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BRIEF SURVEY OF THE MATHEMATICAL LEGACY OF ACADEMICIAN M. KRAVCHUK

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Abstract

This survey is devoted to the memory of the most outstanding mathematician in Ukraine, Academician Mykhailo Kravchuk. A brief summary on his life and mathematical legacy is provided.

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1. Introduction

Mykhailo Pylypovych Kravchuk is an outstanding Ukrainian mathematician and Academician of the Ukrainian Academy of Sciences who made a fundamental contribution to numerous branches of mathematics: algebra and number theory, theory of functions of real and a complex variable, theory of differential and integral equations, probability theory and mathematical statistics, etc. His entire multifaced creative and public life was closely connected with the scientific institutions, institutes of post-secondary and secondary education in Ukraine. In the description given in a speech in 1929 at the Academy of Sciences upon Kravchuk's election as academician, it was fairly remarked that: "...there is no event in the creation of mathematical science (in Ukraine) that has occurred without his involvement... the first Ukrainian secondary schools in the towns and in the countryside, the first university courses, the first Ukrainian universities, both people's university and state-funded university (both in Kyiv), ... mathematical terminology and the language of science... - none of this could have happened without the most active participation of M.P. Kravchuk". Mykhailo Kravchuk was born September 27, 1892, in the village of Chovnytsya (Chovnitsy), now in Kivertsi district in Volyn' (Volhynia) to a land surveyor. In 1901, together with his parents he moved to Luts'k, where in 1910 he graduated from the gymnasium with a gold medal. In 1910 Kravchuk entered the Mathematics Department of the Faculty of Mathematics and Physics of the Kyiv University. There he studied mathematics, physics, and astronomy with great enthusiasm and actively participated in the scientific seminars directed by Professors D.O. Grave and B.Ya. Bukreyev.

M. Kravchuk graduated in 1914 with diploma of the first degree, and remained at the university with a teaching fellowship that enabled him to prepare for scientific research and teaching. In 1915 - 1917 he completed his master's examinations and wrote several articles on the subject of linear algebra, Ukrainian mathematical terminology, and other areas.

The national, cultural and state rebirth of Ukraine following the overthrow of the Czarist regime thrilled the young scientist. He persistently and successfully labored to help in the development of the Ukrainian science and in the creation of secondary and post-secondary Ukrainian schools, and was involved in the education of talented students. In the first years following the revolution, along with his considerable exclusively scientific research, Kravchuk taught mathematics at Ukrainian Gymnasium I and II in Kyiv, and at the Ukrainian People University, served as a member of the Ukrainian Scientific Society in Kyiv and as a member of the Mathematics and Physics Society of Kyiv University, was an associate of the newly created Ukrainian Academy of Sciences, later member of the Committee on Mathematical Terminology of the Institute of the Language of Science of the Ukrainian Academy of Sciences.

During the difficult years of devastation, Kravchuk returned to the countryside. In 1920 - 1921 he was teacher and director of a school in the village of Savarka (now in Bohuslav district near Kyiv).

For a period of many years he taught several mathematics courses at a number of higher educational institutions, including the state university, school of electrical engineering, and at the polytechnic, architectural, veterinarian and zoological, agricultural, and other institutes of Kyiv.

The considerable educational work and public services Kravchuk carried out, were combined with a wide-ranging and diverse scientific creativity. He obtained a number of fundamental results in the theory of alternating matrices and the theory of bilinear and linear transformations. These results laid the foundation for his doctoral dissertation, which Kravchuk brilliantly defended in 1924. In 1925 he was awarded the title of professor.

Kravchuk's great scientific productivity, and the originality and versatility of his thinking, his insistence on exactitude and wealth of scientific accomplishment and his dedication to science inspired his colleagues and associates, and produced an expanded circle of students and followers. In September 1928 Kravchuk participated in the International Mathematical Congress held in Italy in the city of Bologna, where he presented several important papers and on the way back to Ukraine delivered a lecture at a meeting of the French Mathematical Society in Paris. In 1927 Kravchuk was elected a member of the French Mathematical Society. On June 1929, he was unanimously elected an active member of the All-Ukrainian Academy of Sciences. That year and the following eight years were the most fruitful in the creative work of this outstanding mathematician. Kravchuk was also involved in public activities. He was a member of the Council of the Kyiv Institute of Public Education, Dean of the Faculty of occupational education, member of the local council of the Ukrainian Council of the Section of Workers in Science, and other organizations.

The selfless efforts Kravchuk undertook for the sake of the development of science in Ukraine, his extraordinary talent as a teacher and his reputation among students and young mathematicians could not go unnoticed. *On February 21, 1938 he was arrested by the Soviet secret police on fabricated political and spying charges*, accused of involvement in a sort of typical "counter-revolutionary" activities - Ukrainian bourgeois nationalism, foreign spying, etc. - charges that were common in those years.

On September 23, 1938 Kravchuk was sentenced to 20 years of confinement to prison and fine years of exile, and was sent to Kolyma (region in Siberia). He spent in the labor/death camp Gulag three winters and summers of imprisonment, ill, becoming weaker and weaker, *and on March 9, 1942, he departed for eternal life from the frigid wastes of Kolyma. He was posthumously rehabilitated on September 15, 1956.*

But only on the 20th of March 1992, almost 100 years after his birth, Mykhailo Kravchuk was readmitted to membership in the National Academy of Sciences of Ukraine (NANU). The same year his name was entered in the International Calendar of Scientists by UNESCO. The First Kravchuk International Conference was held at Kyiv Polytechnic Institute in 1992. Since that time, there were 12 such conferences.

Recently, three books of M. Kravchuk's works were published in Kyiv. On the 16th of May 2002, the National Technical University of Ukraine (formerly known as Kyiv Polytechnic Institute) named an auditorium in his honor, and on the 20th of May 2003, the NTUU unveiled a statue of M. Kravchuk. A documentary film "Holhofa academika Kravchuka" (The Golgotha of Academician Kravchuk), in Ukranian and English, was released and shown at the conferences and television (the video can be found on the link *http://video.google.com/videoplay?docid=2619247902295160660&q =DVD_VIDEO_RECORDER.MPG&hl=en*).

Mykhailo Kravchuk was the author of more than 180 scientific works, including more than 10 monographs, in a number of branches of mathematics, including algebra and number theory, theory of functions of a real and complex variable, theory of differential and integral equations, mathematical statistics and probability theory, etc.

The results obtained by Kravchuk during his years of scientific creativity concern such fundamental lines of research as the following: - research in the theory of permutation matrices, quadratic and bilinear forms, linear transformations, theory of algebraic and transcendental equations, number theory; - research in the theory of functions of a real and complex variable; discussion of problems of interpolation, extension of the method of least squares to the theory of approximate solution of differential and integral equations; - the creation and mathematical proof of the general method of moments and its application to the approximate solution of ordinary linear differential equations, equations of mathematical physics, integral equations, and other areas; development of the theory of correlation, application of the method of moments in mathematical statistics; - introduction and use of polynomials associated with the binomial distribution, now known in the world literature as Kravchuk's polynomials.

The scientific interests of the young Kravchuk were formed mainly under the influence of the mathematical studies of D.O. Grave, who had taught at Kyiv University from 1902 on. The scientific seminars which Prof. Grave conducted served as a foundation for the future world-famous Kyiv school of algebra, which included such well-known algebraists as M. Kravchuk, O. Schmidt, M. Chebotarev, B. Delone, E. Zhilins'kii, O. Ostrovskii, etc.

2. Algebra and number theory

In his development of the traditions of the general Kyiv school of mathematics, Kravchuk worked on questions in linear algebra, the expansion of algebraic and transcendental equations, and on number theory. His first study ("On the parametric representation of permutation matrices") was written in 1913. In 1905 the German mathematician I. Schur proved that the number of linearly independent matrices of a permutation group of matrices of order n is at most $[n^2/4]+1$. Kravchuk gave a simple proof of Schur's theorem, using what are known as the Frobenius radical groups. He constructed a permutation radical group containing linearly independent matrices, and proved that there does not exist a permutation radical group of order n with greater number of linearly independent matrices (his work [4]). Kravchuk proved Schur's theorem for the general case, establishing in particular, that if a matrix M of rank $(\nu-1)$ belongs to an elementary permutation set of matrices of order n such that $M^{\nu-1}\neq 0$, but $M^{\nu}=0$, the greatest possible number of linearly independent matrices of the corresponding complete set is given by the quantity

$$\left[\frac{(n-\nu)^2}{4}\right]+n=\left[\frac{(n-\nu+2)^2}{4}\right]+\nu-1.$$

But if the highest rank of M is equal to $\nu-1$, the number of linearly independent matrices of the corresponding complete set is equal to n. Kravchuk completely solved the problem of finding all complete permutation sets of matrices of order n whose matrices satisfy a quadratic equation. Since the time of Lagrange, Cauchy, and Jacobi, the problem of linear transformation of quadratic forms of an arbitrary number of variables had been a classical problem possessing important applications in different branches of mathematics and in other sciences. Jacobi, Weierstrass, and Kronecker had achieved considerable results in the development of the theory of bilinear and quadratic forms. The investigations that Kravchuk completed represented a worthy continuation of their work. *His doctoral dissertation, "On quadratic forms and linear transformations"* ([4]), represented a basic study in the theory of permutation matrices and quadratic and bilinear forms. In particular, Kravchuk also gave a generalization and simplification of basic problems from the theory of quadratic and bilinear forms, and solved numerous difficult problems from the theory of linear transformations. In the first section of his dissertation he generalized the Lagrange method of finding the squares of quadratic forms and the methods of Plücker - Gundelfinger, Gundelfinger, and Jacobi for the expansion of a quadratic form into a sum of squares of linear forms. Kravchuk obtained the following convenient formula for the expansion of a quadratic form into a sum of squares of linear form:

$$2f(x) = \frac{1}{A^{(p)}} \sum_{1}^{p} A_y^{(p)} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \frac{1}{(A^{(p)})^2 A^{(p+p')}} \sum_{p+1}^{p+p'} A_y^{(p+p')} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j}$$

$$+ \frac{1}{(A^{(p+p')})^2 A^{(p+p'+p'')}} \sum_{p+p'+1}^{p+p'+p''} A_y^{(p+p'+p'')} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} + \dots$$

$$\left(A^{(p)}, A^{(p+p')}, A^{(p+p'+p'')}, \dots \neq 0 \right),$$

where $2f(x) = \sum_{i,j=1}^{n} a_{ij} x_i x_j$ is a quadratic form, and

$$A^{(p)} = \begin{vmatrix} a_{11}, \dots, a_{1p} \\ \dots\dots\dots\dots \\ a_{p1}, \dots, a_{pp} \end{vmatrix}, \quad A_y^{(p)} = \frac{\partial A^{(p)}}{\partial a_y}.$$

This formula of Kravchuk is also valuable in that it makes to use Hermitian forms in the study of the roots of an algebraic equation. Note that Kravchuk's formula may also be used whenever there are zeros among the leading minors of the discriminant of a quadratic form, in which case Jacobi's formula cannot be applied.

The second section of Kravchuk's dissertation is devoted to the theory of transformations of sheaves of bilinear and quadratic forms to a canonical expression. Using elementary matrix transformations, Kravchuk made an essential simplification in and elegantly set forth the results of Weierstrass and Kronecker, establishing new necessary and sufficient equivalence conditions between two sheaves of bilinear forms.

Kravchuk gave a method of constructing a canonical form for every (singular or nonsingular) sheaf of bilinear forms without enlarging the valuation domain determined by the coefficients of the given forms.

In the third section, Kravchuk for the first time set forth a method of determining the maximal number of linearly independent matrices in a permutation set containing an arbitrary given matrix, and demonstrated a method of constructing all sets that possesses maximal characteristic. He constructed a set of permutation matrices for the case in which a transformation whose characteristic determinant possesses only two nonconstant elementary divisors of degree n_1 and n_2 relative to λ belongs to a complete commutative elementary set of linear transformation of order n. He succeeded in finding the general case of a set that possesses n linearly independent transformations, moreover, he gave the most general case of a set in which the nonconstant elementary divisors of the characteristic polynomial of its linear transformations are all equal to zero.

In the studies [5], [6] and others, Kravchuk considered the theory of unitary and orthogonal transformations, in particular, he gave an expansion of unitary and orthogonal (real and complex) transformations of order n into unitary and orthogonal transformations of order two.

In the number theory, Kravchuk proved several theorems concerning the cubic field and the distribution of primes as classes of permutations. In particular, he gave a simple proof of Delone's theorem concerning a binomial identity of an algebraic number field depending on a cubic root of unity, and simplified the proof of Frobenius' and Chebotarev's theorems concerning the distribution of the primes relative to classes of permutations, [7]-[11]. Specifically, he obtained important results that significantly simplified corresponding proofs given by Chebotarev, who had been one of his examiners in his doctoral dissertation (1927).

3. Theory of functions of real and complex variables

Kravchuk presented his basic results on analytic functions in many journal articles and in a *monograph entitled "Algebraic Studies on Analytic Functions"*. It is well known that the subject of the most general form of analytic functions that satisfy the algebraic addition theorem was first posed and solved by Weierstrass in his course of lectures on elliptic functions, though he did not publish a proof of the theorem. Subsequently, other mathematicians (Phragmen, Schwartz, etc.) gave a proof of this theorem, though it was complicated and involved to the theorem of Picard referred to above and to other results. Kravchuk was the first to prove the following fact:

If $\varphi(u)$ is an algebraic function of u or of $\exp(2\pi i/\omega)$, or, finally, of the Weierstrass function $\rho(u/\omega, \omega')$, a functional equation of the form

$$G\left(\varphi(u), \varphi(v), \varphi(u+v)\right) = 0$$

holds, where G is the symbol for a polynomial.

Note that Weierstrass' theorem has a converse. In his proof of this assertion, Kravchuk relied on one of his earlier studies.

A number of valuable results found by Kravchuk are given in his monograph. On the basis of the studies of A. Markov and T. Stieltjes in the theory of continued fractions and Hadamard's classical results on the poles of meromorphic functions, Kravchuk formulated and proved necessary and sufficient conditions under which m smallest (in modulus) poles of a meromorphic function are real (or positive) and with positive residue. In a generalization of the results of Hermite associated with Sturm's theorem, Kravchuk proved, by means of an appropriate generalization of the quadratic forms, a criterion according to which there exist among the m smallest (in modulus) poles of a meromorphic function $(m-2s)$ real poles with positive residue.

Kravchuk proved that if $f(z) = s_0 + s_1 z + s_2 z^2 + \dots$, and in addition,

$$s_0 > 0, \quad \begin{vmatrix} s_0 s_1 \\ s_1 s_2 \end{vmatrix} > 0, \quad \begin{vmatrix} s_0 s_1 s_2 \\ s_1 s_2 s_3 \\ s_2 s_3 s_4 \end{vmatrix} > 0,$$

then $f(z)$ is a function of the form $R_0 + \frac{\Sigma R}{1-lz}$ $(R_0 \geq 0, R \geq 0, l - \text{real})$, provided that the derivative of the set of singular points of this function is zero. In particular, all the poles of the function $zf(z)$ are simple with negative residues, and every isolated singular point of this function represents a simple real pole.

In his article [11], Kravchuk generalized the Laguerre theorem concerning the zeros of the derivatives of entire functions of finite kind, and obtained a number of significant results, in particular:

If the simple function $f(z) = s_{2k}z + s_{2k+1}z^2 = \dots$ satisfies the conditions

$$s_{2k} \geq 0, \quad \begin{vmatrix} s_{2k} s_{2k+1} \\ s_{2k+1} s_{2k+2} \end{vmatrix} \geq 0, \quad \begin{vmatrix} s_{2k} s_{2k+1} s_{2k+2} \\ s_{2k+1} s_{2k+2} s_{2k+3} \\ s_{2k+2} s_{2k+3} s_{2k+4} \end{vmatrix} \geq 0,$$

then the sum $F(1/x) = f(1/x) + \varphi(1/x)$ has a zero between every two neighboring isolated singular points x' and x'' of the function $f(1/x)$, provided that the function $\varphi(1/x)$ does no have any singular points on the closed interval $[x', x'']$.

If p is not the least of the powers of the polynomials $xP(x)$ and $Q(x)$, and $\varphi(1/x) = F(x)/Q(x)$, the function $F(1/x)$ will have no more than p zeros, except for those that alternate with the poles of $f(1/x)$.

If, moreover, the function $F(x) = H/G$ is meromophic, the two entire functions $H(z)$ and $G(z)$ are of finite order and of the same kind. Hence, if $H = G'$, Laguerre's case follows.

Studying Kravchuk's scientific works, one is inevitably inspired by their singular degree of refinement, grace, and elegant scientific form. A clear illustration is Kravchuk's proof of an existence theorem for the root of an algebraic equation. Kravchuk considered the function

$$\frac{1}{1 + a_1x + a_2x^2 = \cdots + a_px^p} = 1 + b_1x + b_2x^2 + \ldots. \tag{1}$$

Under the condition that of the numbers $|b_{n+1}|, |b_{n+2}|, \ldots, |b_{n+2p-1}|$ none is greater than $|b_{m+i}|$, where i is one of the numbers $1, 2, 3, \ldots, 2p-1$, the following inequality is satisfied:

$$|a_p| \le p\left[\left|\sqrt[n+i]{b_{m+i}}\right|^{n+i/n+p}\right]^p,$$

from which it is clear that the series (1) converges but not for all values of the variable x, and, therefore, the function on the left side of (1) possesses singular points and these singular points may only be poles. But now the poles are the zeros of the polynomial

$$1 + a_1x + \cdots + a_px^p. \tag{2}$$

Thus, the existence of a root for every algebraic equation is proved. Subsequently, Kravchuk presented yet another inequality,

$$\lim_{n\to\infty} \sup \sqrt[n]{b_n} \ge \left|\sqrt[p]{a_p}\right|,$$

establishing moreover, that the modules of all of the zeros of the polynomial (2) are not greater than the quantity $\left|\sqrt[p]{1/a_p}\right|$.

Kravchuk devoted more than 20 studies to questions in the theory of functions of a real variable. He proved some important general interpolation formulas over equal intervals, in particular:

$$\textit{If}\ \ f(z) = \sum_{i=0}^{n} \frac{1}{2\alpha} \int\limits_{x-\alpha}^{x+\alpha} \frac{\sin \pi nz}{\pi n} \frac{(-1)^n f(x_i)}{z - x_i}\, dz \quad (\alpha = n^{2\delta-1},\ \frac{1}{3} < \delta < \frac{1}{2}),$$

then it converges in every interior sub-interval of the open interval $(0, 1)$.

Existence conditions for the leading derivatives of a function of a real variable had interested many mathematicians, in particular Montel, Krylov, Bogolyubov, and Kravchuk himself, who gave an exhaustive answer to this question, that is:

(a) a sufficient existence condition for the derivative d^ky/dx^k on the interval $(0, 1)$ is that the following quantity is bounded:

$$\sqrt{\frac{1}{n}\sum_{i=0}^{n-k}\left(\frac{\Delta^{k+1}y_i}{\Delta x^{k+1}}\right)^2};$$

(b) if the expression

$$(x-a)^{k+r}\sum_{i=0}^{m-k}\frac{(x-\alpha-\overline{i+1h})(x-\alpha-\overline{i+kh})}{k!}$$

$$\times\frac{\Delta y^{k+1}(\alpha+ih)}{h^{k+1}}\quad (0\le\alpha\le a-1,\ \alpha\le x\le 1, 0<r\le 1,\ h=\frac{x-\alpha}{m})$$

is bounded, then the derivative $y^{(k)}(\alpha)$ exists and the function

$$\frac{\Delta y^{(k)}(\alpha)}{h^r}=\frac{y^{(k)}(\alpha+h)-y^{(k)}(\alpha)}{h^r}$$

is bounded, and conversely. Kravchuk obtained analogous results for the derivatives of functions of several variables.

Kravchuk investigated the Green and Stokes transformations from the standpoint of generalizing the continuity conditions imposed on the functions that occur in these transforms, and also for the purpose of generalizing the transformation formulas themselves.

In [13] Kravchuk established the following interesting property of convex functions:

If the function $\varphi(x)$ is increasing and convex, and $A_1<A_2<\dots<A_n$, the sum

$$[\varphi(A_2-A_1)+\varphi(A_n-A_{n-1})]+[\varphi(A_3-A_1)+\varphi(A_4-A_2)+\dots]$$

is the minimum of all sums of the form

$$\varphi(|A_{\alpha_1}-A_{\alpha_2}|)+\varphi(|A_{\alpha_2}-A_{\alpha_3}|)+\dots+\varphi(|A_{\alpha_n}-A_{\alpha_{n-1}}|)+\varphi(|A_{\alpha_1}-A_{\alpha_n}|).$$

In his investigation of the remainder of Lagrange's series, Kravchuk obtained a new formula for the remainder in the case of real variables from which Chebyshev's and Zolotar'ov's formulas follow as special cases.

4. Theory of differential and integral equations

It is worth mentioning Kravchuk's studies in the theory of differential and integral equations, in particular – in methods of approximating the solutions of these equations.

M. Kravchuk's works in the theory of approximate integration of differential and integral equations, together with the studies of M. Krylov, M. Bogolyubov, etc. furthered the active use of variational methods in the approximate solution of a number of problems in applied mathematics and physics. His methods are of special importance today in connection with the development of cybernetics, particularly in the programming of many complex phenomena and processes.

Kravchuk successfully developed the method of least squares in the theory of approximate integration of differential and integral equations. He investigated the method of least squares from two approaches, first attempting to reduce the error of the approximate solutions and, second, proving that the derivatives of these approximations converge if the functions which the approximate solutions involve are chosen appropriately. Most importantly is however that Kravchuk proved the convergence of Krylov's method in the general case. Note that Kravchuk's method combines the advantages of the method of least squares as regards the proof of convergence and the simplicity of the computation of the coefficients of Ritz' approximate method. Kravchuk also presented these results at the International Mathematical Congress held in Bologna in 1928. In particular, in a paper he read to the Congress, he presented the following general theorem:

If the equation

$$L(y) \equiv y^{(k)} + A_1(x)y^{(k-1)} + \cdots + A_k(x)y = f(x)$$

possesses an integral, boundary conditions of the type

$$\sum_{i=0}^{k-1}[\alpha^{(k)}y(0) + \beta_i^{(k)}(1)] = 0$$

will be satisfied, and whenever the functions $\varphi_i(x)$ *satisfy the same conditions, and the functions* $\varphi_i^{(k)}(x)$ *satisfy the complete system, the function*

$$y_m = a_1\varphi_1(x) + a_2\varphi_2(x) + \cdots + a_m\varphi_m(x),$$

determined by the equations

$$\int_0^1 L(y_m)\varphi_i^{(k)}(x)dx = \int_0^1 f(x)\varphi_i^{(k)}(x)dx \quad (i = 1, \ldots, m),$$

will also satisfy the following equations:

$$\lim_{m\to\infty} y_m = y, \ \lim_{m\to\infty} y_m^{'} = y^{'}, \ \ldots, \ \lim_{m\to\infty} y_m^{(k-1)} = y^{(k-1)}.$$

The overwhelming majority of Kravchuk's studies in the theory of approximate integration (as [14], [15], [16], etc) are devoted to the development and application of the method of moments to the approximate solution of ordinary linear differential equations, linear equations of mathematical physics, partial differential equations, and integral equations.

The essential idea of the method of moments is to determinate a function on a given interval (a, b) in terms of the moments on the same interval, that is, in terms of the integrals

$$\int_a^b f(x)\,\varphi_i(x)dx \quad (i = 0, 1, 2, \dots),$$

where $\varphi_i(x)$ are given functions.

His basic results in the theory of moments were published in a fundamental two-volumes monograph, *"Applications of the Method of Moments to the Solution of Linear Differential and Integral Equations"* ([15], [16]).

The first volume of this work represents an investigation of the method of moments applied to the approximate solution of ordinary linear differential equations and systems of ordinary linear differential equations. In the following sections of his monograph, Kravchuk showed how to dispense with Ritz' variational algorithm by adopting a more general method. These results of Kravchuk were of considerable importance, since they markedly enlarged the range of application of the method, and made it possible to extend it to non-selfadjoint problems. By the very nature of the method of selecting the function $\varphi_i(x)$ referred to above, as was underscored by Kravchuk, he was also able to obtain exact solutions for line one-dimensional problems in the form of definite series in which the function $\varphi_i(x)$ plays a role analogous to the role of the fundamental functions, though with the advantage that they may always be taken exactly, moreover, in simple form. In the third section, Kravchuk constructed a complete theory of the approximation method independent even of general theorems for the existence of solutions. This, without a doubt, represented a great scientific unification of theory and applications, and was also of importance in terms of methodology, as it opened the way to new and promising investigations in this area. And in fact, these investigations liberated work in practical applications from the need to perform extraordinarily lengthy calculations involving characteristic numbers and fundamental functions, moreover, they pointed the way towards the creation of functions that could be substituted for characteristic functions in the case of non-selfadjoint problems.

In the second volume of the monograph, linear partial differential equations encountered in the field of mathematical physics were considered. In is well known that Ritz' method produces a convergent results even in the case of the Laplace equations, and Ritz himself proved the convergence of the method bearing his name for the biharmonic equation under boundary conditions of the type

$$z = \theta(s),\ \frac{dz}{dn} = \tilde{\theta}(s)$$

on the closed paths bounding the region in which the function in which the function $z(x, y)$ is defined. Kravchuk generalized Ritz' method, in particular, he proved its convergence for both these types of equations and also for far more general equations. In his investigation, Kravchuk limited his discussion to linear elliptic differential equations with two independent variables, and his method is general in nature, applicable to linear partial differential equations with arbitrary number of independent variables and also to systems of linear differential equations. The method of moments which Kravchuk developed for the approximate integration of linear equations in mathematical physics possesses the flowing truly exceptional property. For example, in the case of ordinary differential equations of order k not does the approximation y_m itself converge, but so do its derivatives though order $(k-1)$-inclusively, and the k-th converges in mean; for equations with two independent variables of order k, the derivatives (though order $(k-2)$ inclusively) of the corresponding approximate solution all converge, and the $(k-1)$-th derivative converges in mean starting with the highest terms of the corresponding differential equation.

Kravchuk established that the method of moments yields an approximation to the integral of the particular problem for a rather broad class of linear differential equations with linear conditions of the same order as the approximation of this integral by a Fourier sum. Subsequently, he extended these results to a more general class of problems and to different methods used in the analytic representation of unknown functions.

In his monograph he also gave applications of the concepts discussed above to integral equations. Specifically, Kravchuk underscored that, at times, the use of methods that involve the substitution of integral equations for a given differential equation often does not simplify problems, and that Hilbert's suggestion to expand linear integral equations by means of series in orthogonal functions may, with appropriate changes, be applied directly to differential equations. Let us, however, consider one example illustrating the application of the method of moments to the approximate solution of integral equations.

Kravchuk skillfully combined into a single general principle the method of moments with the method of the variational algorithm and the method of least squares, both of which rely on the minimum principle, together with a large number of different types of these methods. This principle, moreover, is also obviously a direct generalization of the well-known method of moments.

The monograph also contains many bold thoughts original with Kravchuk concerning the dialectical development of scientific thinking concerning the calculus of variations with reference to the application of the general studies and methods employed in the work to problems in mathematical statistics, particularly the problem of the distribution curve, etc. In the conclusion, he wrote, "The need for internal completeness of method; the unsolved problems that have their roots in the ordinary method of moments; questions related to the use of the minimum principle in problems on mathematical physics that have not been completely answered; the mathematical apparatus of the statistical theory of contemporary physics; the actual evolution of the method in technical applications - it is topics such as these which speak to us of the need for further work beyond the extent outlined in the present monograph".

Mykhailo Kravchuk never learned about the role that his works played in the invention of the first electronic computer. Only recently (2001) Ivan Kachanovski (USA) discovered in the archives of the Iowa State University the fact, that the American scientist *John Atanasoff*[1] (1903-1995) took an interest in Kravchuk's works when he investigated the problem of making electronic computer. Atanasoff wrote to M. Kravchuk on September 9th, 1937:

"*I have found your series of papers on the approximate solution of differential equations very useful in my work. I would like to receive reprints of any of your papers which you have available. I am particularly interested in obtaining copies of those papers which you have published in Ukrainian journals...*"

In view of the terror campaign and mass denunciation of those scientists in USSR who maintained contacts with foreigners as spies, it is no wonder that J. Atanasoff did not receive Kravchuk's reply. On November 16th, 1937

[1] Ed. Note: of Bulgarian origin, see *http://en.wikipedia.org/wiki/John_Vincent_Atanasoff*: John Vincent Atanasoff (October 4, 1903 - June 15, 1995) was a Bulgarian-American physicist. The 1973 decision of the patent suit Honeywell v. Sperry Rand named him *the inventor of the first automatic electronic digital computer*, a special-purpose machine that has come to be called *the Atanasoff-Berry Computer* (ABC). *The son of a Bulgarian immigrant* who became an electrical engineer, Atanasoff held positions as a teaching professor, a governmental wartime research director, and a corporate research executive before being recognized in the 1970s and 1980s for digital electronic computer research he conducted at Iowa State College in the late 1930s and early 1940s.

Atanasoff wrote to the Ukrainian Association for Cultural Relations and asked again to send him the two monographs by Kravchuk. John Atanasoff wrote that the library of the Iowa State University ordered the monographs published by Kravchuk in 1932 and 1936 "as well as all future issues" from a German intermediary but received nothing. The records at the university library indicate that the two-volume book was already in circulation in March 1938. *John Atanasoff, with the help of Eugene Prostov, a librarian at the Iowa State University, translated the two volumes of the Kravchuk book [15], [16] from Ukrainian to English. The book translation remained unpublished.* The first in the world electronic computer created by Atanasoff and Berry in 1937-1942 (*http://en.wikipedia.org/wiki/Atanasoff_Berry_Computer*) is now honored at the history of the Information Age exhibit at the Smithsonian Museum of American History, which attracts many of visitors from around the world. Iowa State University provided $ 300,000 for reconstruction of this computer.

5. Mathematical statistics and probability theory

The depth and diversity of Kravchuk's scientific interests are striking. Thus by 1925, having worked in the areas of probability theory and mathematical statistics for many years already he had obtained a number of deep results related to correlation theory and the normal distribution law for two variable attributes, and had shown useful applications of the method of moments in mathematical statistics and elsewhere.

In his analysis of the correlation problem by means of the method of least squares, Kravchuk established a criterion for the applicability of the bilinear dependence for finding a relation between two dependent attributes. In order to get around the absence of uniqueness in linear regression equation in the case of two variables and to simplify the solution of the linear correlation problem with arbitrary number of variables, Kravchuk arrived at the statement of the conditional extremum problem and a symmetric criterion for the measure of correlation of several variables. With a great skill he made use of an oblique coordinate system, producing a convenient graphical and mechanical method for investigating statistical distributions with two attributes.

In 1929 Kravchuk obtained fundamental results in probability theory associated with the binomial distribution, introducing, in particular, the polynomials of this distribution now referred to in the world literature as *Kravchuk polynomials* (cf. A. Erdélyi (Ed.), "Higher Transcendental Functions", New York etc. (1953), Vol. 2). More than some hundreds of sources from the world scientific literature could be cited, beginning with 1930 and up to the present day, in which Kravchuk polynomials have been studied and generalized, and in which various applications from probability theory through the theory of quantum algebras, theory of special functions, etc. have been demonstrated. By means of these polynomials, Kravchuk proved a new interpolation formula and established finite expansions for certain functions, in particular for the probability distribution in trials of unrotated spheres, e.g. in [17], [18], etc.

The Kravchuk polynomials have the form

$$k_n(x) = \frac{(-1)^n x!(N-x)!}{n! p^x q^{N-x}} \Delta^n \left| \frac{p^x q^{N-x+n}}{(x-n)!(N-x)!} \right|,$$

where $n = 0, 1, \ldots, N;\ p > 0, q > 0, p + q = 1, N$ is a positive integer, and the step function $f(x) = \binom{N}{n} p^x q^{N-x}$ has the orthogonality property

$$\sum_{x=0}^{N} j(x) k_n(x) k_m(x) = \binom{N}{n} p^n q^n \delta_{mn}.$$

The Kravchuk polynomials are expressed explicitly in terms of the Gauss hypergeometric function:

$$k_n(x) = q^n (\frac{x}{n})\, {}_2F_1 \left(-n, x - N; x - n; -\frac{p}{q} \right).$$

The generating function of the Kravchuk polynomials has the form

$$\sum_{n=0}^{N} k_n(x) z^n = (1 + qz)^x (1 - pz)^{N-x}.$$

In many of his studies, Kravchuk successfully applied the results concerning certain conclusions deduced from Lebesgue's inequality, crystallizing the basic assertion of his own general problem of moments. In particular, he established the following important inequality:

$$\rho_{ni} < \frac{1}{n}(UM + B) \int_{\pi/2n+1}^{\pi/2} \frac{\omega(u)}{u}\, du,$$

where ω is the modulus of continuity of the function $p(x)$, U and B are absolute constants, M is the upper bound of the modulus of this function, and $\rho_{ni} (i = 1, 2, \ldots, n)$ are the Christoffel numbers of the n-th Gauss-type quadrature for the irreducible characteristic function $p(x)/\sqrt{1 - x^2}$, with $p(x) < M$. Specifically, he indicated the precision on a given finite interval of an unknown integral of an irreducible function whose first n moments are equal to the corresponding moments of another irreducible bounded function. If $p(x) \geq 0$, and $Q(x) \geq 0$ is a nondecreasing and a bounded function, with the two functions satisfying the equalities

$$\int_{-a}^{+a} x^k \frac{dQ(x)}{\sqrt{a^2 - x^2}} = \int_{-a}^{+a} x^k \frac{p(x)dx}{\sqrt{a^2 - x^2}} + \varepsilon_k \quad (k = 0, 1, 2, \ldots, N-1),$$

where the integrals are taken in Stieltjes sense, then, as Kravchuk showed, the following inequalities

$$\left| \int_{-a}^{+a} \frac{dQ(x)}{\sqrt{a^2 - x^2}} - \int_{-a}^{+a} \frac{p(x)dx}{\sqrt{a^2 - x^2}} \right| < M\{M[p] + Ca_\varepsilon \exp(2N/a)\} \frac{A + B \log N}{N}$$

$$+ \varepsilon \exp(2N/a)(D + E \log N),$$

where $M(p)$ is the upper bound of the function $|p(x)|$ on $(-a, a)$; A, B, C, D and E are absolute constants. If, for example, $a \approx N$ and $\varepsilon \approx 1/N$, the modulus of the difference given above will be less than the quantity

$$\frac{U + \tilde{B} \log N}{N} \{M[p] + G_1\},$$

where U and $\tilde{B}$ are absolute constants, and G_1 is a constant, that is, an approximation of the function $\int_{-a}^{x} \frac{dQ(x)}{\sqrt{a^2-x^2}}$ by the function $\int_{-a}^{x} \frac{p(x)dx}{\sqrt{a^2-x^2}}$ will have order of magnitude $\log N/N$ or higher. Note that Kravchuk expanded upon his initial thought, and investigated the problem of moments in the case of an infinite interval. In fact, Kravchuk was the first who both stated and solved the problem of moments in the context of an approximation, that is:

Suppose that not all, but only some of the first $2n$ moments of two irreducible functions are equal; then, what is the difference between the integrals of these two functions, taken over the same arbitrary limits?

How does this approximation change when the interval over which the problem is solved is increased, or when the equality of the first moments is not exact, but only approximate?

Kravchuk gave concrete results on the subject of approximation in the problem of moments both for the one-dimensional case and for the case of an arbitrary number of variables. There are a number of individual articles and notes by Kravchuk that are devoted to the proof of various inequalities for the estimation of the mean errors of fundamental quantities in correlation theory without special restrictions on the nature of the distribution of the quantities under study, to the proof of Stirling's formula. Kravchuk obtained numerous new results in the area of interpolation and mechanical quadrature. Specifically, he completely solved the problem of determining the mechanical quadrature by means of the coefficients of the quadrature and by the coefficients of quadratures of lower order. Kravchuk's final published scientific study was also a serious study of the distribution of the abscissa of Gaussian-type mechanical quadratures.

The incredible breadth of his scientific range, *the ability to analyze complex questions in philosophy, the history of mathematics and techniques in the teaching of mathematics, new problems in sciences related to mathematics.* In particular, in physics, biology, and chemistry, these aspects of Kravchuk's creativity are especially in evidence in such studies as "Space, time, matter", "Contemporary atomism", "Mathematics in Ukraine", "Euler's influence on the subsequent development of mathematics" , "Mathematics and mathematicians at Kyiv University over the past 100 years", "On the dissolution of crystal and crystallization", "On the growth of organisms", "Mathematics in the service of the economy", as well as a number of other studies.

"In the 1930s Kravchuk also suggested that approximate calculations be widely applied as a starting point in the study of certain new theoretical concepts (irrational numbers, logarithms, maxima, etc.), insisted that students be proved with clear graphic illustrations, and pointed out profound mathematical ideas, and, for this reason alone, had to become an organic part of studies even in secondary school", emphasizing, in particular, that approximate calculations represented a necessary condition if students are to succeeded in gaining an understanding of such complex concepts as that of infinity etc.

Mykhailo Kravchuk was a highly erudite and cultured man. He was fluent in several languages (French, German, Italian, Polish, Russian, etc.) and kept up scientific and, especially, friendly relationship with such world-renowned mathematicians as Hadamard, Hilbert, Courant, Tricomi, Luzin, et al. Meanwhile he composed his own scientific studies in several languages, most of them, however, in his native language, and his use of the Ukrainian language represented a superb example of Ukrainian mathematical and scientific style. Through his published studies and the papers that he delivered in lectures, Kravchuk played an important role in raising the scientific and technical level of the presentation of mathematics in secondary and post-secondary schools. *He was the organizer of the first mathemati-*

cal olympiad of high school students in the city of Kyiv (1935). Kravchuk held a mathematics chair at the Kiev Polytechnic Institute. Possessing an extraordinary pedagogical skill and a brilliant gift for the dissemination of scientific ideas, Kravchuk gathered around himself a group of young mathematicians, inspiring them every day with his own creative enthusiasm, and knew how to direct their own mathematical research. *Many of his students later became well-known mathematicians. His students included also Sergey Korolev, Arkhip Lyulka, and Vladimir Chelomei, future leading rocket and jet engine designers.*

Even this brief survey of far from all of Kravchuk's mathematical studies persuasively testifies to his unique mathematical talents, great erudition, and manifold scientific interests. Not only he poses and solves many different problems, but also pointed out highly promising problems for future research. It would be of interest, and important, to trace out the subsequent development of the manifold scientific ideas put forward by Kravchuk, however this is already a subject for a special study outside the scope of the present condensed survey.

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Editorial Note: This is an invited survey on the life and achievements of the Ukrainian mathematician M. Kravchuk, among them - the so-called Kravchuk polynomials, orthogonal polynomials of hypergeometric type, being important class of special functions with various applications. Prof. Nina A. Virchenko dedicated many efforts and years to populirize M. Kravchuk's heritage, organizing in Kiev annual Kravchuk International Conferences. More details can be found, e.g. in Wikipedia,

http://en.wikipedia.org/wiki/Mikhail_Kravchuk,
http://en.wikipedia.org/wiki/Kravchuk_polynomial.