The United Solution Set to 3D Linear System with Symmetric Interval Matrix

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1 Introduction

Consider the linear algebraic system

$$
\begin{aligned}
A(p)\cdot x \;&=\; b(p), & p &= (p_1,\ldots,p_m)^\top, & (1)\\
a_{ij}(p) &:= a_{ij,0}+\sum_{\mu=1}^{m} a_{ij,\mu}p_\mu, & b_i(p) &:= b_{i,0}+\sum_{\nu=1}^{m} b_{i,\mu}p_\mu, & (2)\\
a_{ij,\mu}, b_{i,\mu} &\in \mathbb{R}, & \mu &= 0,\ldots,m,\ i,j=1,\ldots,n. &
\end{aligned}
$$

The parameters p_μ, $\mu = 1,\ldots,m$ are uncertain and varying within given intervals

$$p \;\in\; [p] = ([p]_1,\ldots,[p]_m)^\top. \tag{3}$$

A set of solutions to (1)–(3), called *united parametric solution set*, is

$$\Sigma^p = \Sigma\left(A(p), b(p), [p]\right) \quad := \quad \{x\in\mathbb{R}^n \mid \exists p\in[p], A(p)x = b(p)\}. \tag{4}$$

Characterizing the solution set (4) by inequalities not involving the interval parameters is a fundamental problem useful for visualizing the solution set, exploring its properties and for computing componentwise boundaries. Apart from quantifier elimination, the only known general way of describing the parametric solution set is a lengthy and non-unique Fourier-Motzkin-type parameter elimination process presented in [1]. The special cases of symmetric and skew-symmetric solution sets

$$\Sigma_{sym} := \{x\in\mathbb{R}^n \mid Ax = b, A = A^\top, A\in[A], b\in[b]\}, \quad (A = -A^\top \text{ for } \Sigma_{skew})$$

are studied most exhaustively, see [1, 2] and the references given therein. Following a different approach than the Fourier-Motzkin-type parameter elimination, M. Hladík provided in [2] explicit descriptions of the symmetric and skew-symmetric solution sets which have the smallest, known by now, number of characterizing inequalities.
Basing on an improved Fourier-Motzkin-type parameter elimination process [3, Theorem 3.1] and some sufficient conditions (proven therein) for detecting superfluous characterizing inequalities, here we study the parameter elimination process and its properties for linear parametric systems involving 3×3 symmetric matrix. The consideration answers the open question, see [1], about the uniqueness of the parameter elimination process for the symmetric solution set. The obtained explicit description of the 3D symmetric solution set involves two times less number of characterizing inequalities than that reported in [2].

Let $\mathbb{R}^n, \mathbb{R}^{n\times m}$ be the set of real vectors with n components and the set of real $n \times m$ matrices, respectively. A real compact interval is $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n \times m$ matrices, respectively. Define mid-point $\dot{a} := (a^- + a^+)/2$ and radius $\hat{a} := (a^+ - a^-)/2$. These functionals are applied to interval vectors and matrices componentwise.

Definition 1. A parameter p_μ, $1 \leq \mu \leq m$, is of 1st class *if it occurs in only one equation of the system (1).*

Definition 2. A parameter p_μ, $1 \leq \mu \leq m$, is of 2nd class *if it is involved in more than one equation of the system (1).*

The elimination of 2nd class parameters from two inequality pairs is studied in [3].

Theorem 1 ([3]). *For two arbitrary inequality pairs (α) and (β)*

$$f_{0,\lambda}(x) + \sum_{\mu=1}^{m+s} f_{\mu,\lambda}(x)p_\mu \quad \leq 0 \leq \quad f_{0,\lambda}(x) + \sum_{\mu=1}^{m+s} f_{\mu,\lambda}(x)p_\mu, \qquad \lambda \in \{\alpha, \beta\},$$

involving $m+s$ interval parameters p_μ such that $f_{\mu,\lambda}(x) \not\equiv 0$ for all $\lambda \in \{\alpha,\beta\}$, $\mu \in M$, $Card(M) = m$ and $f_{\mu,\lambda}(x) \not\equiv 0$ for exactly one $\lambda \in \{\alpha,\beta\}$, $\mu \in S$, $Card(S) = s$, the elimination of all parameters yields the inequalities

$$\left| f_{0,\lambda}(x) + \sum_{\mu=1}^{m+s} f_{\mu,\lambda}(x)\dot{p}_\mu \right| \quad \leq \quad \sum_{\mu=1}^{m+s} |f_{\mu,\lambda}(x)|\hat{p}_\mu, \qquad \lambda \in \{\alpha,\beta\} \tag{5}$$

$$\left| \Delta_{0,i}(x) + \sum_{\mu=1,\mu\neq i}^{m+s} \Delta_{\mu,i}(x)\dot{p}_\mu \right| \quad \leq \quad \sum_{\mu=1,\mu\neq i}^{m+s} |\Delta_{\mu,i}(x)|\hat{p}_\mu, \qquad i \in M, \tag{6}$$

where $\Delta_{u,v}(x) := f_{u,\alpha}(x)f_{v,\beta}(x) - f_{u,\beta}(x)f_{v,\alpha}(x)$.

The proof of Theorem 1 was constructive showing which combination of inequalities is superfluous/redundant[1] and with respect to which cross inequality.

Corollary 1 (constructive). *For two inequality pairs $(\alpha), (\beta)$ involving the parameter p_1 in both inequality pairs and the parameter p_2 in only one, the elimination of p_1 generates the cross inequality (α,β) involving p_2. Then, in the elimination of p_2*

$\alpha \times (\alpha,\beta)$ *is* **superfluous** *to the inequality* (β) *if p_2 is involved in* (α)
$\beta \times (\alpha,\beta)$ *is* **superfluous** *to the inequality* (α) *if p_2 is involved in* (β).

The above cross inequalities are **redundant** *to (β), resp. (α), if more than one 2nd class or 1st class parameter have been eliminated before the elimination of p_2.*

[1] A cross inequality which is equivalent to another cross inequality is called superfluous, while a cross inequality which does not contribute to the boundary of the solution set is called redundant.

Corollary 2 (constructive). *For two inequality pairs $(\alpha),(\beta)$ involving the parameters p_1, p_2 in both inequality pairs, the elimination of p_1 generates the cross inequality[2] (α_1, β_1) involving p_2. Then the elimination of p_2 generates three more cross inequality pairs: $(\alpha_1, \beta_1)_2$, $\alpha_{1,2} \times (\alpha_1, \beta_1)_2$, $\beta_{1,2} \times (\alpha_1, \beta_1)_2$. The cross inequality pairs $\alpha_{1,2}(\alpha_1, \beta_1)_2/f_{1\alpha}(x)$ and $\beta_{1,2}(\alpha_1, \beta_1)_2/f_{1\beta}(x)$ (where $f_{1\alpha}(x)$, $f_{1\beta}(x)$ are the coefficient functions of p_1 in the inequalities (α), resp. (β)) are equivalent and therefore one of them is superfluous with respect to the other. The cross inequality pair $(\alpha_1, \beta_1)_2$ is either superfluous or redundant to $\alpha_{1,2}(\alpha_1, \beta_1)_2/f_{1\alpha}(x)$, resp. $\beta_{1,2}(\alpha_1, \beta_1)_2/f_{1\beta}(x)$ which yields only one active cross inequality (6) in the elimination of p_2 instead of 3.*

2 Parameter Elimination in 3D Symmetric System

Consider the the following slight generalization of the classical linear system with symmetric matrix, considered in [1, 2].

$$A(p)x = b(p), \qquad \text{where} \tag{7}$$

$$A(p) := A + B(q), \quad A = (a_{ij}) - \mathrm{diag}(a_{ii}) \in \mathbb{R}^{n\times n}, \ A = A^\top \tag{8}$$

$$B(q) = B^{(0)} + \sum_{\nu=1}^{s} B^{(\nu)} q_\nu \in \mathbb{R}^{n\times n}, \quad b(q) = b^{(0)} + \sum_{\nu=1}^{s} b^{(\nu)} q_\nu \in \mathbb{R}^{n\times 1}, \tag{9}$$

$$q = (q_1, \ldots, q_s) \ := \ (a_{11}, \ldots, a_{nn}, q_{n+1}, \ldots, q_s), \tag{10}$$

every q_ν, $\nu = 1, \ldots, s$, is involved in exactly one equation of the system and

$$p = (p_1, \ldots, p_m) := (a_{12}, \ldots, a_{n-1,n}, q_1, \ldots, q_s), \qquad p \in ([p_1], \ldots, [p_m]), \tag{11}$$

where $m = n(n-1)/2 + s$. The system (7)–(11) can contain in the diagonal elements of the matrix and in the right-hand side vector numerical values and an arbitrary but fixed number of 1st class parameters q_ν. We call this system *quasi-symmetric* and search for a description of its solution set by the improved Fourier-Motzkin-type elimination of parameters.

Consider a 3×3 quasi-symmetric system and assume that all 1st class parameters are eliminated from the trivial set of inequality pairs characterizing the solution set. Since all 1st class parameters behave the same way, without loss of generality we assume that each equation involves only one 1st class parameter q_i whose coefficient vector is $g_i(x)$, $i = 1, \ldots, n$. Let $\mathcal{N} = \{1, 2, 3\}$ be the index set of the three characterizing inequality pairs. For any $i \in \mathcal{N}$ and $\mathcal{N}_i = \{1, \ldots, n\} \setminus \{i\}$, the inequality pair (e_i) is

$$f_{i0}(x) + g_i(x)\dot{q}_i \mp |g_i(x)|\hat{q}_i + \sum_{j\in\mathcal{N}_i} x_j a_{ij} \ \le 0 \le \cdots, \tag{e_i}$$

where $f_{i0}(x) = A_{i\bullet 0}x - b_{i0}$ and "$\cdots$" denotes the whole expression in the left inequality with the bottom sign of $\mp$. For arbitrary $\alpha, \beta, \gamma \in \mathcal{N}$ we perform the elimination

[2] The subscript in the notation of the cross inequalities denotes which parameter is eliminated.

$a_{\alpha\beta}, a_{\alpha\gamma}, a_{\beta\gamma}$. The elimination of $a_{\alpha\beta}$ generates the following cross inequality pair

$$\begin{aligned} f_{\alpha 0}x_\alpha - f_{\beta 0}x_\beta + g_\alpha x_\alpha \dot{q}_\alpha - g_\beta x_\beta \dot{q}_\beta \mp |g_\alpha x_\alpha|\hat{q}_\alpha \mp |g_\beta x_\beta|\hat{q}_\beta \\ + \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta\}} x_\alpha x_j a_{\alpha j} - \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha\}} x_\beta x_j a_{\beta j} \ \le 0 \le \cdots \end{aligned} \qquad (e_{(\alpha,\beta)})$$

and updates the initial characterizing inequalities (e_α), (e_β) by combining the latter with the end-point inequalities for the parameter $a_{\alpha\beta}$.
The parameter $a_{\alpha\gamma}$ is involved in the updated inequality (e_α) and the inequalities $(e_\gamma), (e_{(\alpha,\beta)})$. So, in the elimination of $a_{\alpha\gamma}$ we have to consider all cross inequalities between these three inequality pairs. The cross between $(e_\alpha), (e_\gamma)$ gives

$$\begin{aligned} f_{\alpha 0}x_\alpha - f_{\gamma 0}x_\gamma + g_\alpha x_\alpha \dot{q}_\alpha - g_\gamma x_\gamma \dot{q}_\gamma \mp |g_\alpha x_\alpha|\hat{q}_\alpha \mp |g_\gamma x_\gamma|\hat{q}_\gamma + x_\alpha x_\beta \dot{a}_{\alpha\beta} \mp |x_\alpha x_\beta|\hat{a}_{\alpha\beta} \\ + \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta,\gamma\}} x_\alpha x_j a_{\alpha j} - \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha\}} x_\gamma x_j a_{\gamma j} \ \le 0 \le \cdots . \end{aligned} \qquad (e_{(\alpha,\gamma)})$$

The cross between $(e_\alpha), (e_{(\alpha,\beta)})$ is redundant to (e_β) by the constructive Corollary 1. Eliminating $a_{\alpha\gamma}$, the cross between $(e_\gamma), (e_{(\alpha,\beta)})$ gives

$$\begin{aligned} f_{\gamma 0}x_\gamma - f_{\alpha 0}x_\alpha + f_{\beta 0}x_\beta + g_\gamma x_\gamma \dot{q}_\gamma - g_\alpha x_\alpha \dot{q}_\alpha + g_\beta x_\beta \dot{q}_\beta \mp |g_\gamma x_\gamma|\hat{q}_\gamma \mp |g_\alpha x_\alpha|\hat{q}_\alpha \mp |g_\beta x_\beta|\hat{q}_\beta \\ + \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha\}} x_j x_\gamma a_{\gamma j} - \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta,\gamma\}} x_\alpha x_j a_{\alpha j} + \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha\}} x_\beta x_j a_{\beta j} \ \le 0 \le \cdots \end{aligned} \quad (e_{\gamma_\alpha(\alpha,\beta)})$$

The parameter $a_{\beta\gamma}$ is involved in the inequalities $(e_\beta), (e_\gamma), (e_{(\alpha,\beta)}), (e_{(\alpha,\gamma)}), (e_{\gamma_\alpha(\alpha,\beta)})$. In the elimination of $a_{\beta\gamma}$ we have to consider all cross inequalities between these five inequality pairs updated by combining them with the end-point inequalities for $a_{\alpha\beta}$ and $a_{\alpha\gamma}$. The cross between $(e_\beta), (e_\gamma)$ gives

$$\begin{aligned} f_{\beta 0}x_\beta - f_{\gamma 0}x_\gamma + g_\beta x_\beta \dot{q}_\beta \mp |g_\beta x_\beta|\hat{q}_\beta - g_\gamma x_\gamma \dot{q}_\gamma \mp |g_\gamma x_\gamma|\hat{q}_\gamma + x_\alpha x_\beta a_{\beta\alpha} \mp |x_\alpha x_\beta| a_{\beta\alpha} \\ -x_\alpha x_\gamma \dot{a}_{\gamma\alpha} \mp |x_\alpha x_\gamma|\hat{a}_{\gamma\alpha} + \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha,\gamma\}} x_\beta x_j a_{\beta j} - \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha,\beta\}} x_\gamma x_j a_{\gamma j} \le 0 \le \cdots . \end{aligned} \quad (e_{(\beta,\gamma)})$$

The cross between $(e_\beta), (e_{(\alpha,\beta)})$ is redundant to (e_α) by the constructive Corollary 1. Eliminating $a_{\beta\gamma}$, the cross between the updated $(e_\gamma), (e_{(\alpha,\beta)})$ gives

$$\begin{aligned} -f_{\gamma 0}x_\gamma - f_{\alpha 0}x_\alpha + f_{\beta 0}x_\beta - g_\gamma x_\gamma \dot{q}_\gamma - g_\alpha x_\alpha \dot{q}_\alpha + g_\beta x_\beta \dot{q}_\beta \\ \mp |g_\gamma x_\gamma|\hat{q}_\gamma \mp |g_\alpha x_\alpha|\hat{q}_\alpha \mp |g_\beta x_\beta|\hat{q}_\beta - 2x_\alpha x_\gamma \dot{a}_{\gamma\alpha} \mp 2|x_\alpha x_\gamma|\hat{a}_{\gamma\alpha} \\ - \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha,\beta\}} x_\gamma x_j a_{\gamma j} - \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta,\gamma\}} x_\alpha x_j a_{\alpha j} + \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha,\gamma\}} x_\beta x_j a_{\beta j} \ \le 0 \le \cdots . \end{aligned} \qquad (e_{\gamma_\beta(\alpha\beta)})$$

Eliminating $a_{\beta\gamma}$, the cross between the updated $(e_\beta), (e_{(\alpha,\gamma)})$ gives

$$\begin{aligned} -f_{\beta 0}x_\beta - f_{\alpha 0}x_\alpha + f_{\gamma 0}x_\gamma - g_\beta x_\beta \dot{q}_\beta - g_\alpha x_\alpha \dot{q}_\alpha + g_\gamma x_\gamma \dot{q}_\gamma \\ \mp |g_\beta x_\beta|\hat{q}_\beta \mp |g_\alpha x_\alpha|\hat{q}_\alpha \mp |g_\gamma x_\gamma|\hat{q}_\gamma - 2x_\alpha x_\beta \dot{a}_{\alpha\beta} \mp 2|x_\alpha x_\beta|\hat{a}_{\alpha\beta} \\ - \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha,\gamma\}} x_\beta x_j a_{\beta j} - \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta,\gamma\}} x_\alpha x_j a_{\alpha j} + \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha,\beta\}} x_\gamma x_j a_{\gamma j} \le 0 \le \cdots . \end{aligned} \qquad (e_{\beta_\gamma(\alpha\gamma)})$$

The cross between the updated $(e_\gamma),(e_{(\alpha,\gamma)})$ is redundant to (e_α) by Corollary 1. Eliminating $a_{\beta\gamma}$, the cross between the updated $(e_\beta),(e_{\gamma_\alpha(\alpha,\beta)})$ gives

$$\begin{aligned} &f_{\beta 0}x_\beta - f_{\gamma 0}x_\gamma + f_{\alpha 0}x_\alpha + g_\beta x_\beta \dot{q}_\beta - g_\gamma x_\gamma \dot{q}_\gamma + g_\alpha x_\alpha \dot{q}_\alpha \\ &\mp 3|g_\beta x_\beta|\hat{q}_\beta \mp |g_\gamma x_\gamma|\hat{q}_\gamma \mp |g_\alpha x_\alpha|\hat{q}_\alpha + 2x_\alpha x_\beta \dot{a}_{\alpha\beta} \mp 2|x_\alpha x_\beta|\hat{a}_{\alpha\beta} \\ &+ \sum_{j\in\mathcal{N}_\beta\setminus\{\alpha,\gamma\}} x_\beta x_j a_{\beta j} - \sum_{j\in\mathcal{N}_\gamma\setminus\{\alpha,\beta\}} x_j x_\gamma a_{\gamma j} + \sum_{j\in\mathcal{N}_\alpha\setminus\{\beta,\gamma\}} x_\alpha x_j a_{\alpha j} \ \le 0 \le \cdots . \end{aligned}$$

This inequality pair is superfluous to $(-1)(e_{\beta_\gamma(\alpha,\gamma)})$ if there are no 1st class parameters, or redundant to the latter otherwise (due to the extra positive term $2|g_\beta x_\beta|\hat{q}_\beta$). Analogously, eliminating $a_{\beta\gamma}$, we prove that the following cross inequalities are superfluous (if there are no 1st class parameters) or redundant (otherwise). The cross inequality pair between $(e_\gamma),(e_{\gamma_\alpha(\alpha,\beta)})$ is superfluous/redundant to $(-1)(e_{\gamma_\beta(\alpha,\beta)})$ by Corollary 2. The cross inequality pair between the updated $(e_{(\alpha,\beta)}),(e_{(\alpha,\gamma)})$ is superfluous/redundant to $(-1)(e_{(\beta,\gamma)})$. The cross inequality pair between $(e_{(\alpha,\beta)})$, $(e_{\gamma_\alpha(\alpha,\beta)})$ is superfluous/redundant to $(e_{\gamma_\beta(\alpha,\beta)})$ by Corollary 2. The cross inequality pair between $(e_{(\alpha,\gamma)})$, $(e_{\gamma_\alpha(\alpha,\beta)})$ is superfluous/redundant to $(e_{\beta_\gamma(\alpha,\gamma)})$.

Thus, for a 3-dimensional quasi-symmetric linear system (7)–(11), we proved above by Corollaries 1, 2 and by a direct evaluation that the improved Fourier-Motzkin-type parameter elimination of $a_{\alpha\beta}, a_{\alpha\gamma}, a_{\beta\gamma}$ yields 6 active characterizing cross inequalities

$$e_{(\alpha,\beta)},\ e_{(\alpha,\gamma)},\ e_{\gamma_\alpha(\alpha,\beta)},\ e_{(\beta,\gamma)},\ e_{\beta_\gamma(\alpha,\gamma)},\ e_{\gamma_\beta(\alpha,\beta)},$$

and 6 superfluous/redundant cross inequalities. The active characterizing cross inequalities are two times less than those reported in [2]. Since α,β,γ are taken arbitrary from the set $\{1,2,3\}$, for different orders of parameter elimination the active characterizing cross inequalities are the same (up to the order of their generation). Exchanging the order of inequalities or the order of parameter elimination we have

$$e_{(\alpha,\beta)} = -e_{(\beta,\alpha)} \tag{12}$$

$$e_{\gamma_\alpha(\alpha,\beta)} = \begin{cases} e_{\beta_\alpha(\alpha,\gamma)} & \text{if } \alpha<\gamma \\ -e_{\beta_\alpha(\gamma,\alpha)} & \text{if } \alpha>\gamma \end{cases} \tag{13}$$

$$e_{\gamma_\beta(\alpha,\beta)} \overset{(12)}{=} (-1)e_{\gamma_\beta(\beta,\alpha)} \overset{(13)}{=} \begin{cases} (-1)e_{\alpha_\beta(\beta,\gamma)} & \text{if } \beta<\gamma \\ e_{\alpha_\beta(\gamma,\beta)} & \text{if } \beta>\gamma. \end{cases} \tag{14}$$

Table 5: Elimination schema for the 2nd class parameters in a 3D system with symmetric matrix. Rows, labeled in the second column, represent the characterizing inequality pairs. The occurrence of the parameters is represented column-wise: “v” denotes the occurrence of a non-eliminated parameter in the corresponding inequality, “!” denotes the occurrence of an eliminated parameter, dashes denote lack of a corresponding parameter.

	ineqs	a_{12}	a_{13}	a_{23}	
	(1)	v	v	–	
	(2)	v	–	v	
el.par.:	(3)	–	v	v	superfluous ineqs
a_{12}	(1,2)	–	v	v	
a_{13}	(1,3)	!	–	v	
	$3_1(1,2)$	–	–	v	
					$1_3(1,2)$ by Corollary 1
a_{23}	(2,3)	!	!	–	
	$3_2(1,2)$	–	!	–	
					$3_2(1,3)$ by Corollary 1
					$3_2 \times 3_1(1,2)$ by Corollary 2
					$2_3(1,2)$ by Corollary 1
	$2_3(1,3)$	!	–	–	
					$2_3 \times 3_1(1,2) \overset{(13)}{\equiv} 2_3 \times 2_1(1,3)$ red. to $2_3(1,3)$
					$(1,2)_3 \times (1,3) \equiv 3_2 \times 3_1(1,2)$ red. to $3_2(1,2)$
					$(1,2)_3 \times 3_1(1,2)$ by Corollary 2

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