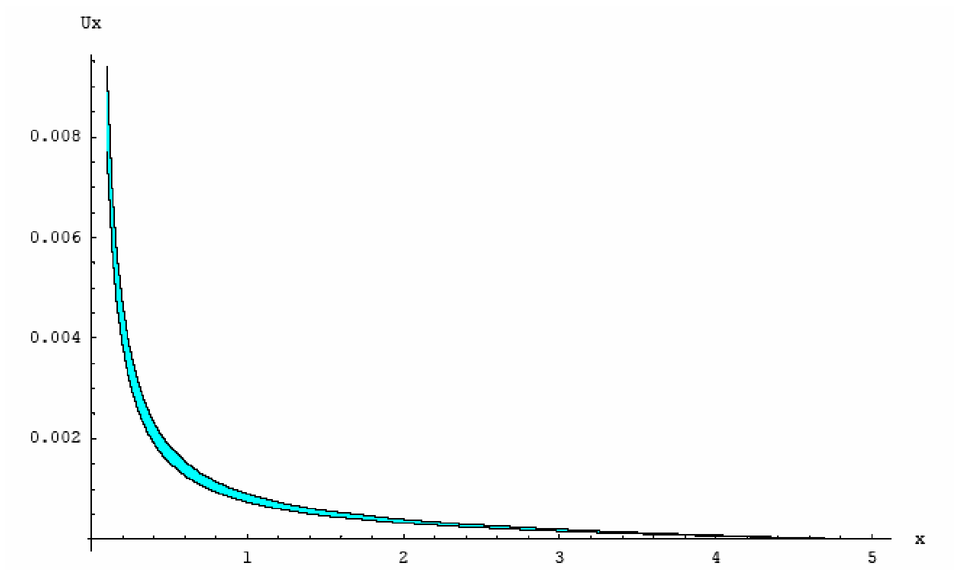

Fig. 4: Interval band for the displacements u_x along the line $y=0$.

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