A Complete Proof System for First Order Interval Temporal Logic with Projection

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Abstract

This paper presents an ω-complete proof system for the extension of first order Interval Temporal Logic (ITL, [12, 15, 4]) by a projection operator [16, 11]. Alternative earlier approaches to the axiomatisation of projection in ITL are briefly presented and discussed. An extension of the proof system which is complete for the extension of Duration Calculus (DC, [24]) by projection is also given.

Introduction

First Order Interval Temporal Logic (ITL) was introduced in [12, 15] as a tool for the formal specification and verification of hardware systems. The completeness of a proof system for ITL with respect to an abstractly defined class of frames was first presented in [4]. Numerous extensions of ITL have been shown to be useful in the specification of various kinds of software and hardware systems. Among these are the real time based Duration Calculus (DC) [24] and various extensions of it, which include iteration and more general fixed point operators [5, 18, 8], higher-order quantifiers [18, 22, 8] and expanding modalities [23, 19].

A binary modal operator called projection and denoted by Π was first introduced to discrete time ITL in [12]. This operator subjects its second formula operand to evaluation at an interval of time which is obtained by keeping only some of the points of the reference interval, including the end points. The points to be kept are those which satisfy the first operand of Π. In [16, 17], another variant of projection was introduced. It was denoted by $\mathbf{proj}$ in [16] and by $\triangle$ [17], and differs from Π in the way the first operand determines the time points from the reference interval to be selected. A similar projection operator, denoted by $(.\backslash\backslash.)$, was introduced to DC in [11]. DC is a real-time logic and the introduction of projection made it necessary to admit finite nonempty sets of time points as "intervals" along with the "ordinary" closed and bounded real-time intervals, which are the possible worlds in DC from the view point of Kripke semantics.

All the projection operators mentioned above have been introduced to enable the concise and flexible specification of *time granularity*, which is essential for the conve-

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nience of choosing different levels of abstraction for the specification of hardware behaviour. In discrete-time ITL, the operators Π and $\mathbf{proj}$ provide access from discrete-time intervals of finer granularity to discrete-time intervals of coarser granularity, thus providing expressiveness for the specification of systems in which the initial step of discretisation of time has already been made. The generalisation proposed in [11] allows the reasoning about this initial time discretisation step itself to be formalised within the logic. Along with the convenience of reasoning about the behaviour of single discrete-time systems at different levels of time granularity, this more general form of projection facilitates the specification of systems which have multiple discrete-time components with possibly independent clock rates. That is why the projection operator from [11] significantly enhances the expressive power of DC by enabling combined specifications of dense- and discrete-time properties of the modelled systems. Another interesting real-time projection operator, which can be regarded as a real-time variant of Π, has been studied in [9, 10].

In this paper we present a complete proof system for the extension of abstract time ITL as introduced in [4] by a projection operator that we denote by $(.\backslash\backslash.)$. The operator we study here is a straighforward generalisation of that from [11], except that it allows *vacuous* projection, like $\triangle$. This operator leads to a system of abstract-and-discrete time ITL. Both DC and discrete time ITL with projection can be regarded as special cases of this system. The variants of projection in ITL with and without vacuous projection are interdefinable. The one with vacuous projection is technically more convenient and therefore we develop our proof system and its completeness argument for this variant of the projection operator.

The proof system we present is ω-complete and contains one infinitary rule, which is related to the ITL modal operator called *iteration*. Iteration is known in ITL and DC and can be regarded as a special case of projection. We obtained this proof system by extending the complete proof system for abstract time ITL known from [4]. Our completeness argument is an extension of the one given in that article too, and follows closely the version of that completeness argument for the ω-complete proof system for abstract time DC developed in [6].

1 Preliminaries on ITL with projection

In this paper $ITL_{\backslash\backslash}$ stands for the extension of abstract time ITL by the projection operator $(.\backslash\backslash.)$ to be defined below. The model of time for ITL from [4] includes a linearly ordered set of time points, and a measure function which maps bounded intervals of time to their durations. Durations themselves are required to constitute an appropriate kind of linearly ordered semigroup. This is what we mean by *abstract time* in this paper too. In fact, frames for $ITL_{\backslash\backslash}$ are ITL frames with the measure function straightforwardly extended to include the durations of discrete-time intervals. In this section we give a brief formal introduction to $ITL_{\backslash\backslash}$.

1.1 Languages

An $ITL_{\backslash\backslash}$ language is essentially a first order language extended by the ITL-specific binary modality $(.;.)$ and the $ITL_{\backslash\backslash}$-specific binary modality $(.\backslash\backslash.)$. The vocabulary of an $ITL_{\backslash\backslash}$ language consists of *constant symbols* c, d, $\ldots$, *function symbols* f, g, …and *relation symbols* R, S, $\ldots$. Function symbols and relation symbols have *arity* to indicate the number of arguments their occurrences take in terms and atomic formulas, respectively. Furthermore, every $ITL_{\backslash\backslash}$ vocabulary contains countably many *individual variables* x, y, $\ldots$. Non-logical symbols are either *flexible* or *rigid*, depending on whether their interpretations depend on reference intervals or not, respectively, as it becomes clear below.

Given the vocabulary of an $ITL_{\backslash\backslash}$ language, its *terms* t and *formulas* φ are defined by the following BNFs:

$t ::= c \mid x \mid f(t, \ldots, t)$

$\varphi ::= \bot \mid R(t, \ldots, t) \mid \varphi \Rightarrow \varphi \mid (\varphi;\varphi) \mid \varphi^* \mid (\varphi\backslash\backslash\varphi) \mid \exists x\varphi$

Terms and formulas that are built using no modalities and flexible symbols are called *rigid*. The other terms and formulas are called *flexible*. Every $ITL_{\backslash\backslash}$ vocabulary contains the rigid constant 0, the flexible constant ℓ, the rigid binary function symbol $+$ and equality $=$. In this paper we consider only vocabularies which contain infinitely many 0-ary flexible relation symbols, because of their special role in our proof system for $ITL_{\backslash\backslash}$.

The two binary operators $(.;.)$ and $(.\backslash\backslash.)$ are known as *chop* and *projection*, respectively. Although *iteration* $(.)^*$ is definable in $ITL_{\backslash\backslash}$, we prefer to regard it as a separate modality in our presentation, because of its role in our proof system. We always use parentheses in formulas involving chop and projection. The diversity of notation in the literature, especially concerning projection, indicates that the unambiguity that our convention offers is important.

1.2 Frames, models and satisfaction

Definition 1 A *time domain* is a linearly ordered set. Let $\langle T, \leq\rangle$ be a time domain. We denote the set $\{[\tau_1, \tau_2] : \tau_1, \tau_2 \in T, \tau_1 \leq \tau_2\}$ by $\mathbf{I}(T)$ and the set of the nonempty finite subsets of T by $\mathbf{P}_{\mathit{fin}}(T)$, respectively. We denote $\mathbf{I}(T) \cup \mathbf{P}_{\mathit{fin}}(T)$ by $\mathbf{I}_{\backslash\backslash}(T)$. The elements of $\mathbf{I}(T)$, $\mathbf{P}_{\mathit{fin}}(T)$ and $\mathbf{I}_{\backslash\backslash}(T)$ are called *ordinary intervals*, *discrete intervals* and just *intervals*, respectively.

A *duration domain* is a system of the kind $\langle D, 0^{(0)}, +^{(2)}\rangle$ which satisfies the following axioms:

$(D1)$ $x + (y + z) = (x + y) + z$

$(D2)$ $x + 0 = x$,
$0 + x = x$

$(D3)$ $x + y = x + z \Rightarrow y = z$,
$x + z = y + z \Rightarrow x = y$

$(D4)$ $x + y = 0 \Rightarrow x = 0$

$(D5)$ $\exists z(x + z = y \vee y + z = x)$,
$\exists z(z + x = y \vee z + y = x)$

Given a time domain $\langle T, \leq\rangle$ and a duration domain $\langle D, 0, +\rangle$, a function m :

$\mathbf{I}_{\backslash\backslash}(T) \to D$ is called a *measure function*, if the following properties hold for all $\sigma, \sigma' \in \mathbf{I}_{\backslash\backslash}(T)$:

$(M1)$	$\min\sigma = \min\sigma' \wedge m(\sigma) = m(\sigma') \Rightarrow \max\sigma = \max\sigma'$
$(M2)$	if $\sigma \cup \sigma' \in \mathbf{I}_{\backslash\backslash}(T)$, then $\max\sigma = \min\sigma' \Rightarrow m(\sigma) + m(\sigma') = m(\sigma \cup \sigma')$
$(M3)$	$m(\sigma) = x + y \Rightarrow \exists\tau \in \sigma\ m([\min\sigma, \tau]) = x$ for $\sigma \in \mathbf{I}(T)$
$(M_{\backslash\backslash})$	$m(\sigma) = m([\min\sigma, \max\sigma])$ for $\sigma \in \mathbf{P}_{\mathit{fin}}(T)$

Here, as usual, $\min\sigma$ and $\max\sigma$ stand for the least and the greatest time point of the interval σ, respectively, in the sense of the ordering $\leq$ on T.

A linear ordering can be defined on duration domains by putting

$$x \leq y \text{ iff } \exists z(x + z = y).$$

The only $ITL_{\backslash\backslash}$-specific axiom about measure functions here is $M_{\backslash\backslash}$. It postulates that removing all but finitely many internal points from an interval does not affect its duration. This choice of extending m to discrete intervals differs from the one in [16] where all intervals are discrete and the duration of an interval is defined by means of the number of its points. In the sequel we denote unions $\sigma \cup \sigma'$ of intervals σ, σ' such that $\min\sigma' = \max\sigma$ and either $\sigma, \sigma' \in \mathbf{I}(T)$ or $\sigma, \sigma' \in \mathbf{P}_{\mathit{fin}}(T)$ by $\sigma; \sigma'$. We tacitly assume $\min\sigma' = \max\sigma$ wherever we use $\sigma; \sigma'$. For example, in this notation $M2$ can be abbreviated to $m(\sigma) + m(\sigma') = m(\sigma; \sigma')$

Definition 2 An $ITL_{\backslash\backslash}$ *frame* is a tuple of the form $\langle\langle T, \leq\rangle, \langle D, 0, +\rangle, m\rangle$ where $\langle T, \leq\rangle$ is a time domain, $\langle D, 0, +\rangle$ is a duration domain and $m : \mathbf{I}_{\backslash\backslash}(T) \to D$ is a measure function.

The real-time based frame $F_{\mathbf{R}} = \langle\langle \mathbf{R}, \leq_{\mathbf{R}}\rangle, \langle \mathbf{R}_+, 0_{\mathbf{R}}, +_{\mathbf{R}}\rangle, \lambda\sigma. \max\sigma - \min\sigma\rangle$ where $\mathbf{R}_+$ stands for the set of the non-negative reals is undoubtedly the most interesting $ITL_{\backslash\backslash}$ frame. Since $\mathbf{P}_{\mathit{fin}}(\mathbf{R})$ consists of the finite sets of reals, the frame $F_{\mathbf{R}}$ embeds the practically significant model of *timed state sequences* where states are labelled with real-valued time stamps. Correspondence between the validity of DC formulas on timed state sequences and their validity in real time has been studied in [2]. Another interesting $ITL_{\backslash\backslash}$ frame is $F_{\mathbf{Z}} = \langle\langle \mathbf{Z}, \leq_{\mathbf{Z}}\rangle, \langle \mathbf{N}, 0_{\mathbf{N}}, +_{\mathbf{N}}\rangle, \lambda\sigma. \max\sigma - \min\sigma\rangle$. Since bounded intervals of integers are finite, we have $\mathbf{P}_{\mathit{fin}}(\mathbf{Z}) = \mathbf{I}(\mathbf{Z})$.

Definition 3 Given an $ITL_{\backslash\backslash}$ language $\mathbf{L}$ and an $ITL_{\backslash\backslash}$ frame F with its components named as above, a function I on the vocabulary of $\mathbf{L}$ is an *interpretation of* $\mathbf{L}$ *into* F, if it satisfies the following conditions:

- $I(c) \in D$, $I(f) : D^n \to D$, and $I(R) : D^n \to \{0, 1\}$ for rigid constant symbols c, and n-ary rigid function symbols f and relation symbols R
- $I(c) : \mathbf{I}_{\backslash\backslash}(T) \to D$, $I(f) : \mathbf{I}_{\backslash\backslash}(T) \times D^n \to D$, and $I(R) : \mathbf{I}_{\backslash\backslash}(T) \times D^n \to \{0, 1\}$ for the corresponding kinds of flexible symbols;
- $I(x) \in D$ for individual variables x;
- $I(=)$ is $=$, $I(0) = 0$, $I(+) = +$, and $I(\ell) = m$.

Given an $ITL_{\backslash\backslash}$ language $\mathbf{L}$, an $ITL_{\backslash\backslash}$ *model for* $\mathbf{L}$ is a pair of the form $\langle F, I\rangle$ where F is an $ITL_{\backslash\backslash}$ frame and I is an interpretation of $\mathbf{L}$ into I.

Given a frame F, we denote its components by $\langle T_F, \leq_F \rangle$, $\langle D_F, 0_F, +_F \rangle$ and m_F, respectively. The same applies to models. We denote the frame and the interpretation of a model M by F_M and I_M, respectively.

Definition 4 Given an $ITL_{\backslash\backslash}$ language **L**, a model $M = \langle F, I \rangle$ for it, and an interval $\sigma \in \mathbf{I}_{\backslash\backslash}(T_F)$, the value $I_\sigma(t)$ of a term t in **L** is defined by induction on the construction of t as follows:

$I_\sigma(c)$	$=$	$I(c)$	for rigid constants c
$I_\sigma(c)$	$=$	$I(c)(\sigma)$	for flexible constants c
$I_\sigma(f(t_1, \dots, t_n))$	$=$	$I(f)(I_\sigma(t_1), \dots, I_\sigma(t_n))$	for rigid n-ary f
$I_\sigma(f(t_1, \dots, t_n))$	$=$	$I(f)(\sigma, I_\sigma(t_1), \dots, I_\sigma(t_n))$	for flexible n-ary f

Given an interpretation I of an $ITL_{\backslash\backslash}$ language **L** into a frame F, a symbol s from **L** and an object a of the type of $I(s)$ in F, we denote the interpretation which assigns a to s and is equal to I for all the other symbols from the vocabulary of **L** by I_s^a.

Definition 5 We define the relation $M, \sigma \models \varphi$ where $M = \langle F, I \rangle$ is an $ITL_{\backslash\backslash}$ model for some language **L**, $\sigma \in \mathbf{I}_{\backslash\backslash}(T_F)$ and φ is a formula in **L** by induction on the construction of φ as follows:

$M, \sigma \not\models \bot$		
$M, \sigma \models R(t_1, \dots, t_n)$	iff	$I(R)(I_\sigma(t_1), \dots, I_\sigma(t_n)) = 1$, for rigid n-ary R
$M, \sigma \models R(t_1, \dots, t_n)$	iff	$I(R)(\sigma, I_\sigma(t_1), \dots, I_\sigma(t_n)) = 1$, for flexible n-ary R
$M, \sigma \models \varphi \Rightarrow \psi$	iff	either $M, \sigma \models \psi$ or $M, \sigma \not\models \varphi$
$M, \sigma \models (\varphi; \psi)$	iff	$M, \sigma_1 \models \varphi$ and $M, \sigma_2 \models \psi$
		for some $\sigma_1, \sigma_2 \in \mathbf{I}_{\backslash\backslash}(T_F)$ such that $\sigma_1; \sigma_2 = \sigma$
$M, \sigma \models \varphi^*$	iff	either $\min \sigma = \max \sigma$
		or there exist an $n > 0$ and $\sigma_1, \dots, \sigma_n \in \mathbf{I}_{\backslash\backslash}(T_F)$
		such that $\sigma_1; \dots; \sigma_n = \sigma$ and $M, \sigma_i \models \varphi$, $i = 1, \dots, n$
$M, \sigma \models (\psi \backslash\backslash \varphi)$	iff	either $\min \sigma = \max \sigma$ and $M, \sigma \models \varphi$,
		or there exist an $n > 0$ and $\sigma_1, \dots, \sigma_n \in \mathbf{I}_{\backslash\backslash}(T_F)$
		such that $\sigma_1; \dots; \sigma_n = \sigma$, $M, \sigma_i \models \psi$, $i = 1, \dots, n$,
		and $M, \{\min \sigma_1, \max \sigma_1, \dots, \max \sigma_n\} \models \varphi$
$M, \sigma \models \exists x \varphi$	iff	$\langle F, I_x^a \rangle, \sigma \models \varphi$ for some $a \in D_F$

Note that the clause about the satisfaction of formulas of the kind $(\psi \backslash\backslash \varphi)$ always refers to satisfaction of φ at a discrete interval, while in other clauses intervals on the right side are of the same kind as that on the left side.

1.3 Abbreviations

First order logic abbreviations and infix notation are used in $ITL_{\backslash\backslash}$ in the ordinary way. These include the constant $\top$, the connectives $\neg$, $\wedge$, $\vee$ and $\Leftrightarrow$ and quantifier $\forall$. We use $t_1 \leq t_2$ to abbreviate the formula $\exists x(t_1 + x = t_2)$ which defines the standard linear ordering on durations. The following abbreviations are specific to the modal operators $(.;.)$, $(.\backslash\backslash.)$ and $(.)^*$:

$\Diamond\varphi \rightleftharpoons (\top; \varphi; \top)$, $\Box\varphi \rightleftharpoons \neg\Diamond\neg\varphi$

$\varphi^0 \rightleftharpoons \ell = 0$, $\varphi^{k+1} \rightleftharpoons (\varphi^k; \varphi)$ for $k < \omega$

$\varphi^+ \rightleftharpoons (\varphi^*;\varphi)$
$(\varphi_1\backslash\backslash\varphi_2\backslash\backslash\ldots\backslash\backslash\varphi_n) \rightleftharpoons (\varphi_1\backslash\backslash\ldots\backslash\backslash(\varphi_{n-1}\backslash\backslash\varphi_n)\ldots)$
$(\varphi_1;\varphi_2;\ldots;\varphi_n) \rightleftharpoons (\varphi_1;\ldots;(\varphi_{n-1};\varphi_n)\ldots)$

Note that we use $\Diamond$ and $\Box$ to abbreviate formulas in the way that has been adopted in the literature on DC, which is different from their use as discrete-time ITL abbreviations. Iteration can be defined using $(.\backslash\backslash.)$ by the equivalence $\varphi^* \Leftrightarrow (\varphi\backslash\backslash\top)$. However, we prefer to regard it as an independent operator.

The possibility for $M,\sigma \models (\varphi\backslash\backslash\psi)$ to hold due to $\min\sigma = \max\sigma$ and $M,\sigma \models \psi$ is called *vacuous* projection. This possibility is ruled out in the version of the operator in [11]. The projection of φ and ψ there is equivalent to $(\varphi\backslash\backslash\psi) \wedge \varphi^+$.

1.4 A proof system for ITL

ITL (without projection) can be defined by restricting the above definitions to $\mathbf{I}(T)$ wherever $\mathbf{I}_{\backslash\backslash}(T)$ is involved and disregarding the clauses which refer to discrete intervals or the operators $(.\backslash\backslash.)$ and $(.)^*$. Here follows the proof system for ITL which was proved complete in [4].

$(A1_l)$	$(\varphi;\psi) \wedge \neg(\chi;\psi) \Rightarrow (\varphi \wedge \neg\chi;\psi)$	(MP)	$\dfrac{\varphi\ \varphi \Rightarrow \psi}{\psi}$
$(A1_r)$	$(\varphi;\psi) \wedge \neg(\varphi;\chi) \Rightarrow (\varphi;\psi \wedge \neg\chi)$	(G)	$\dfrac{\varphi}{\forall x\varphi}$
$(A2)$	$((\varphi;\psi);\chi) \Leftrightarrow (\varphi;(\psi;\chi))$	(N_l)	$\dfrac{\varphi}{\neg(\neg\varphi;\psi)}$
(R_l)	$(\varphi;\psi) \Rightarrow \varphi$ if φ is rigid	(N_r)	$\dfrac{\varphi}{\neg(\psi;\neg\varphi)}$
(R_r)	$(\varphi;\psi) \Rightarrow \psi$ if ψ is rigid	$(Mono_l)$	$\dfrac{\varphi \Rightarrow \psi}{(\varphi;\chi) \Rightarrow (\psi;\chi)}$
(B_l)	$(\exists x\varphi;\psi) \Rightarrow \exists x(\varphi;\psi)$ if x is not free in ψ	$(Mono_r)$	$\dfrac{\varphi \Rightarrow \psi}{(\chi;\varphi) \Rightarrow (\chi;\psi)}$
(B_r)	$(\varphi;\exists x\psi) \Rightarrow \exists x(\varphi;\psi)$ if x is not free in φ		
$(L1_l)$	$(\ell = x;\varphi) \Rightarrow \neg(\ell = x;\neg\varphi)$		
$(L1_r)$	$(\varphi;\ell = x) \Rightarrow \neg(\neg\varphi;\ell = x)$		
$(L2)$	$(\ell = x;\ell = y) \Leftrightarrow \ell = x + y$		
$(L3_l)$	$\varphi \Rightarrow (\ell = 0;\varphi)$		
$(L3_r)$	$\varphi \Rightarrow (\varphi;\ell = 0)$		

This system also includes first order logic axioms, equality axioms and the axioms $D1$-$D5$ about duration domains. Substitution $[t/x]\varphi$ of individual variable x by term t in formula φ is allowed in axiom instances only if t is rigid or x does not occur in the scope of modal operators in φ. This system is sound with respect to the semantics of $ITL_{\backslash\backslash}$ too, except for the axiom $L2$, which fails in the case of discrete intervals.

2 A proof system for $ITL_{\backslash\backslash}$

In order to present our proof system for $ITL_{\backslash\backslash}$, we need to introduce some special abbreviations. Given two formulas φ and ψ, we define a formula φ^ψ. The new formula is meant to be equivalent to $(\psi\backslash\backslash\varphi)$, provided that ψ has the property defined below in the considered $ITL_{\backslash\backslash}$ models and intervals.

Definition 6 Let M be a model for some $ITL_{\backslash\backslash}$ language $\mathbf{L}$ and ψ be a formula in $\mathbf{L}$. Let $\sigma \in \mathbf{I}_{\backslash\backslash}(T_M)$. Then ψ has *the unique partition property at the interval σ in the model M* if either $\min\sigma = \max\sigma$ and $M,\sigma \not\models \psi$, or there exist a finite set of time points $\tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n$ in σ such that $\tau_0 = \min\sigma$, $\tau_n = \max\sigma$ and the only subintervals σ' of σ which satisfy $M, \sigma' \models \psi$ are $\sigma \cap [\tau_{i-1}, \tau_i]$, $i = 1, \ldots, n$.

A formula ψ which has the unique partition property at some interval σ in a model M can be used to unambiguously specify the discrete "sub"interval $\{\tau_0, \tau_1, \ldots, \tau_{n-1}, \tau_n\}$ of a nonzero length interval σ, where $\tau_0, \ldots, \tau_n$, are the time points in σ which occur in the definition of the unique partition property above. In case σ is a 0-length interval, its only subinterval is σ itself, both discrete and ordinary. The discrete interval which can be specified this way is of the kind that appears in the definition of $M, \sigma \models (\psi\backslash\backslash\varphi)$ for arbitrary φ. That is why a ψ with the unique partition property can be used as a *witness* for the satisfaction of formulas of the kind $(\chi\backslash\backslash\varphi)$, provided that $M, \sigma \models \Box(\psi \Rightarrow \chi)$. The unique partition property can be characterised by an axiom as follows:

Lemma 7 *Let M be a model for some $ITL_{\backslash\backslash}$ language $\mathbf{L}$, $\sigma \in \mathbf{I}_{\backslash\backslash}(T_M)$ and ψ be a formula in $\mathbf{L}$. Then ψ has the unique partition property at σ in M iff*
$M, \sigma \models \psi^* \wedge \neg(\psi^*; \psi \wedge ((\psi; \ell \neq 0) \vee \ell = 0); \psi^*) \wedge \neg(\top; \psi; \neg\psi^*) \wedge \neg(\neg\psi^*; \psi; \top)$
.

Proof: If ψ has the unique partition property at σ, then a direct check shows that the above formula holds for ψ at σ.

For the opposite implication, consider the case $\min\sigma = \max\sigma$ first. In this case

$$M, \sigma \models \neg(\psi^*; \psi \wedge ((\psi; \ell \neq 0) \vee \ell = 0); \psi^*)$$

is equivalent to

$$M, \sigma \models \neg(\psi \wedge ((\psi; \ell \neq 0) \vee \ell = 0))$$

and this implies $M, \sigma \not\models \psi$, because $M, \sigma \models \ell = 0$.

In case $\min\sigma \neq \max\sigma$, $M, \sigma \models \psi^*$ implies that there is a subinterval σ' of σ such that $M, \sigma' \models \psi$. Now

$$M, \sigma \models \neg(\top; \psi; \neg\psi^*) \wedge \neg(\neg\psi^*; \psi; \top)$$

implies that $M, \sigma \cap [\min\sigma, \min\sigma'] \models \psi^*$ and $M, \sigma \cap [\max\sigma', \max\sigma] \models \psi^*$. This means that there is a finite set of time points $\tau_0 < \tau_1 < \ldots < \tau_{n-1} < \tau_n$ in σ such that $\min\sigma = \tau_0$, $\max\sigma = \tau_n$, $M, \sigma \cap [\tau_{i-1}, \tau_i] \models \psi$ for $i = 1, \ldots, n$ and $\sigma' = \sigma \cap [\tau_{i-1}, \tau_i]$ for some $i \in \{1, \ldots, n\}$. For the sake of contradiction, assume that there exists another finite set of time points $\tau'_0 < \tau_1 < \ldots < \tau'_{m-1} < \tau_m$ in σ such that $\min\sigma = \tau'_0$, $\max\sigma = \tau'_m$, $M, \sigma \cap [\tau'_{i-1}, \tau'_i] \models \psi$ for $i = 1, \ldots, m$. Then there is a least i such that $\tau_i \neq \tau'_i$. Clearly, $0 < i \leq \min(m, n)$. Then

$$M, \sigma \cap [\tau_{i-1}, \max(\tau_i, \tau'_i)] \models \psi \wedge ((\psi; \ell \neq 0) \vee \ell = 0).$$

This implies that

$$M, \sigma \models (\psi^*; \psi \wedge ((\psi; \ell \neq 0) \vee \ell = 0); \psi^*),$$

which is a contradiction. $\dashv$

The formula φ^ψ we mention in the beginning of this section is defined by induction on the construction of φ as follows:

$\bot^\psi$	$\rightleftharpoons$	$\bot$	
$(R(t_1,\ldots,t_n))^\psi$	$\rightleftharpoons$	$R(t_1,\ldots,t_n)\wedge\psi^*$	for rigid $R(t_1,\ldots,t_n)$
$(R(t_1,\ldots,t_n))^\psi$	$\rightleftharpoons$	$(\ell=0\wedge R(t_1,\ldots,t_n))\vee$	for flexible
		$(\ell\neq 0\wedge(\psi\backslash\backslash R(t_1,\ldots,t_n))\wedge\psi^*)$	$R(t_1,\ldots,t_n)$
$(\varphi_1\Rightarrow\varphi_2)^\psi$	$\rightleftharpoons$	$(\varphi_1^\psi\Rightarrow\varphi_2^\psi)\wedge\psi^*$	
$(\varphi_1;\varphi_2)^\psi$	$\rightleftharpoons$	$(\psi^*\wedge\varphi_1^\psi;\psi^*\wedge\varphi_2^\psi)$	
$(\varphi^*)^\psi$	$\rightleftharpoons$	$(\psi^*\wedge\varphi^\psi)^*$	
$(\varphi_1\backslash\backslash\varphi_2)^\psi$	$\rightleftharpoons$	$(\varphi_1^\psi\backslash\backslash\varphi_2)$	
$(\exists x\varphi)^\psi$	$\rightleftharpoons$	$\exists x(\varphi^\psi)$	

Following our notational conventions, we regards formulas of the form $(t_1=t_2)^\psi$ as instances of $(R(t_1,t_2))^\psi$ here.

To denote the formula which occurs in Lemma 7 concisely, we use the abbreviations

$\Diamond^\psi\varphi\rightleftharpoons(\psi^*;\varphi;\psi^*)$

$\Box^\psi\varphi\rightleftharpoons\neg\Diamond^\psi\neg\varphi$

$\overline{\psi}^*\rightleftharpoons\psi^*\wedge\Box^\psi(\psi\Rightarrow(\ell\neq 0\wedge\neg(\psi;\ell\neq 0)))\wedge\neg(\top;\psi;\neg\psi^*)\wedge\neg(\neg\psi^*;\psi;\top)$

It can easily be shown by induction on the construction on φ that, if $M,\sigma\models\overline{\psi}^*$, then

$M,\sigma\models\Box^\psi((\psi\backslash\backslash\varphi)\Leftrightarrow\varphi^\psi)$.

We use formulas like φ^ψ and $\overline{\psi}^*$ extensively to present our axioms and rules for $ITL_{\backslash\backslash}$ concisely. For technical reasons, we also introduce abbreviations for some kinds of formulas with nested occurrences of $(.\backslash\backslash.)$. Let $\varphi_0,\ \ldots,\ \varphi_n,\ R_1,\ \ldots,\ R_n$, and $t_{i,j}$, $i=1,\ldots,n$, $j=1,2$, be $ITL_{\backslash\backslash}$ formulas, 0-ary (flexible) relation symbols and terms, respectively. We introduce the sequence of formulas

$\mathsf{proj}(\varphi_k,t_{k+1,1},t_{k+1,2},R_{k+1},\varphi_{k+1},\ldots,\varphi_{n-1},t_{n,1},t_{n,2},R_n,\varphi_n)$, $k=0,\ldots,n$.

The kth member of this sequence takes $4(n-k)+1$ arguments. These are φ_k, …, φ_n, with $t_{i,1},t_{i,2},R_i$, inserted between φ_{i-1} and φ_i, $i=k+1,\ldots,n$. We define the formulas $\mathsf{proj}(\ldots)$ by the clauses:

$\mathsf{proj}(\varphi_n)\rightleftharpoons\varphi_n$

$\mathsf{proj}(\varphi_{k-1},t_{k,1},t_{k,2},R_k,\varphi_k,\ldots,\varphi_{n-1},t_{n,1},t_{n,2},R_n,\varphi_n)\rightleftharpoons$

$(\ell=t_{k,1};\overline{R_k}^*\wedge\Box(R_k\Rightarrow\varphi_{k-1})\wedge\mathsf{proj}(\varphi_k,\ldots,\varphi_{n-1},t_{n,1},t_{n,2},R_n,\varphi_n)^{R_k};\ell=t_{k,2})$

$4(n-k)+1$-ary $\mathsf{proj}(\ldots)$ is meant to correspond to $n-k$ projections, each preceded by the selection of a subinterval, in the way φ^ψ corresponds to a single projection. The 0-ary relation symbols R_1, …, R_n here determine the particular subinterval partitions involved in satisfying the considered projections. The terms $t_{i,j}$, $i=1,\ldots,n$, determine the subintervals involved in the considered subinterval selections. We introduce concise notation for the formulas which express these $n-k$ projections and subinterval selections by straightforward use of the operator $(.\backslash\backslash.)$ as follows:

$(\psi\backslash\backslash_{t_1}^{t_2}\varphi)\rightleftharpoons(\ell=t_1;(\psi\backslash\backslash\varphi),\ell=t_2)$, for arbitrary φ, ψ, t_1 and t_2

$(\varphi_{k-1}\backslash\backslash_{t_{k,1}}^{t_{k,2}}\varphi_k\backslash\backslash_{t_{k+1,1}}^{t_{k+1,2}}\ldots\backslash\backslash_{t_{n,1}}^{t_{n,2}}\varphi_n)\rightleftharpoons(\varphi_{k-1}\backslash\backslash_{t_{k,1}}^{t_{k,2}}(\varphi_k\backslash\backslash_{t_{k+1,1}}^{t_{k+1,2}}\ldots\backslash\backslash_{t_{n,1}}^{t_{n,2}}\varphi_n))$

It can be shown that, if R_1,…,R_n do not occur in φ_0, …, φ_n, then

$$(\varphi_0\backslash\backslash_{t_{1,1}}^{t_{1,2}}\varphi_1\ldots\varphi_{n-1}\backslash\backslash_{t_{n,1}}^{t_{n,2}}\varphi_n)$$

is equivalent to

$$\exists R_1 \dots \exists R_n \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \varphi_1, \dots, \varphi_{n-1}, t_{n,1}, t_{n,2}, R_n, \varphi_n),$$

where the quantifier prefix $\exists R_1 \dots \exists R_n$ is interpreted in the ordinary way. However, such quantification is not allowed in $ITL_{\backslash\backslash}$. To axiomatise this equivalence without involving such quantification explicitly is the main idea behind the proof system for $ITL_{\backslash\backslash}$ that we present in this paper.

In order to present our proof system we also need a class of $ITL_{\backslash\backslash}$ formulas that we call *plain chop formulas*. These are formulas of the kind

$(\ell = c_1 \wedge R; \dots ; \ell = c_n \wedge R)$

where R stands for a (flexible) 0-ary relation symbol and $c_1, \dots, c_n$, are rigid constants. For the case of n being 1 the above formula is just $\ell = c_1 \wedge R$.

The last abbreviation we introduce here is

$\delta \rightleftharpoons \neg(\ell \neq 0; \ell \neq 0)$

Clearly, $M, \sigma \models \delta$, iff σ is either 0-length or discrete with no internal points, that is, iff $\sigma = \{\min \sigma, \max \sigma\}$. Hence $M, \sigma \models \delta^*$ iff $\sigma \in \mathbf{P}_{fin}(T_M)$. For an example of the use of δ we make below, let φ be $(\ell = c_1 \wedge R; \dots ; \ell = c_n \wedge R)$. Then $M, \sigma \models [\delta/R]\varphi$ iff σ is a discrete interval which can be represented in the form $\sigma_1; \dots ; \sigma_n$ where $M, \sigma_i \models \ell = c_i$, and none of the intervals $\sigma_1, \dots, \sigma_n$ has internal points.

2.1 The system

The proof system we propose for $ITL_{\backslash\backslash}$ consists of the axioms and rules from the proof system for ITL with the exception of the axiom $L2$, and axioms and rules about iteration, discrete intervals and $(.\backslash\backslash.)$:

Iteration

$(I1)$ $\quad \ell = 0 \vee (\varphi^*; \varphi) \Rightarrow \varphi^*$

$(I2)$ $\quad \dfrac{\forall k < \omega\ (\chi_1; \varphi^k; \chi_2) \Rightarrow \psi}{(\chi_1; \varphi^*; \chi_2) \Rightarrow \psi}$

Discrete Intervals

$(L2_{\Leftarrow})$ $\quad \Box(\ell = x + y \Rightarrow (\ell = x; \ell = y)) \vee \delta^*$

$(L2_{\Rightarrow})$ $\quad (\ell = x; \ell = y) \Rightarrow \ell = x + y$

$(DI1)$ $\quad \delta^* \Rightarrow (\varphi \Leftrightarrow (\delta \backslash\backslash \varphi))$

$(DI2)$ $\quad [\top/R]\varphi \Rightarrow (\top \backslash\backslash [\delta/R]\varphi)$, if φ is a plain chop formula and R occurs in it

Projection

(PR) $\quad (\varphi\backslash\backslash\psi) \Leftrightarrow \varphi^* \wedge \psi$ for rigid formulas ψ

$(P\ell)$ $\quad \varphi^* \wedge \ell = x \Leftrightarrow (\varphi\backslash\backslash\ell = x)$

$(P1)$ $\quad \overline{\psi}^* \wedge \Box^\psi(\psi \Rightarrow \chi) \wedge \varphi^\psi \Rightarrow (\chi\backslash\backslash\varphi)$

$(P2)$ $\quad \dfrac{\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \varphi_1, t_{2,1}, t_{2,2}, R_2, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta}{(\varphi_0\backslash\backslash_{t_{1,1}}^{t_{1,2}}\varphi_1\backslash\backslash_{t_{2,1}}^{t_{2,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}}\varphi_n) \Rightarrow \theta}$

$(P3)$ $\quad \overline{\psi}^* \Rightarrow ((\psi\backslash\backslash\varphi) \Leftrightarrow \varphi^\psi)$

$(P4)$ $\quad (\varphi\backslash\backslash(\psi\backslash\backslash\chi)) \Leftrightarrow ((\varphi\backslash\backslash\psi)\backslash\backslash\chi)$ (**PJ7** from [17], $(\backslash\backslash$-1) from [11])

(PN) $\quad \dfrac{\varphi}{\psi^* \Rightarrow (\psi\backslash\backslash\varphi)}$

Instances of $P2$ are allowed only if the 0-ary flexible relation symbols R_i, $i = 1, \ldots, n$, are distinct and do not occur in $\varphi_1, \ldots, \varphi_n$ and θ. In the rest of the paper we mostly apply a special case of $P2$, which is obtained by putting $n = 1$ and choosing both $t_{1,1}$ and $t_{1,2}$ to be 0. This instance of $P2$ is equivalent to the rule:

$(P2_0)$ $\quad \dfrac{\overline{R_1}^* \wedge \Box(R_1 \Rightarrow \varphi_0) \wedge \varphi_1^{R_1} \Rightarrow \theta}{(\varphi_0\backslash\backslash\varphi_1) \Rightarrow \theta}$, if R_1 does not occur in φ_0, φ_1 and θ.

This rule is probably the most appropriate to illustrate the meaning of $P2$. Let an interval σ in a model M satisfy $\varphi_1^{R_1} \Rightarrow \theta$ for *any* R_1, provided that R_1 has the unique partition property at σ in M and the partition $\tau_0 < \ldots < \tau_n$ of σ which satisfies $M, \sigma \cap [\tau_{i-1}, \tau_i] \models R_1$, $i = 1, \ldots, n$ is such that $M, \sigma \cap [\tau_{i-1}, \tau_i] \models \varphi_0$, $i = 1, \ldots, n$ as well. Then $M, \sigma \models (\varphi_0\backslash\backslash\varphi_1) \Rightarrow \theta$. The side condition on R_1 not to occur in φ_0, φ_1 and θ corresponds to the requirement for the premiss of the rule to hold for any R_1 with the properties described in the rule. The role of this side condition becomes still clearer, if we note that $(\varphi_0\backslash\backslash\varphi_1)$ is equivalent to

$\exists R_1(\overline{R_1}^* \wedge \Box(R_1 \Rightarrow \varphi_0) \wedge \varphi_1^{R_1})$.

This makes our rule resemble a left introduction rule for $\exists R_1$. Similarly, the intended meaning of $P1$ can be written as

$\overline{\psi}^* \wedge \Box^\psi(\psi \Rightarrow \chi) \wedge \varphi^\psi \Rightarrow \exists R(\overline{R}^* \wedge \Box(R \Rightarrow \chi) \wedge \varphi^R)$

and provides a way to introduce positive occurrences of $(.\backslash\backslash.)$. However, as we already mentioned, the quantifier prefix $\exists R$ is not allowed in our language. The instances of $P2$ for $n > 1$ are convenient for technical reasons to become clear later.

The axiom $L2_\Leftarrow$ says that ordinary intervals can be "chopped" at any time point within their boundaries. Axiom $DI1$ is based on the fact that δ defines a partition of the reference interval which involves all of its time points, in case this interval is discrete. Hence, the interval obtained by taking the chopping points involved in this partition is the reference interval itself and therefore satisfies the same formulas. $DI2$ says that, given an arbitrary finite set of points in the reference interval, the discrete interval which consists of these points, together with the boundaries of the reference interval, can be accessed by means of $(.\backslash\backslash.)$ from the reference interval. The validity of $P\ell$ follows from $M_{\backslash\backslash}$. The meanings of the other axioms and rules are straightforward.

We call a formula φ an $ITL_{\backslash\backslash}$ *theorem* iff it can be obtained from instances of axioms of the proof system for $ITL_{\backslash\backslash}$ by means of the rules of this system. We use the expression $\vdash_{ITL_{\backslash\backslash}} \varphi$ to denote that φ is an $ITL_{\backslash\backslash}$ theorem.

3 Completeness of the proof system

In this section we present a Henkin-style completeness argument for our proof system. We closely follow the argument about ITL from [4]. In what follows we assume that the extension of the ITL proof system by $I1$ and $I2$ only is ω-complete for ITL^*, the extension of ITL by iteration only. (Although iteration is generally regarded as a basic operator in ITL, the result in [4] does not cover it.) Completeness arguments for systems with an infinitary rule of this kind for logics with Kleene star are known in the literature (cf. e.g. [20].) We do not include this proof in this paper, in order to avoid a lengthy, yet routine presentation, and concentrate on the aspects of the proof system which are specific to the new operator $(.\backslash\backslash.)$. For the same reason, we mark fragments of $ITL_{\backslash\backslash}$ deductions below by ITL^* and skip the details, in case no $ITL_{\backslash\backslash}$-specific axioms or rules besides $I1$ and $I2$, are involved in them. Similarly, we mark purely first order predicate logic fragments of deduction by PC.

The completeness argument is divided in two major parts. In the first part we establish the properties of $ITL_{\backslash\backslash}$ theories which are needed for their use in the construction of a model to show the satisfiability of an arbitrary given consistent set of $ITL_{\backslash\backslash}$ formulas. The second part is the construction of this model itself. In both of these major parts we use the $ITL_{\backslash\backslash}$ theorems which are listed and derived below:

Theorem 8

$(T1)$	$\vdash_{ITL_{\backslash\backslash}} \neg(\varphi\backslash\backslash\bot)$	
$(T2)$	$\vdash_{ITL_{\backslash\backslash}} (\varphi\backslash\backslash\exists x\psi) \Rightarrow \exists x(\varphi\backslash\backslash\psi)$ *if* $x \notin FV(\varphi)$	(**FOLJ1** from [17], $(\backslash\backslash$-6) from [11])
$(T3)$	$\vdash_{ITL_{\backslash\backslash}} \ell = 0 \Rightarrow ((\varphi\backslash\backslash\psi) \Leftrightarrow \psi)$	
$(T4)$	$\vdash_{ITL_{\backslash\backslash}} \overline{\varphi}^* \wedge (\varphi\backslash\backslash\psi_1 \Rightarrow \psi_2) \wedge (\varphi\backslash\backslash\psi_1) \Rightarrow (\varphi\backslash\backslash\psi_2)$	
$(T5)$	$\vdash_{ITL_{\backslash\backslash}} \Box(\varphi_1 \Rightarrow \varphi_2) \Rightarrow ((\varphi_1\backslash\backslash\psi) \Rightarrow (\varphi_2\backslash\backslash\psi))$	($(\backslash\backslash$-7) from [11])

Proof: In the $ITL_{\backslash\backslash}$ deductions for the above theorems we assume that R is a 0-ary flexible relation symbol which does not occur in φ, ψ, φ_i, ψ_i, $i = 1, 2$.

$T1$:

1	$\overline{R}^* \wedge \Box^R(R \Rightarrow \varphi) \wedge \bot^R \Rightarrow \bot$	PC, the definition of $\bot^R$
2	$(\varphi\backslash\backslash\bot) \Rightarrow \bot$	1, $P2_0$

$T2$:

1	$(\exists x\psi)^R \Leftrightarrow \exists x(\psi^R)$	ITL, the definition of $(\exists x\psi)^R$
2	$\overline{R}^* \wedge \Box^R(R \Rightarrow \varphi) \wedge \psi^R \Rightarrow (\varphi\backslash\backslash\psi)$	$P1$
3	$\overline{R}^* \wedge \Box^R(R \Rightarrow \varphi) \wedge \exists x(\psi^R) \Rightarrow \exists x(\varphi\backslash\backslash\psi)$	2, PC
4	$\overline{R}^* \wedge \Box^R(R \Rightarrow \varphi) \wedge (\exists x\psi)^R \Rightarrow \exists x(\varphi\backslash\backslash\psi)$	1, 3, PC
5	$\Box(R \Rightarrow \varphi) \Rightarrow \Box^R(R \Rightarrow \varphi)$	ITL^*
6	$\overline{R}^* \wedge \Box(R \Rightarrow \varphi) \wedge (\exists x\psi)^R \Rightarrow \exists x(\varphi\backslash\backslash\psi)$	4, 5, PC
7	$(\varphi\backslash\backslash\exists x\psi) \Rightarrow \exists x(\varphi\backslash\backslash\psi)$	6, $P2_0$

$T3$:

1	$\ell = 0 \Rightarrow \delta^*$	ITL^*
2	$\ell = 0 \Rightarrow \Box(\varphi \Rightarrow \delta)$	ITL^*
3	$\delta^* \Rightarrow ((\delta\backslash\backslash\psi) \Leftrightarrow \psi)$	$DI1$
4	$\ell = 0 \Rightarrow ((\delta\backslash\backslash\psi) \Leftrightarrow \psi)$	1, 3, PC
5	$\Box(\varphi \Rightarrow \delta) \Rightarrow ((\varphi\backslash\backslash\psi) \Rightarrow (\delta\backslash\backslash\psi))$	$T5$
6	$\ell = 0 \Rightarrow ((\varphi\backslash\backslash\psi) \Rightarrow \psi)$	2, 4, 5, PC
7	$\overline{\varphi}^* \wedge \Box^{\varphi}(\varphi \Rightarrow \varphi) \wedge \psi^{\varphi} \Rightarrow (\varphi\backslash\backslash\psi)$	$P1$
8	$\ell = 0 \Rightarrow \overline{\varphi}^*$	ITL^*
9	$\Box^{\varphi}(\varphi \Rightarrow \varphi)$	ITL^*
10	$\ell = 0 \wedge \psi \Rightarrow \psi^{\varphi}$	plain induction on the construction of ψ
11	$\ell = 0 \Rightarrow (\psi \Rightarrow (\varphi\backslash\backslash\psi))$	7-10, PC
12	$\ell = 0 \Rightarrow ((\varphi\backslash\backslash\psi) \Leftrightarrow \psi)$	6, 11, PC

$T4$:

1	$\overline{\varphi}^* \Rightarrow ((\varphi\backslash\backslash\psi_1 \Rightarrow \psi_2) \Leftrightarrow (\psi_1 \Rightarrow \psi_2)^{\varphi})$	$P3$
2	$\overline{\varphi}^* \Rightarrow ((\varphi\backslash\backslash\psi_1) \Leftrightarrow \psi_1^{\varphi})$	$P3$
3	$\overline{\varphi}^* \Rightarrow \Box^{\varphi}(\varphi \Rightarrow \varphi)$	ITL^*
4	$(\psi_1 \Rightarrow \psi_2)^{\varphi} \Rightarrow (\psi_1^{\varphi} \Rightarrow \psi_2^{\varphi})$	the definition of $(\psi_1 \Rightarrow \psi_2)^{\varphi}$
5	$\overline{\varphi}^* \wedge (\varphi\backslash\backslash\psi_1 \Rightarrow \psi_2) \wedge (\varphi\backslash\backslash\psi_1) \Rightarrow \Box^{\varphi}(\varphi \Rightarrow \varphi) \wedge \psi_2^{\varphi}$	1-4, PC
6	$\overline{\varphi}^* \wedge \Box^{\varphi}(\varphi \Rightarrow \varphi) \wedge \psi_2^{\varphi} \Rightarrow (\varphi\backslash\backslash\psi_2)$	$P1$
7	$\overline{\varphi}^* \wedge (\varphi\backslash\backslash\psi_1 \Rightarrow \psi_2) \wedge (\varphi\backslash\backslash\psi_1) \Rightarrow (\varphi\backslash\backslash\psi_2)$	5, 6, PC

$T5$:

1	$\overline{R}^* \wedge \Box(R \Rightarrow \varphi_2) \wedge \psi^R \Rightarrow (\varphi_2\backslash\backslash\psi)$	$P1$
2	$\Box(R \Rightarrow \varphi_1) \wedge \Box(\varphi_1 \Rightarrow \varphi_2) \Rightarrow \Box(R \Rightarrow \varphi_2)$	ITL
3	$\overline{R}^* \wedge \Box(R \Rightarrow \varphi_1) \wedge \psi^R \Rightarrow ((\varphi_2\backslash\backslash\psi) \vee \neg\Box(\varphi_1 \Rightarrow \varphi_2))$	1,2, PC
4	$(\varphi_1\backslash\backslash\psi) \Rightarrow ((\varphi_2\backslash\backslash\psi) \vee \neg\Box(\varphi_1 \Rightarrow \varphi_2))$	3, $P2_0$
5	$\Box(\varphi_1 \Rightarrow \varphi_2) \Rightarrow ((\varphi_1\backslash\backslash\psi) \Rightarrow (\varphi_2\backslash\backslash\psi))$	4, PC

$\dashv$

3.1 $ITL_{\backslash\backslash}$ theories

Throughout this section $\mathbf{L}$ denotes an $ITL_{\backslash\backslash}$ language. We identify $\mathbf{L}$ with the set of its terms and its formulas. Thus, $t \in \mathbf{L}$ stands for t *is a term in* $\mathbf{L}$ and $\varphi \in \mathbf{L}$ stands for φ *is a formula in* $\mathbf{L}$. Similarly, $\Gamma \subseteq \mathbf{L}$ stands for Γ *is a set of formulas in* $\mathbf{L}$. We assume that the vocabulary of $\mathbf{L}$ contains no more than countably many symbols of every kind. Given a set C of rigid constants and flexible 0-ary relation symbols, the $ITL_{\backslash\backslash}$ language obtained by adding the symbols from C to the vocabulary of $\mathbf{L}$ is denoted by $\mathbf{L}(C)$.

To introduce the notion of an $ITL_{\backslash\backslash}$ theory in $\mathbf{L}$, we need to define closedness of sets of formulas $\Gamma \subseteq \mathbf{L}$ under the rules of our proof system for $ITL_{\backslash\backslash}$. Closedness under MP and $I2$ only can be defined straightforwardly: a set $\Gamma \subseteq \mathbf{L}$ is closed under these rules, if whenever Γ contains all the premisses of an instance of a rule, Γ also contains the conclusion of the instance of the rule. The definition becomes a little more complicated if $P2$ gets involved, because this rule has a side condition. Roughly speaking, a derivation by an instance

$$\frac{\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \varphi_1, t_{2,1}, t_{2,2}, R_2, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta}{(\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \varphi_1 \backslash\backslash_{t_{2,1}}^{t_{2,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta}$$

of $P2$ from a set of formulas Γ can be unsound, unless the restriction not to contain occurrences of $R_1, \ldots, R_n$ is imposed not only on the formulas $\varphi_0, \ldots, \varphi_n$ and θ, but on the formulas from Γ too. This is so, because such a derivation is equivalent to an application of the rule

$$\frac{\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \varphi_1, t_{2,1}, t_{2,2}, R_2, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow (\bigwedge \Gamma \Rightarrow \theta)}{(\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \varphi_1 \backslash\backslash_{t_{2,1}}^{t_{2,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow (\bigwedge \Gamma \Rightarrow \theta)}$$

Yet such a rule cannot be formulated in $ITL_{\backslash\backslash}$, because Γ may be infinite. Closedness under first order logic quantifier-related rules requires some special care too.

To define closedness under these rules soundly, we introduce a relation of derivability as follows:

Definition 9 We define $\Gamma \vdash_{\mathbf{L}} \varphi$ as the smallest relation between sets of formulas Γ, languages $\mathbf{L}$ and formulas φ which satisfies the following conditions:

- If $\varphi \in \mathbf{L}$ and $\vdash_{ITL_{\backslash\backslash}} \varphi$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
- If $\varphi \in \Gamma$ and $\Gamma \subseteq \mathbf{L}$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
- If $\Gamma \vdash_{\mathbf{L}} \psi \Rightarrow \varphi$ and $\Gamma \vdash_{\mathbf{L}} \psi$ for some formula $\psi \in \mathbf{L}$, then $\Gamma \vdash_{\mathbf{L}} \varphi$.
- If $\Gamma \vdash_{\mathbf{L}} (\chi_1; \varphi^k; \chi_2) \Rightarrow \psi$ for all $k < \omega$, then $\Gamma \vdash_{\mathbf{L}} (\chi_1; \varphi^*; \chi_2) \Rightarrow \psi$.
- If $\Gamma \vdash_{\mathbf{L}(\{c\})} [c/x]\varphi \Rightarrow \psi$ for some rigid constant c which is not in the vocabulary of $\mathbf{L}$, nor occurs in φ, ψ, then $\Gamma \vdash_{\mathbf{L}} \exists x \varphi \Rightarrow \psi$.
- If $\Gamma \vdash_{\mathbf{L}(\{R_1,\ldots,R_n\})} \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \varphi_1, t_{2,1}, t_{2,2}, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta$ for some distinct 0-ary relation symbols $R_1, \ldots, R_n$ which are not in $\mathbf{L}$, nor occur in $\varphi_0, \ldots, \varphi_n, \theta$, then $\Gamma \vdash_{\mathbf{L}} (\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \varphi_1 \backslash\backslash_{t_{2,1}}^{t_{2,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta$.

Note that the relation $\vdash$ introduced here is distinct from $\vdash_{ITL_{\backslash\backslash}}$, which is used to denote theoremhood in $ITL_{\backslash\backslash}$. Clearly, $\vdash_{ITL_{\backslash\backslash}} \varphi$ is equivalent to $\emptyset \vdash_{\mathbf{L}} \varphi$ and $\varphi \in \mathbf{L}$ for some $ITL_{\backslash\backslash}$ language $\mathbf{L}$. Besides, if $\mathbf{L}' \subseteq \mathbf{L}''$ and $\Gamma \vdash_{\mathbf{L}'} \varphi$, then obviously $\Gamma \vdash_{\mathbf{L}''} \varphi$.

Now let us characterise $\vdash$ by an inductive definition, in order to be able to reason about it by means of transfinite induction.

Definition 10 Let $\Gamma \vdash_{\mathbf{L}}^{\alpha} \varphi$ be a relation between sets of $ITL_{\backslash\backslash}$ formulas Γ, $ITL_{\backslash\backslash}$ languages $\mathbf{L}$, ordinals α and $ITL_{\backslash\backslash}$ formulas φ. Let this relation be defined by induction on α as follows:

$\Gamma \vdash_{\mathbf{L}}^{0} \varphi$ iff $\varphi \in \mathbf{L}$, $\Gamma \subseteq \mathbf{L}$ and either $\vdash_{ITL_{\backslash\backslash}} \varphi$ or $\varphi \in \Gamma$. For $\alpha \neq 0$, $\Gamma \vdash_{\mathbf{L}}^{\alpha} \varphi$ iff at least one of the following holds:

- $\Gamma \vdash^{\beta_1}_{\mathbf{L}} \psi \Rightarrow \varphi$ and $\Gamma \vdash^{\beta_2}_{\mathbf{L}} \psi$ for some $\beta_1, \beta_2 < \alpha$ and some $\psi \in \mathbf{L}$.
- The formula φ is $\exists x \psi \Rightarrow \chi$ and $\Gamma \vdash^{\beta}_{\mathbf{L}(\{c\})} [c/x]\psi \Rightarrow \chi$ for some $\beta < \alpha$ and some rigid constant c which is not in the vocabulary of $\mathbf{L}$, nor occurs in φ.
- The formula φ is $(\varphi_0 \backslash\backslash^{t_{1,2}}_{t_{1,1}} \dots \backslash\backslash^{t_{n,2}}_{t_{n,1}} \varphi_n) \Rightarrow \theta$ and

  $$\Gamma \vdash^{\beta}_{\mathbf{L}(\{R_1,\dots,R_n\})} \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta$$

  for some $\beta < \alpha$ and some distinct 0-ary flexible relation symbols $R_1, \dots, R_n$ which are not in the vocabulary of $\mathbf{L}$ and do not occur in φ.
- The formula φ is $(\chi_1; \psi^*; \chi_2) \Rightarrow \theta$ and for every $k < \omega$ there exists a $\beta_k < \alpha$ such that $\Gamma \vdash^{\beta_k}_{\mathbf{L}} (\chi_1; \psi^k; \chi_2) \Rightarrow \theta$.

Induction Principle. $\Gamma \vdash_{\mathbf{L}} \varphi$ *is equivalent to* $\exists \alpha \in \mathrm{Ord}\ \Gamma \vdash^{\alpha}_{\mathbf{L}} \varphi$.

Proof: Obviously $\exists \alpha \in \mathrm{Ord}\ \Gamma \vdash^{\alpha}_{\mathbf{L}} \varphi$ defines a relation which satisfies the closedness conditions on $\vdash$. Suppose that the relation defined this way is not the smallest one that satisfies these closedness conditions for the sake of contradiction. Choosing α_0 to be the least ordinal such that there exist a language $\mathbf{L}$, a set of formulas Γ and a formula φ which satisfy $\Gamma \vdash^{\alpha_0}_{\mathbf{L}} \varphi$ and $\Gamma \not\vdash_{\mathbf{L}} \varphi$ immediately brings a contradiction. $\dashv$

Lemma 11 $\Gamma \vdash_{\mathbf{L}} \varphi$ *implies* $\varphi \in \mathbf{L}$ *and* $\Gamma \subseteq \mathbf{L}$.

Proof: Induction on $\alpha \in \mathrm{Ord}$ for α satisfying $\Gamma \vdash^{\alpha}_{\mathbf{L}} \varphi$. $\dashv$

Given a set $\Gamma \subseteq \mathbf{L}$, we denote the set $\{\varphi \in \mathbf{L} : \Gamma \vdash_{\mathbf{L}} \varphi\}$ by $Cn_{\mathbf{L}}(\Gamma)$. $Cn_{\mathbf{L}}(\Gamma)$ consists of the logical consequences of Γ in $\mathbf{L}$ which can be derived using our axioms and rules. Given Γ and a formula φ, we denote the set $\{\psi : \Gamma \vdash_{\mathbf{L}} \varphi \Rightarrow \psi\}$ by $\Gamma +_{\mathbf{L}} \varphi$.

Lemma 12 $\Gamma +_{\mathbf{L}} \varphi = Cn_{\mathbf{L}}(\Gamma \cup \{\varphi\})$.

Proof: The inclusion $\Gamma +_{\mathbf{L}} \varphi \subseteq Cn_{\mathbf{L}}(\Gamma \cup \{\varphi\})$ follows trivially from the closedness of $\vdash$ under *MP*. To show that $\Gamma \cup \{\varphi\} \vdash_{\mathbf{L}} \psi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi \Rightarrow \psi$, we use induction on $\alpha \in \mathrm{Ord}$ for α satisfying $\Gamma \cup \{\varphi\} \vdash^{\alpha}_{\mathbf{L}} \psi$. We prove that if $\Gamma \cup \{\varphi\} \vdash^{\alpha}_{\mathbf{L}} \psi$, then there exists an ordinal α' such that $\Gamma \vdash^{\alpha'}_{\mathbf{L}} \varphi \Rightarrow \psi$ by induction on α. We give details on the case of ψ being of the form $(\chi_1; \psi^*; \chi_2) \Rightarrow \theta$ and $\Gamma \cup \{\varphi\} \vdash^{\alpha}_{\mathbf{L}} \psi$ being true because of the existence of some $\beta_k < \alpha$ such that $\Gamma \cup \{\varphi\} \vdash^{\beta_k}_{\mathbf{L}} (\chi_1; \psi^k; \chi_2) \Rightarrow \theta$, $k < \omega$, here. In this case the induction hypothesis implies that for every $k < \omega$ there exists an ordinal β'_k such that $\Gamma \vdash^{\beta'_k}_{\mathbf{L}} \varphi \Rightarrow ((\chi_1; \psi^k; \chi_2) \Rightarrow \theta)$. Using that

$$\vdash_{ITL_{\backslash\backslash}} (\varphi \Rightarrow ((\chi_1; \psi^k; \chi_2) \Rightarrow \theta)) \Rightarrow ((\chi_1; \psi^k; \chi_2) \Rightarrow (\varphi \Rightarrow \theta))$$

and the definition of $\vdash^{(.)}$, we infer that $\Gamma \vdash^{\beta'_k+1}_{\mathbf{L}} (\chi_1; \psi^k; \chi_2) \Rightarrow (\varphi \Rightarrow \theta)$. Let α' be an ordinal that is greater than $\sup\{\beta'_k + 1 : k < \omega\}$. Then $\Gamma \vdash^{\alpha'}_{\mathbf{L}} (\chi_1; \psi^*; \chi_2) \Rightarrow (\varphi \Rightarrow \theta)$ by the definition of $\vdash^{(.)}$. Just like above, this implies that $\Gamma \vdash^{\alpha'+1}_{\mathbf{L}} \varphi \Rightarrow ((\chi_1; \psi^*; \chi_2) \Rightarrow \theta)$. The cases of $\Gamma \cup \{\varphi\} \vdash^{\alpha}_{\mathbf{L}} \psi$ holding for some other of the possible reasons according to the definition are dealt with in similar ways. $\dashv$

Definition 13 A set $\Gamma \subseteq \mathbf{L}$ is called *theory in* $\mathbf{L}$, if $\Gamma = Cn_{\mathbf{L}}(\Gamma)$. A theory Γ is called *consistent*, if $\perp \not\in \Gamma$. A set $\Gamma \subseteq \mathbf{L}$ is called *consistent in* $\mathbf{L}$ if some consistent theory in $\mathbf{L}$ contains it. A theory in $\mathbf{L}$ is called *maximal in* $\mathbf{L}$ if it is consistent and it is not a proper subset of any consistent theory in $\mathbf{L}$. A theory Γ is called *complete in* $\mathbf{L}$, if either $\varphi \in \Gamma$ or $\neg\varphi \in \Gamma$ for every $\varphi \in \mathbf{L}$.

The conditions $\perp \in Cn_{\mathbf{L}'}(\Gamma)$ and $\perp \in Cn_{\mathbf{L}''}(\Gamma)$ are equivalent for any two languages $\mathbf{L}'$ and $\mathbf{L}''$ which contain Γ. That is why consistency can be regarded as a property of sets of formulas Γ regardless of the language $\mathbf{L}$ involved in the definition of this property. Using Lemma 12, one can easily show that a theory is complete iff it is maximal in its language.

The following adaptation of the notion of *Henkin theory*, which is specific to our proof system for $ITL_{\backslash\backslash}$, plays a key role in further constructions.

Definition 14 A theory Γ in $\mathbf{L}(C)$ is called a *Henkin theory with witnesses in* C, if

- $\exists x\varphi \in \Gamma$ implies $[c/x]\varphi \in \Gamma$ for some $c \in C$.
- Given a natural number $n \geq 1$ and $\varphi_i \in \mathbf{L}(C)$, $i = 0, \ldots, n$, $t_{i,1}, t_{i,2} \in \mathbf{L}(C)$, $i = 1, \ldots, n$, such that $(\varphi_0 \backslash\backslash^{t_{1,2}}_{t_{1,1}} \ldots \backslash\backslash^{t_{n,2}}_{t_{n,1}} \varphi_n) \in \Gamma$, there exist $R_1, \ldots, R_n \in C$ such that $\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \in \Gamma$.

For the rest of this section we assume that C consists of countably many rigid constants and countably many flexible 0-ary relation symbols, none of which is in the vocabulary of $\mathbf{L}$.

Theorem 15 (Lindenbaum lemma) *Let* $\Gamma_0 \subset \mathbf{L}$ *be consistent. Then there exists a maximal Henkin theory* $\Gamma \subset \mathbf{L}(C)$ *with witnesses in* C *such that* $\Gamma_0 \subseteq \Gamma$.

Proof: This proof follows a general pattern known from numerous modal logics.

Let the set of all the formulas from $\mathbf{L}(C)$ be $\{\varphi_k : k < \omega\}$. We define the sequence of consistent sets $\Gamma_k \subset \mathbf{L}(C)$ so that $\Gamma_k \setminus \Gamma_0$ is finite for every $k < \omega$. This entails that formulas from Γ_k have occurrences of only finitely many elements of C for every k. Given a set of formulas $\Delta \subset \mathbf{L}(C)$, we use $\mathbf{L}(\Delta)$ to denote the $ITL_{\backslash\backslash}$ language obtained by extending the vocabulary of $\mathbf{L}$ with the symbols from C which occur in the formulas from Δ. The definition of Γ_k, $k < \omega$, is as follows:

Γ_0 is as given in the theorem. Assume that Γ_k has been defined for some $k < \omega$. If $\Gamma_k \cup \{\varphi_k\}$ is not consistent, then $\Gamma_{k+1} = \Gamma_k$. Otherwise we consider the following cases:

1. φ_k is $\exists x\psi$ for some $\psi \in \mathbf{L}(C)$. Then we choose a rigid constant $c \in C$ which does not occur in formulas from Γ_k, nor in φ_k, and put $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k, [c/x]\psi\}$. Assume that Γ_{k+1} is inconsistent for the sake of contradiction. This implies

$$\Gamma_k \vdash_{\mathbf{L}(\Gamma_k \cup \{\varphi_k\})} \varphi_k \Rightarrow \perp$$

by Lemma 12, which contradicts the consistency of $\Gamma_k \cup \{\varphi_k\}$. Hence Γ_{k+1} is consistent.

2. φ_k is $(\chi_1; \psi^*; \chi_2)$ for some $\chi_1, \chi_2, \psi \in \mathbf{L}(C)$. Then there exists an $n < \omega$ such that $\Gamma_k \cup \{\varphi_k, (\chi_1; \psi^n; \chi_2)\}$ is consistent too. Assuming the contrary would bring a

contradiction with the given consistency of Γ_k. We choose an n with the above property and put $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k, (\chi_1; \psi^n; \chi_2)\}$.
3. φ_k is $(\psi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \psi_n)$ for some $\psi_i \in \mathbf{L}(C)$, $i = 0, \ldots, n$, and $t_{i,1}, t_{i,2} \in \mathbf{L}(C)$, $i = 1, \ldots, n$, such that ψ_n is not of the form $(\ell = t'; (\psi' \backslash\backslash \psi''); \ell = t'')$. The restriction on the form of ψ_n here is to ensure that φ_k does not satisfy the stated condition for more than one n. Let

$\chi_j \rightleftharpoons (\psi_j \backslash\backslash_{t_{j+1,1}}^{t_{j+1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \psi_n)$, $j = 0, \ldots, n-1$, $\chi_n \rightleftharpoons \psi_n$.

Then φ_k can be also recorded as $(\psi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{j,1}}^{t_{j,2}} \chi_j)$ for $j = 1, \ldots, n$. Let R_i^j, $j = 1, \ldots, n$, $i = 1, \ldots, j$, be distinct 0-ary flexible relation symbols from C which do not occur in $\mathbf{L}(\Gamma_k \cup \{\varphi_k\})$. Let

$$\Gamma_{k+1}^p = \Gamma_k \cup \{\varphi_k\} \cup \{\mathsf{proj}(\psi_0, t_{1,1}, t_{1,2}, R_1^j, \ldots, t_{j,1}, t_{j,2}, R_j^j, \chi_j) : j = 1, \ldots, p\}$$

for $p = 1, \ldots, n$, $\Gamma_{k+1}^0 = \Gamma_k \cup \{\varphi_k\}$ and Γ_{k+1} be Γ_{k+1}^n. We must prove that Γ_{k+1} is consistent. We prove that Γ_{k+1}^p is consistent by induction on p. Γ_{k+1}^0 is consistent by assumption. Assume that Γ_{k+1}^p is consistent and Γ_{k+1}^{p+1} is inconsistent for some $p < n$ for the sake of contradiction. Then Lemma 12 entails that

$$\mathsf{proj}(\psi_0, t_{1,1}, t_{1,2}, R_1^{p+1}, \ldots, t_{p+1,1}, t_{p+1,2}, R_{p+1}^{p+1}, \chi_{p+1}) \Rightarrow \bot$$

is a member of $Cn_{\mathbf{L}(\Gamma_{k+1}^p)(\{R_1^{p+1}, \ldots, R_{p+1}^{p+1}\})}(\Gamma_{k+1}^p)$, whence $\Gamma_{k+1}^p \vdash_{\mathbf{L}(\Gamma_{k+1}^p)} \varphi_k \Rightarrow \bot$ by the clause about closedness under $P2$ in the definition of $\vdash$, because $R_1^{p+1}, \ldots, R_{p+1}^{p+1}$ do not occur in $\mathbf{L}(\Gamma_{k+1}^p)$. This entails that Γ_{k+1}^p is inconsistent too, which is a contradiction.
4. None of the above cases holds. Then $\Gamma_{k+1} = \Gamma_k \cup \{\varphi_k\}$.
A standard argument shows that $\Gamma = \bigcup\limits_{k<\omega} \Gamma_k$ is a maximal Henkin theory in $\mathbf{L}(C)$ with witnesses in C. Clearly $\Gamma \supset \Gamma_0$. $\dashv$

Lemma 16 *Let Γ be a complete theory in $\mathbf{L}(C)$ and $(\chi_1; \varphi^*; \chi_2) \in \Gamma$. Then there exists a $k < \omega$ such that $(\chi_1; \varphi^k; \chi_2) \in \Gamma$.*

Proof: Assume that $(\chi_1; \varphi^k; \chi_2) \notin \Gamma$ for all $k < \omega$ for the sake of contradiction. Then $(\chi_1; \varphi^k; \chi_2) \Rightarrow \bot \in \Gamma$ for all $k < \omega$, because Γ is complete. Hence, $(\chi_1; \varphi^*; \chi_2) \Rightarrow \bot \in \Gamma$ by $I2$, because Γ is a theory. This is a contradiction. $\dashv$

Given sets of formulas $\Gamma, \Gamma_1, \Gamma_2 \subseteq \mathbf{L}(C)$ and a formula $\varphi \in \mathbf{L}(C)$, we denote the set $\{(\varphi; \psi) : \varphi \in \Gamma_1, \psi \in \Gamma_2\}$ by $\Gamma_1; \Gamma_2$.

Lemma 17 *Let Γ_1, Γ_2 and Γ be complete theories in $\mathbf{L}(C)$. Let $\ell = c_i \in \Gamma_i$ for some $c_i \in C$, $i = 1, 2$, and $\Gamma_1; \Gamma_2 \subset \Gamma$. Then $\varphi \in \Gamma$, $\varphi \in \Gamma_1$ and $\varphi \in \Gamma_2$ are equivalent for rigid $\varphi \in \mathbf{L}(C)$. Similarly, $\psi^* \wedge \varphi \in \Gamma$ is equivalent to $(\psi \backslash\backslash \varphi) \in \Gamma$ for rigid $\varphi \in \mathbf{L}(C)$.*

Proof: The first part follows from axioms R_l and R_r. The second follows from axiom PR. $\dashv$

The following lemma presents some $ITL_{\backslash\backslash}$ theorems which are essentially theorems in ITL^*.

Lemma 18

$(T6) \quad \vdash_{ITL_{\backslash\backslash}} \overline{R_0}^* \wedge \overline{R}^* \wedge \Box(R \Rightarrow R_0) \Rightarrow \Box(R \Leftrightarrow R_0)$

$(T7) \quad \vdash_{ITL_{\backslash\backslash}} \overline{R_0}^* \wedge (R_0\backslash\backslash\overline{R}^*) \Rightarrow \overline{(R_0\backslash\backslash R)}^*$

Proof: Let M be a model for an ITL^* language which contains R and R_0. Then Lemma 7 entails that $M, \sigma \models \Box(R \Leftrightarrow R_0)$ for all intervals $\sigma \in \mathbf{I}_{\backslash\backslash}(T_M)$ such that $M, \sigma \models \overline{R_0}^*, \overline{R}^*, \Box(R \Rightarrow R_0)$. Hence the formula $T6$ is valid in ITL^*. $ITL_{\backslash\backslash}$ is a conservative extension of ITL^* and we assume that the extension of the proof system of ITL by $I1$ and $I2$ is complete for ITL^*. Hence the formula $T6$ is a theorem in $ITL_{\backslash\backslash}$. The formula $T7$ is equivalent to $\overline{R_0}^* \wedge (\overline{R}^*)^{R_0} \Rightarrow \overline{(R_0\backslash\backslash R)}^*$ by $P3$, which is valid in ITL^*. Hence $T7$ is an $ITL_{\backslash\backslash}$ theorem too. $\dashv$

Lemma 19 *Let Γ be a set of formulas in* $\mathbf{L}$, $\Gamma \neq \emptyset$, *and* $\Gamma \vdash_{\mathbf{L}} \varphi$. *Let c_1, c_2 be two rigid constants in the vocabulary of* $\mathbf{L}$. *Then* $\{(\ell = c_1; \psi; \ell = c_2) : \psi \in \Gamma\} \vdash_{\mathbf{L}} (\ell = c_1; \varphi; \ell = c_2)$.

Proof: Let Δ denote $\{(\ell = c_1; \psi; \ell = c_2) : \psi \in \Gamma\}$. We do the proof by induction on $\alpha \in \mathrm{Ord}$ for α satisfying $\Gamma \vdash^{\alpha}_{\mathbf{L}} \varphi$. According to the definition of $\Gamma \vdash^{\alpha}_{\mathbf{L}} \varphi$, at least one of the following cases holds:

1. φ is $ITL_{\backslash\backslash}$ theorem and $\varphi \in \mathbf{L}$. Let $\psi \in \Gamma$. Then $\psi \Rightarrow \varphi$ is an $ITL_{\backslash\backslash}$ theorem and $\psi \Rightarrow \varphi \in \mathbf{L}$. Hence $(\ell = c_1; \psi; \ell = c_2) \Rightarrow (\ell = c_1; \varphi; \ell = c_2)$ is an $ITL_{\backslash\backslash}$ theorem in $\mathbf{L}$ too by $Mono_l$ and $Mono_r$. Now $\Delta \vdash_{\mathbf{L}} (\ell = c_1; \varphi; \ell = c_2)$ follows from $\Delta \vdash_{\mathbf{L}} (\ell = c_1; \psi; \ell = c_2)$ and $\Delta \vdash_{\mathbf{L}} (\ell = c_1; \psi; \ell = c_2) \Rightarrow (\ell = c_1; \varphi; \ell = c_2)$ by MP.
2. $\varphi \in \Gamma$. Then $(\ell = c_1; \varphi; \ell = c_2) \in \Delta$, whence $\Delta \vdash_{\mathbf{L}} (\ell = c_1; \varphi; \ell = c_2)$.
3. There exist $\beta_1, \beta_2 < \alpha$ and $\psi \in \mathbf{L}$ such that $\Gamma \vdash^{\beta_1}_{\mathbf{L}} \psi \Rightarrow \varphi$ and $\Gamma \vdash^{\beta_2}_{\mathbf{L}} \psi$. This case is dealt with using that

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; \psi \Rightarrow \varphi; \ell = c_2)$ and $\Delta \vdash_{\mathbf{L}} (\ell = c_1; \psi; \ell = c_2)$

by the induction hypothesis, and

$\vdash_{ITL_{\backslash\backslash}} (\ell = c_1; \psi; \ell = c_2) \Rightarrow ((\ell = c_1; \psi \Rightarrow \varphi; \ell = c_2) \Rightarrow (\ell = c_1; \varphi; \ell = c_2))$.

4. φ is $\exists x\psi \Rightarrow \theta$ where $\psi, \theta \in \mathbf{L}$ and $\Gamma \vdash^{\beta}_{\mathbf{L}(\{c\})} [c/x]\psi \Rightarrow \theta$ for some $\beta < \alpha$, where c is a rigid constant which is not in the vocabulary of $\mathbf{L}$. Then, by the induction hypothesis,

$\Delta \vdash_{\mathbf{L}(\{c\})} (\ell = c_1; [c/x]\psi \Rightarrow \theta; \ell = c_2)$.

This entails that

$\Delta \vdash_{\mathbf{L}(\{c\})} [c/x](\ell = c_1; \psi; \ell = c_2) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$.

Hence

$\Delta \vdash_{\mathbf{L}} \exists x(\ell = c_1; \psi; \ell = c_2) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$

which implies

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; \exists x\psi; \ell = c_2) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$

by B_l and B_r, and finally

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; \exists x\psi \Rightarrow \theta; \ell = c_2)$

5. φ is $(\varphi_0\backslash\backslash^{t_{1,2}}_{t_{1,1}} \ldots \backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n) \Rightarrow \theta$ where $\varphi_0, \ldots, \varphi_n, t_{1,1}, t_{1,2}, \ldots, t_{n,1}, t_{n,2} \in \mathbf{L}$, and

$\Gamma \vdash^{\beta}_{\mathbf{L}(\{R_1,\ldots,R_n\})} \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta$

for some $\beta < \alpha$ and some $R_1, \ldots, R_n$ that are distinct 0-ary relation symbols out of the vocabulary of **L**. Then, by the induction hypothesis,

$\Delta \vdash_{\mathbf{L}(\{R_1,\ldots,R_n\})} (\ell = c_1; \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta; \ell = c_2)$.

This entails that

$\Delta \vdash_{\mathbf{L}(\{R_1,\ldots,R_n\})} \mathsf{proj}(\varphi_0, c_1 + t_{1,1}, t_{1,2} + c_2, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$

by the definition of proj. Hence

$\Delta \vdash_{\mathbf{L}} (\varphi_0 \backslash\backslash_{c_1+t_{1,1}}^{t_{1,2}+c_2} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$,

which is equivalent to

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; (\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta; \ell = c_2)$.

6. φ is $(\chi_1; \psi^*; \chi_2) \Rightarrow \theta$ where $\psi, \theta \in \mathbf{L}$, and for every $k < \omega$ there exists a $\beta_k < \alpha$ such that

$\Gamma \vdash_{\mathbf{L}}^{\beta_k} (\chi_1; \psi^k; \chi_2) \Rightarrow \theta$.

Then

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; (\chi_1; \psi^k; \chi_2) \Rightarrow \theta; \ell = c_2)$,

whence, by $A2$, $Mono_l$ and $Mono_r$

$\Delta \vdash_{\mathbf{L}} ((\ell = c_1; \chi_1); \psi^k; (\chi_2; \ell = c_2)) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$

for all $k < \omega$ by the induction hypothesis. Hence

$\Delta \vdash_{\mathbf{L}} ((\ell = c_1; \chi_1); \psi^*; (\chi_2; \ell = c_2)) \Rightarrow (\ell = c_1; \theta; \ell = c_2)$.

by $I2$. This is equivalent to

$\Delta \vdash_{\mathbf{L}} (\ell = c_1; (\chi_1; \psi^*; \chi_2) \Rightarrow \theta; \ell = c_2)$

by $A2$, $Mono_l$ and $Mono_r$ again. This concludes the proof. $\dashv$

Given a set of formulas Γ and a formula φ, $(\varphi\backslash\backslash\Gamma)$ stands for $\{(\varphi\backslash\backslash\psi) : \psi \in \Gamma\}$.

Lemma 20 *Let Γ be a set of formulas in* **L**, *and $\Gamma \vdash_{\mathbf{L}} \varphi$. Let R be a 0-ary flexible relation symbol in* **L**. *Then $(R\backslash\backslash\Gamma) \cup \{\overline{R}^*\} \vdash_{\mathbf{L}} (R\backslash\backslash\varphi)$.*

Proof: Transfinite induction on $\alpha \in \mathrm{Ord}$ such that $\Gamma \vdash_{\mathbf{L}}^{\alpha} \varphi$, like in the proof of Lemma 19. Let Δ denote $(R\backslash\backslash\Gamma) \cup \{\overline{R}^*\}$. We consider the following cases:

1. φ is an $ITL_{\backslash\backslash}$ theorem and $\varphi \in \mathbf{L}$. Then $R^* \Rightarrow (R\backslash\backslash\varphi)$ is an $ITL_{\backslash\backslash}$ theorem in **L** too, by the rule PN. Since $\overline{R}^* \in \Delta$, $\Delta \vdash_{\mathbf{L}} (R\backslash\backslash\varphi)$ by MP from $\Delta \vdash_{\mathbf{L}} R^* \Rightarrow (R\backslash\backslash\varphi)$.

2. $\varphi \in \Gamma$, then $(R\backslash\backslash\varphi) \in \Delta$, whence $\Delta \vdash_{\mathbf{L}} (R\backslash\backslash\varphi)$.

3. There exist $\beta_1, \beta_2 < \alpha$ and $\psi \in \mathbf{L}$ such that $\Gamma \vdash_{\mathbf{L}}^{\beta_1} \psi \Rightarrow \varphi$ and $\Gamma \vdash_{\mathbf{L}}^{\beta_2} \psi$. Then, by the induction hypothesis,

$\Delta \vdash_{\mathbf{L}} (R\backslash\backslash\psi \Rightarrow \varphi)$ and $\Delta \vdash_{\mathbf{L}} (R\backslash\backslash\psi)$.

Now, using $T4$ of Theorem 8 and $\overline{R}^* \in \Delta$, we can infer that

$\Delta \vdash_{\mathbf{L}} (R\backslash\backslash\varphi)$.

4. φ is $\exists x\psi \Rightarrow \theta$ and $\Gamma \vdash_{\mathbf{L}(\{c\})}^{\beta} [c/x]\psi \Rightarrow \theta$ for some $\beta < \alpha$, where c is a rigid constant which is not in the vocabulary of **L**. Then, by the induction hypothesis,

$\Delta \vdash_{\mathbf{L}(\{c\})} (R\backslash\backslash[c/x]\psi \Rightarrow \theta)$.

This entails that

$\Delta \vdash_{\mathbf{L}(\{c\})} ([c/x]\psi)^R \Rightarrow \theta^R$

by $P3$ and the definition of $(.)^R$. Then

$\Delta \vdash_{\mathbf{L}(\{c\})} [c/x](\psi^R) \Rightarrow \theta^R$
by the definition of $(.)^R$. Hence
$\Delta \vdash_{\mathbf{L}} \exists x(\psi^R) \Rightarrow \theta^R$ and $\Delta \vdash_{\mathbf{L}} (\exists x\psi \Rightarrow \theta)^R$
by the definition of $(.)^R$ again, whence, using $\overline{R}^* \in \Delta$ and $P3$, we obtain
$\Delta \vdash_{\mathbf{L}} (R \backslash\backslash \exists x\psi \Rightarrow \theta)$
5. φ is $(\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta$ and
$\Gamma \vdash^{\beta}_{\mathbf{L}(\{R_1,\ldots,R_n\})} \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \Rightarrow \theta$
for some $\beta < \alpha$ and some distinct 0-ary relation symbols $R_1, \ldots, R_n$ which are not in the vocabulary of $\mathbf{L}$. Let P stand for $\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n)$ in the rest of this proof for the sake of brevity. Then, by the induction hypothesis,
$\Delta \vdash_{\mathbf{L}(\{R_1,\ldots,R_n\})} (R \backslash\backslash P \Rightarrow \theta)$.
Like in the previous cases, this entails that
$\Delta \vdash_{\mathbf{L}(\{R_1,\ldots,R_n\})} P^R \Rightarrow \theta^R$
by $P3$ and the definition of $(.)^R$. Let R_0 be a 0-ary flexible relation symbol not in the vocabulary of $\mathbf{L}(\{R_1, \ldots, R_n\})$. Then
$\vdash_{ITL_{\backslash\backslash}} \overline{R}^* \Rightarrow (\overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \Rightarrow \Box(R \Leftrightarrow R_0))$.
by $T6$ of Lemma 18. Since $T5$ of Theorem 8 implies that
$\vdash_{ITL_{\backslash\backslash}} \Box(R \Leftrightarrow R_0) \Rightarrow ((R \backslash\backslash P) \Leftrightarrow (R_0 \backslash\backslash P))$,
and $P3$ implies that
$\vdash_{ITL_{\backslash\backslash}} \overline{R}^* \Rightarrow ((R \backslash\backslash P) \Leftrightarrow P^R)$ and $\vdash_{ITL_{\backslash\backslash}} \overline{R_0}^* \Rightarrow ((R_0 \backslash\backslash P) \Leftrightarrow P^{R_0})$,
we have
$\Delta \vdash_{\mathbf{L}(\{R_0,R_1,\ldots,R_n\})} \overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \Rightarrow (P^R \Leftrightarrow P^{R_0})$
This entails that
$\Delta \vdash_{\mathbf{L}(\{R_0,R_1,\ldots,R_n\})} \overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \wedge P^{R_0} \Rightarrow \theta^R$.
Yet
$\vdash_{ITL_{\backslash\backslash}} (\overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \wedge P^{R_0}) \Leftrightarrow$
$\mathsf{proj}(R, 0, 0, R_0, \varphi_0, t_{1,1}, t_{1,2}, R_1, \ldots, t_{n,1}, t_{n,2}, R_n, \varphi_n)$
by the definition of proj, $L3_l$ and $L3_r$. Hence
$\Delta \vdash_{\mathbf{L}} (R \backslash\backslash_0^0 \varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta^R$.
by $P2$. This entails that
$\Delta \vdash_{\mathbf{L}} (R \backslash\backslash (\varphi_0 \backslash\backslash_{t_{1,1}}^{t_{1,2}} \ldots \backslash\backslash_{t_{n,1}}^{t_{n,2}} \varphi_n) \Rightarrow \theta)$
by $P3$, $L3_l$ and $L3_r$ again.
6. φ is $(\chi_1; \psi^*; \chi_2) \Rightarrow \theta$ and for every $k < \omega$ there exists a $\beta_k < \alpha$ such that
$\Gamma \vdash^{\beta_k}_{\mathbf{L}} (\chi_1; \psi^k; \chi_2) \Rightarrow \theta$.
Then
$\Delta \vdash_{\mathbf{L}} (R \backslash\backslash (\chi_1; \psi^k; \chi_2) \Rightarrow \theta)$
by the induction hypothesis, whence
$\Delta \vdash_{\mathbf{L}} ((\chi_1; \psi^k; \chi_2) \Rightarrow \theta)^R$
by $P3$ for all $k < \omega$. Now note that $((\chi_1; \psi^k; \chi_2) \Rightarrow \theta)^R$ stands for
$((\chi_1^R \wedge R^*; (\psi^R \wedge R^*)^k; \chi_2^R \wedge R^*) \Rightarrow \theta^R) \wedge R^*$.
Hence
$\Delta \vdash_{\mathbf{L}} (\chi_1^R \wedge R^*; (\psi^R \wedge R^*)^k; \chi_2^R \wedge R^*) \Rightarrow \theta^R$
for all $k < \omega$, which implies that
$\Delta \vdash_{\mathbf{L}} ((\chi_1^R \wedge R^*; (\psi^R \wedge R^*)^*; \chi_2^R \wedge R^*) \Rightarrow \theta^R) \wedge R^*$.

by $I2$. Yet the above formula is exactly $((\chi_1;\psi^*;\chi_2) \Rightarrow \theta)^R$. This, together with $\overline{R}^* \in \Delta$ and $P3$, implies

$\Delta \vdash_{\mathbf{L}} (R\backslash\backslash(\chi_1;\psi^*;\chi_2) \Rightarrow \theta)$.

This concludes the proof. $\dashv$

Lemma 21 *Let Γ be a complete Henkin theory in $\mathbf{L}(C)$ with witnesses in C. Let $c_1, c_2 \in C$ be such that $\Delta = \{\varphi \in \mathbf{L}(C) : (\ell = c_1;\varphi;\ell = c_2) \in \Gamma\}$ is nonempty. Then Δ is a complete Henkin theory in $\mathbf{L}(C)$ with witnesses in C.*

Proof: Let $\Delta \vdash_{\mathbf{L}(C)} \varphi$. Then Lemma 19 entails that $\Gamma \vdash_{\mathbf{L}(C)} (\ell = c_1;\varphi;\ell = c_2)$. Hence $\varphi \in \Delta$. This shows that $\Delta = Cn_{\mathbf{L}(C)}(\Delta)$.

Let $\varphi \in \mathbf{L}(C)$. Then either $(\ell = c_1;\varphi;\ell = c_2) \in \Gamma$, or $\neg(\ell = c_1;\varphi;\ell = c_2) \in \Gamma$, because Γ is complete. If $(\ell = c_1;\varphi;\ell = c_2) \in \Gamma$, then $\varphi \in \Delta$ by the definition of Δ. If $\neg(\ell = c_1;\varphi;\ell = c_2) \in \Gamma$, then $(\ell = c_1;\neg\varphi;\ell = c_2) \in \Gamma$ by several applications of $A1_l$, $A1_r$, $Mono_l$ and $Mono_r$. In this case $\neg\varphi \in \Delta$. Hence, Δ is a complete theory in $\mathbf{L}(C)$.

Let $\exists x\varphi \in \Delta$, that is, $(\ell = c_1;\exists x\varphi;\ell = c_2) \in \Gamma$. Then $\exists x(\ell = c_1;\varphi;\ell = c_2) \in \Gamma$ by B_r, B_l and $Mono_l$. Since Γ is a Henkin theory with witnesses in C, there exists a $c \in C$ such that $[c/x](\ell = c_1;\varphi;\ell = c_2) \in \Gamma$. This implies, that $[c/x]\varphi \in \Delta$.

Let $(\varphi_0\backslash\backslash^{t_{1,2}}_{t_{1,1}}\dots\backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n) \in \Delta$, that is, $(\ell = c_1;(\varphi_0\backslash\backslash^{t_{1,2}}_{t_{1,1}}\dots\backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n);\ell = c_2) \in \Gamma$. Then $(\varphi_0\backslash\backslash^{t_{1,2}+c_2}_{c_1+t_{1,1}}\dots\backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n) \in \Gamma$ by $A2$, $L2_\Rightarrow$, $Mono_r$ and $Mono_l$. Since Γ is a Henkin theory with witnesses in C, there exist $R_1,\dots,R_n \in C$ such that $\mathsf{proj}(\varphi_0, c_1 + t_{1,1}, t_{1,2} + c_2, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \in \Gamma$. Since

$$\neg\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \in \Delta$$

entails

$$\neg\mathsf{proj}(\varphi_0, c_1 + t_{1,1}, t_{1,2} + c_2, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \in \Gamma,$$

which is a contradiction by means of $A1_l$ and $A1_r$, and Δ is a complete theory in $\mathbf{L}(C)$, we have $\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n) \in \Delta$. This concludes the proof that Δ is a Henkin theory with witnesses in C. $\dashv$

Corollary 22 *Let $\Gamma_1, \Gamma_2 \subseteq \mathbf{L}(C)$. Let Γ be a complete Henkin theory in $\mathbf{L}(C)$ with witnesses in C. Let $\Gamma_1;\Gamma_2 \subset \Gamma$ and $c_1, c_2 \in C$ be such that $\ell = c_i \in \Gamma_i$, $i = 1, 2$. Then there exist two complete Henkin theories $\Gamma'_i \supseteq \Gamma_i$, $i = 1, 2$, in $\mathbf{L}(C)$ with witnesses in C, such that $\Gamma'_1;\Gamma'_2 \subset \Gamma$.*

Proof: Choose

$$\Gamma'_1 = \{\varphi \in \mathbf{L}(C) : (\varphi;\ell = c_2) \in \Gamma\} \text{ and } \Gamma'_2 = \{\varphi \in \mathbf{L}(C) : (\ell = c_1;\varphi) \in \Gamma\}.$$

$\dashv$

Lemma 23 *Let Γ be a complete Henkin theory in $\mathbf{L}(C)$ with witnesses in C. Let $R \in C$ be such that $\overline{R}^* \in \Gamma$ and Δ be $\{\varphi : (R\backslash\backslash\varphi) \in \Gamma\}$. Then Δ is a complete Henkin theory in $\mathbf{L}(C)$ with witnesses in C.*

Proof: Like in the proof of Lemma 21, we can show that $\Delta = Cn_{\mathbf{L}(C)}(\Delta)$, yet using Lemma 20 instead of Lemma 19.

Let $\varphi \in \mathbf{L}(C)$. Then either $(R\backslash\backslash\varphi) \in \Gamma$, or $\neg(R\backslash\backslash\varphi) \in \Gamma$, because Γ is complete. If $(R\backslash\backslash\varphi) \in \Gamma$, then $\varphi \in \Delta$. If $\neg(R\backslash\backslash\varphi) \in \Gamma$, then $(R\backslash\backslash\neg\varphi) \in \Gamma$, because $\overline{R}^* \in \Gamma$, $\overline{R}^* \Rightarrow ((R\backslash\backslash\neg\varphi) \Leftrightarrow (\neg\varphi)^R) \in \Gamma$ as an instance of $P3$, and $(\neg\varphi)^R$ is equivalent to $R^* \wedge \neg\varphi^R$ by the definition of $(.)^R$. Hence, Δ is a complete theory in $\mathbf{L}(C)$.

Consistency of Δ follows from $T1$ of Theorem 8. Let $\exists x\varphi \in \Delta$, that is, $(R\backslash\backslash\exists x\varphi) \in \Gamma$. Then $\exists x(R\backslash\backslash\varphi) \in \Gamma$, because $\overline{R}^* \Rightarrow ((R\backslash\backslash\exists x\varphi) \Leftrightarrow (\exists x\varphi)^R) \in \Gamma$ as a consequence of $P3$, and $(\exists x\varphi)^R$ is equivalent to $\exists x\varphi^R$ by the definition of $(.)^R$. Since Γ is a Henkin theory with witnesses in C, there exists a $c \in C$ such that $[c/x](R\backslash\backslash\varphi) \in \Gamma$. This implies, that $[c/x]\varphi \in \Delta$.

Let $(R\backslash\backslash(\varphi_0\backslash\backslash^{t_{1,2}}_{t_{1,1}} \dots \backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n)) \in \Gamma$. Then

$$(R\backslash\backslash^0_0\varphi_0\backslash\backslash^{t_{1,2}}_{t_{1,1}} \dots \backslash\backslash^{t_{n,2}}_{t_{n,1}}\varphi_n) \in \Gamma$$

by $A2$, $L3_l$, $L3_r$, $Mono_r$ and $Mono_l$. Since Γ is a Henkin theory with witnesses in C, there exist $R_0, \dots, R_n \in C$ such that

$$(\ell = 0; \overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \wedge \mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n)^{R_0}; \ell = 0) \in \Gamma,$$

Let P stand for $\mathsf{proj}(\varphi_0, t_{1,1}, t_{1,2}, R_1, \dots, t_{n,1}, t_{n,2}, R_n, \varphi_n)$ for the rest of the proof for the sake of brevity. Now $(\ell = 0; \overline{R_0}^* \wedge \Box(R_0 \Rightarrow R) \wedge P^{R_0}; \ell = 0) \in \Gamma$ implies $\overline{R_0}^*, \Box(R_0 \Rightarrow R), P^{R_0} \in \Gamma$ by $L3_l$ and $L3_r$. We need to prove that $(R\backslash\backslash P) \in \Gamma$. From $\overline{R_0}^* \in \Gamma$ and the consequence $\overline{R_0}^* \Rightarrow ((R_0\backslash\backslash P) \Leftrightarrow P^{R_0})$ of $P3$, we obtain $(R_0\backslash\backslash P) \in \Gamma$. From $\Box(R_0 \Rightarrow R), (R_0\backslash\backslash P) \in \Gamma$ we obtain $(R\backslash\backslash P) \in \Gamma$ by $T5$ of Theorem 8. Hence $P \in \Delta$. This concludes the proof that Δ is a Henkin theory with witnesses in C. $\dashv$

3.2 The canonical $ITL_{\backslash\backslash}$ model

In this section we carry out the actual construction of a canonical model for $ITL_{\backslash\backslash}$ to conclude our completeness argument. Given an $ITL_{\backslash\backslash}$ language $\mathbf{L}$, we start from a given complete Henkin theory Γ in an extension $\mathbf{L}(C)$ of $\mathbf{L}$ with witnesses in some set C of countably many rigid constants and countably many 0-ary flexible relation symbols, none of which is in the vocabulary of C. Every consistent set of $\mathbf{L}$ formulas can be extended to such a theory by Theorem 15.

3.2.1 The canonical frame

Let $c_1 \equiv c_2$ iff $c_1 = c_2 \in \Gamma$ for rigid constants $c_1, c_2 \in C$. Clearly, $\equiv$ is an equivalence relation. Given $c \in C$, we denote $\{c' \in C : c \equiv c'\}$ by $[c]$.

Our first step is to define the duration domain $\langle D, 0, +\rangle$ of the canonical frame. Let D be $\{[c] : c \in C\}$, $0 = \{c \in C : c = 0 \in \Gamma\}$, and let the binary operation $+$ be defined on D by the equality $[c_1] + [c_2] = \{c : c = c_1 + c_2 \in \Gamma\}$.

Proposition 24 *The above definition of $+$ is correct and $\langle D, 0, +\rangle$ is a duration domain.*

Proof: Standard argument in first order predicate logic Henkin style completeness proofs. Cf. e.g. [21]. $\dashv$

Our next step is to define the time domain $\langle T, \leq \rangle$. We represent time points as pairs $\langle [c'], [c''] \rangle$ such that $c' + c'' = \ell \in \Gamma$. Time points can be represented more economically as classes $[c']$ such that $c' \leq \ell \in \Gamma$ too, because the existence of a $c'' \in C$ such that $c' + c'' = \ell \in \Gamma$ can be derived using that Γ is a Henkin theory. Yet having both $[c']$ and $[c'']$ explicitly occurring in our representation makes it more convenient to describe the rest of our construction and carry out the relevant proofs. We define T as the set $\{\langle [c'], [c''] \rangle : c', c'' \in C, \ell = c' + c'' \in \Gamma\}$. We define the relation $\leq$ on T by the equivalence $\langle [c'_1], [c''_1] \rangle \leq \langle [c'_2], [c''_2] \rangle \leftrightarrow \exists x (c'_1 + x = c'_2) \in \Gamma$.

Proposition 25 *The above definition of $\leq$ is correct and $\langle T, \leq \rangle$ is a time domain.*

Proof: Direct check. $\dashv$

Note that $\ell = c_1 + c_2 \in \Gamma$ need not imply $(\ell = c_1; \ell = c_2) \in \Gamma$, because Γ itself may happen to be the theory of a discrete interval.

Given the time domain $\langle T, \leq \rangle$, an element of $\mathbf{I}(T)$ can be straightforwardly denoted by $[\langle [c'_1], [c''_1] \rangle, \langle [c'_2], [c''_2] \rangle]$. However, we prefer the more concise form $\langle [c'_1], [c''_2] \rangle$ for $[\langle [c'_1], [c''_1] \rangle, \langle [c'_2], [c''_2] \rangle]$, because the classes $[c'_1]$ and $[c''_2]$ are unambiguously determined by the conditions $\ell = c'_1 + c''_1, \ell = c'_2 + c''_2 \in \Gamma$. Hence, if $\exists x (c + x + d = \ell) \in \Gamma$, the pair $\langle [c], [d] \rangle$ can be used to denote the ordinary interval $[\langle [c], [c'] \rangle, \langle [d'], [d] \rangle]$, where $c', d' \in C$ are such that $c + c' = \ell, d' + d = \ell \in \Gamma$. In case $c + d = \ell \in \Gamma$, the pair $\langle [c], [d] \rangle$ denotes both the 0-length interval $[\langle [c], [d] \rangle, \langle [c], [d] \rangle]$ and the unique time point in this interval. The intended meaning will always be clear from the context.

Similarly, we concisely denote discrete intervals $\{\langle [c'_1], [c''_1] \rangle, \ldots, \langle [c'_n], [c''_n] \rangle\}$ where $\langle [c'_i], [c''_i] \rangle \leq \langle [c'_{i+1}], [c''_{i+1}] \rangle$, $i = 1, \ldots, n-1$, by $\langle [d_0], \ldots, [d_n] \rangle$, where the constants $d_i \in C$, $i = 1, \ldots, n$ are determined by the conditions

$$d_0 = c'_1 \in \Gamma,\ c'_i + d_i = c'_{i+1} \in \Gamma,\ i = 1, \ldots, n-1,\ d_n = c''_n \in \Gamma.$$

The conjunction of these conditions implies that $d_0 + \ldots + d_n = \ell \in \Gamma$. Every sequence $d_0, \ldots, d_n$ which satisfies the latter condition represents a unique discrete interval in T. Obviously the sequence $d_0, \ldots, d_n$ represents the same discrete interval as the sequence $d'_0, \ldots, d'_m$ iff $d_0 = d'_0, d_n = d'_m \in \Gamma$ and $d'_0, \ldots, d'_m$ can be obtained from $d_0, \ldots, d_n$ by deleting and/or inserting constants c such that $c = 0 \in \Gamma$. Of course, we can avoid this ambiguity by allowing only sequences $d_0, \ldots, d_n$ that satisfy $d_1 \neq 0, \ldots, d_{n-1} \neq 0 \in \Gamma$. However such a convention would only make our proofs look more complicated, because of the need to consider more special cases, in order to follow it. That is why we assume that, for example, $\langle [d_0], [0], [d_1], [d_2], [d_3] \rangle$ and $\langle [d_0], [d_1], [d_2], [0], [0], [d_3] \rangle$ stand for the same interval, which consists of three points iff $d_1 \neq 0, d_2 \neq 0 \in \Gamma$.

Given a discrete interval denoted by $\langle [d_0], \ldots, [d_n] \rangle$, the smallest ordinary interval which contains it can be denoted by $\langle [d_0], [d_n] \rangle$. For 0-length intervals, which are both ordinary and discrete, the above notation is $\langle [d_0], [d_1] \rangle$ where $d_0 + d_1 = \ell \in \Gamma$, according to both conventions.

Note that T itself is an ordinary interval, and it can be denoted by $\langle 0, 0 \rangle$. In our notation, $\mathbf{P}_{\mathit{fin}}(T)$ can be represented as $\{\langle [d_0], \ldots, [d_n] \rangle : 1 \leq n < \omega, d_0, \ldots, d_n \in C, \ell = d_0 + \ldots + d_n \in \Gamma\}$.

We define the function $m : \mathbf{I}_{\backslash\backslash}(T) \to D$ by putting

$$m(\langle [c_0], \ldots, [c_n] \rangle) = \{c \in C : \ell = c_0 + c + c_n \in \Gamma\}.$$

Note that this equality is meaningful for both discrete and ordinary intervals. In the latter case there is nothing in place of "...". For discrete intervals, one can easily find out that

$$m(\langle [c_0], [c_1], \ldots, [c_{n-1}], [c_n] \rangle) = \{c \in C : c = c_1 + \ldots + c_{n-1} \in \Gamma\}.$$

For example, $m(\langle [d_0], [d_1], [d_2], [d_3] \rangle) = \{c \in C : c = d_1 + d_2 \in \Gamma\}$.

Proposition 26 *The above definition of m is correct and m is a measure function on* $\mathbf{I}_{\backslash\backslash}(T)$.

Proof: Direct check, using the instances of the axiom $P\ell$ in Γ. $\dashv$

Propositions 24, 25 and 26 entail that

$$F = \langle \langle T, \leq \rangle, \langle D, 0, + \rangle, m \rangle$$

is an $ITL_{\backslash\backslash}$ frame.

3.2.2 The reference interval

Let

$$\sigma_0 = \{\langle [c_1], [c_2] \rangle : (\ell = c_1; \ell = c_2) \in \Gamma\}.$$

Proposition 27 $\sigma_0 \in \mathbf{I}_{\backslash\backslash}(T)$.

Proof: Since Γ is a complete theory, either $\ell = c_1 + c_2 \Leftrightarrow (\ell = c_1; \ell = c_2) \in \Gamma$ for all $c_1, c_2 \in C$, or $\delta^* \in \Gamma$. This follows from the instances of $L2_{\Leftarrow}$ and $L2_{\Rightarrow}$ in Γ by a purely-ITL argument.

Let $\delta^* \in \Gamma$. Then $\delta^k \in \Gamma$ for some $k < \omega$ by Lemma 16. Since Γ has witnesses in C, there exist $c_1, \ldots, c_k \in C$ such that $(\delta \wedge \ell = c_1; \ldots; \delta \wedge \ell = c_k) \in \Gamma$. This implies that $\sigma_0 = \langle 0, [c_1], \ldots, [c_n], 0 \rangle$.

Let $\ell = c_1 + c_2 \Leftrightarrow (\ell = c_1; \ell = c_2) \in \Gamma$ hold for all $c_1, c_2 \in C$. Then clearly $\sigma_0 = \langle 0, 0 \rangle = T$. $\dashv$

3.2.3 The canonical interpretation

Now let us construct an interpretation I of $\mathbf{L}(C)$ into F such that $\langle F, I \rangle, \sigma_0 \models \varphi$ for all $\varphi \in \Gamma$. To do this, we first associate a complete Henkin theory $\mu(\sigma)$ in $\mathbf{L}(C)$ with witnesses in C with every interval $\sigma \in \mathbf{I}_{\backslash\backslash}(T)$ such that $\sigma \subseteq \sigma_0$.

Definition 28 Let $\langle [c_1], [c_2] \rangle \in \mathbf{I}(T)$. Then we denote the set

$$\{\varphi \in \mathbf{L}(C) : (\ell = c_1; \varphi; \ell = c_2) \in \Gamma\}$$

by $\mu(\langle [c_1], [c_2] \rangle)$.

Definition 29 Let $\sigma \in \mathbf{P}_{fin}(T)$ and $\sigma = \langle [c_0], \dots, [c_n] \rangle$. Let $(\ell = c_0; \dots; \ell = c_n) \in \Gamma$. Let $R \in C$ be such that

$$(\ell = c_0; \overline{R}^* \wedge (\ell = c_1 \wedge \delta; \dots; \ell = c_{n-1} \wedge \delta)^R; \ell = c_n) \in \Gamma.$$

Then $\mu^R(\sigma)$ denotes the set $\{\varphi : (\ell = c_0; (R \backslash\backslash \varphi); \ell = c_n) \in \Gamma\}$.

The condition $(\ell = c_0; \dots; \ell = c_n) \in \Gamma$ in Definition 29 is not an immediate consequence of $\sigma \in \mathbf{P}_{fin}(T)$, because, for example, $\langle T, \leq \rangle$ may be dense, and σ_0 still may be discrete, which makes possible $\sigma \not\subseteq \sigma_0$. However, if this condition is satisfied, then $(\ell = c_0; (\top \backslash\backslash (\ell = c_1 \wedge \delta; \dots; \ell = c_{n-1} \wedge \delta)); \ell = c_n) \in \Gamma$ by $DI2$, and

$$(\ell = c_0; \overline{R}^* \wedge (\ell = c_1 \wedge \delta; \dots; \ell = c_{n-1} \wedge \delta)^R; \ell = c_n) \in \Gamma$$

is guaranteed to hold for some $R \in C$, because Γ is a Henkin theory with witnesses in C. Hence, for every discrete interval $\sigma \subseteq \sigma_0$ there exists an $R \in C$ such that $\mu^R(\sigma)$ is defined.

It can easily be established that, if the sequence $c'_0, \dots, c'_m$ of rigid constants in C can be obtained from the sequence $c_0, \dots, c_n$ by inserting and/or deleting rigid constants $c \in C$ such that $c = 0 \in \Gamma$, and $c_0 = c'_0, c_n = c'_m \in \Gamma$, then

$$\overline{R}^* \wedge (\ell = c_1 \wedge \delta; \dots; \ell = c_{n-1} \wedge \delta)^R \text{ and } \overline{R}^* \wedge (\ell = c'_1 \wedge \delta; \dots; \ell = c'_{n-1} \wedge \delta)^R$$

are equivalent. Hence, the definition of $\mu^R(\sigma)$ does not depend on the choice of the sequence $c_0, \dots, c_n$ used to represent σ.

The following proposition entails that $\mu^R(\sigma)$ does not depend on the choice of R and, since appropriate R are always available in C, we can define $\mu(\sigma)$ as $\mu^R(\sigma)$ for some arbitrarily chosen R with the required properties.

Proposition 30 *Let R_1 and R_2 both satisfy the requirements on R from Definition 29 for some discrete interval σ. Then $\mu^{R_1}(\sigma) = \mu^{R_2}(\sigma)$.*

Proof: Let $\sigma = \langle [c_0], \dots, [c_n] \rangle$. Since removing the elements c_i satisfying $c_i = 0 \in \Gamma$ from the sequence $c_1, \dots, c_{n-1}$ has no effect on $\mu^R(\sigma)$ for appropriate R, we can assume that $c_1 \neq 0, \dots, c_{n-1} \neq 0 \in \Gamma$ without loss of generality. Let P denote $(\ell = c_1 \wedge \delta; \dots; \ell = c_{n-1} \wedge \delta)$ for the sake of brevity. The definition of δ^R implies that

$$\vdash_{ITL_{\backslash\backslash}} c_i \neq 0 \wedge (\ell = c_i \wedge \delta)^R \Rightarrow \ell = c_i \wedge R \wedge \delta,\ i = 1, \dots, n-1.$$

That is why the formula

$$\bigwedge_{i=1}^{n-1} c_i \neq 0 \wedge \overline{R_1}^* \wedge P^{R_1} \wedge \overline{R_2}^* \wedge P^{R_2} \Rightarrow \Box(R_1 \Leftrightarrow R_2)$$

has an equivalent one which is valid in the ITL^*-subset of $ITL_{\backslash\backslash}$. This implies that the above formula is an $ITL_{\backslash\backslash}$ theorem. Hence $(\ell = c_0; \overline{R_i}^* \wedge P^{R_i}; \ell = c_n) \in \Gamma$, $i = 1, 2$, entails that $(\ell = c_0; \Box(R_1 \Leftrightarrow R_2); \ell = c_n) \in \Gamma$. This means that

$$(\ell = c_0; (R_1 \backslash\backslash \varphi); \ell = c_n) \in \Gamma \text{ and } (\ell = c_0; (R_2 \backslash\backslash \varphi); \ell = c_n) \in \Gamma$$

are equivalent for every $\varphi \in \mathbf{L}(C)$ by $T5$ of Theorem 8, $Mono_l$ and $Mono_r$. $\dashv$

The only remaining case in which $\mu(\sigma)$ is over-defined occurs if σ is both an ordinary and a discrete interval. This is possible for 0-length intervals σ, and in case the time domain $\langle T, \leq \rangle$ itself is discrete. The following proposition shows that the two definitions of μ agree in this case too:

Proposition 31 *Let $c_0, \ldots, c_n \in C$ be such that $c_1 \neq 0, \ldots, c_{n-1} \neq 0 \in \Gamma$ and*

$$(\ell = c_0; \ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta; \ell = c_n) \in \Gamma.$$

Let $R \in C$ be such that

$$(\ell = c_0; \overline{R}^* \wedge (\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)^R; \ell = c_n) \in \Gamma.$$

Then $\mu(\langle [c_0], [c_n] \rangle) = \mu^R(\langle [c_0], [c_1], \ldots, [c_{n-1}], [c_n] \rangle)$.

Proof: Both $\mu(\langle [c_0], [c_n] \rangle)$ and $\mu^R(\langle [c_0], [c_1], \ldots, [c_{n-1}], [c_n] \rangle)$ are complete theories by Lemmata 21 and 23. Hence it is sufficient to prove that $\mu^R(\langle [c_0], \ldots, [c_n] \rangle) \subseteq \mu(\langle [c_0], [c_n] \rangle)$.

Using that δ^R stands for $R^* \wedge \neg(\ell \neq 0 \wedge R^*; \ell \neq 0 \wedge R^*)$, we obtain

$$(\ell = c_0; \overline{R}^* \wedge (\ell = c_1 \wedge R; \ldots; \ell = c_{n-1} \wedge R); \ell = c_n) \in \Gamma$$

from $(\ell = c_0; \overline{R}^* \wedge (\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)^R; \ell = c_n) \in \Gamma$ by a purely-ITL^* deduction. Together with $(\ell = c_0; \ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta; \ell = c_n) \in \Gamma$, this implies $(\ell = c_0; \Box(R \Rightarrow \delta); \ell = c_n) \in \Gamma$ by a purely-ITL^* deduction again. Hence $\Box(R \Rightarrow \delta) \in \mu(\langle [c_0], [c_n] \rangle)$, and, given a φ such that $(R \backslash\backslash \varphi) \in \mu(\langle [c_0], [c_n] \rangle)$, $T5$ of Theorem 8 implies that we have $(\delta \backslash\backslash \varphi) \in \mu(\langle [c_0], [c_n] \rangle)$ too. Since $(\ell = c_0; \ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta; \ell = c_n) \in \Gamma$ implies $\delta^* \in \mu(\langle [c_0], [c_n] \rangle)$, $(\delta \backslash\backslash \varphi) \in \mu(\langle [c_0], [c_n] \rangle)$ implies $\varphi \in \mu(\langle [c_0], [c_n] \rangle)$ by $DI1$. This concludes the proof, because $\varphi \in \mu^R(\langle [c_0], \ldots, [c_n] \rangle)$ is equivalent to $(R \backslash\backslash \varphi) \in \mu(\langle [c_0], [c_n] \rangle)$ by the definition of $\mu^R(\langle [c_0], \ldots, [c_n] \rangle)$. $\dashv$

We define the canonical interpretation I of $\mathbf{L}(C)$ into the canonical frame F by the following (standard) clauses:

$I(x) = [c]$	if $x = c \in \Gamma$	for individual variables x
$I(d) = [c]$	if $d = c \in \Gamma$	for rigid constants d
$I(f)([c_1], \ldots, [c_n]) = [c]$	if $f(c_1, \ldots, c_n) = c \in \Gamma$	for n-ary rigid function symbols f
$I(R)([c_1], \ldots, [c_n]) = 1$	if $R(c_1, \ldots, c_n) \in \Gamma$	for n-ary rigid relation symbols R
$I(d)(\sigma) = [c]$	if $d = c \in \mu(\sigma)$	for flexible constants d
$I(f)(\sigma, [c_1], \ldots, [c_n]) = [c]$	if $f(c_1, \ldots, c_n) = c \in \mu(\sigma)$	for n-ary flexible function symbols f
$I(R)(\sigma, [c_1], \ldots, [c_n]) = 1$	if $R(c_1, \ldots, c_n) \in \mu(\sigma)$	for n-ary flexible relation symbols R

The above definitions for flexible symbols' interpretations apply only to $\sigma \subseteq \sigma_0$. The values of I on these symbols for other $\sigma \in \mathbf{I}_{\backslash\backslash}(T)$ are irrelevant to the properties of $\langle F, I\rangle$ that we need to establish. Besides, Γ provides no information on how to determine these values. That is why we leave them unspecified. The only exception is $I(\ell)$:

$I(\ell)(\sigma) = m(\sigma)$ for *all* $\sigma \in \mathbf{I}_{\backslash\backslash}(T)$.

This definition agrees with the clause about flexible constant symbols in general, which applies to ℓ too.

Clearly, $M = \langle F, I\rangle$ is a model for $\mathbf{L}(C)$.

3.2.4 The truth lemma

Now let us prove that $M, \sigma_0 \models \varphi$ iff $\varphi \in \Gamma$, for all $\varphi \in \mathbf{L}(C)$. To do this, we use the auxiliary propositions below.

Proposition 32 *Let $\sigma_1, \sigma_2 \subseteq \sigma_0$. Let either $\sigma_1, \sigma_2 \in \mathbf{I}(T)$ or $\sigma_1, \sigma_2 \in \mathbf{P}_{fin}(T)$. Let $\max \sigma_1 = \min \sigma_2$. Then $\mu(\sigma_1); \mu(\sigma_2) \subset \mu(\sigma_1; \sigma_2)$.*

Proof: We do the cases $\sigma_1, \sigma_2 \in \mathbf{I}(T)$ and $\sigma_1, \sigma_2 \in \mathbf{P}_{fin}(T)$ separately.

Let $\sigma_1, \sigma_2 \in \mathbf{I}(T)$. Then there exist $c_1', c_1'', c_2', c_2'' \in C$ such that

$$\mu(\sigma_i) = \{\varphi \in \mathbf{L}(C) : (\ell = c_i'; \varphi; \ell = c_i'') \in \Gamma\},\ i = 1, 2,$$

and $\max \sigma_1 = \min \sigma_2$ implies that $c_1' \leq c_2', c_2'' \leq c_1'', c_2' + c_1'' = \ell \in \Gamma$. It can be shown by an ITL deduction from this that

$$(\ell = c_1'; \varphi_1; \ell = c_1'') \wedge (\ell = c_2'; \varphi_2; \ell = c_2'') \Rightarrow (\ell = c_1'; \varphi_1; \varphi_2; \ell = c_2'') \in \Gamma$$

Since obviously $\mu(\sigma_1; \sigma_2) = \{\varphi \in \mathbf{L}(C) : (\ell = c_1'; \varphi; \ell = c_2'') \in \Gamma\}$, this entails that $\mu(\sigma_1); \mu(\sigma_2) \subset \mu(\sigma_1; \sigma_2)$.

Now let $\sigma_1, \sigma_2 \in \mathbf{P}_{fin}(T)$. Let $\sigma_1; \sigma_2 = \langle [c_0], \ldots, [c_n]\rangle$. Then obviously there exists a $k \in \{1, \ldots, n-1\}$ such that σ_1 and σ_2 can be represented as $\langle [c_0], \ldots, [c_k], [c']\rangle$ and $\langle [c''], [c_{k+1}], \ldots, [c_n]\rangle$, respectively, where $c', c'' \in C$ are such that

$$c'' = c_0 + \cdots + c_{k-1}, c' = c_0 + \cdots + c_k \in \Gamma.$$

Let $R \in C$ satisfy the requirements of Definition 29 with respect to $\sigma_1; \sigma_2$ and $\mu(\sigma_1; \sigma_2) = \mu^R(\sigma_1; \sigma_2)$. The above representations of σ_1 and σ_2 imply that R satisfies these requirements with respect to σ_1 and σ_2 too, and $\mu(\sigma_i) = \mu^R(\sigma_i)$, $i = 1, 2$. Just like in the previous case,

$$\left(\begin{array}{l} (\ell = c_0; (R\backslash\backslash\varphi_1) \wedge \overline{R}^*; \ell = c')\wedge \\ (\ell = c''; (R\backslash\backslash\varphi_2) \wedge \overline{R}^*; \ell = c_n) \end{array}\right) \Rightarrow (\ell = c_0; (R\backslash\backslash\varphi_1) \wedge \overline{R}^*; (R\backslash\backslash\varphi_2) \wedge \overline{R}^*; \ell = c_n)$$

is a member of Γ. Besides, the choice of R entails that $(\ell = c_0; \overline{R}^*; \ell = c_n) \in \Gamma$. Hence, it is sufficient to show that

$$\vdash_{ITL_{\backslash\backslash}} \big((R\backslash\backslash\varphi_1) \wedge \overline{R}^*; (R\backslash\backslash\varphi_2) \wedge \overline{R}^*\big) \wedge \overline{R}^* \Rightarrow (R\backslash\backslash(\varphi_1; \varphi_2)).$$

It follows from the deduction below:

1	$(R\backslash\backslash\varphi_i) \wedge \overline{R}^* \Rightarrow \varphi_i^R \wedge R^*$	$P3$, ITL^*, $i = 1, 2$
2	$((R\backslash\backslash\varphi_1) \wedge \overline{R}^*; (R\backslash\backslash\varphi_2) \wedge \overline{R}^*) \Rightarrow (\varphi_1; \varphi_2)^R$	1, $Mono_l$, $Mono_r$
3	$\overline{R}^* \Rightarrow (R \Rightarrow R)^R$	ITL
4	$(\varphi_1; \varphi_2)^R \wedge (R \Rightarrow R)^R \wedge \overline{R}^* \Rightarrow (R\backslash\backslash(\varphi_1; \varphi_2))$	$P1$
5	$\big((R\backslash\backslash\varphi_1) \wedge \overline{R}^*; (R\backslash\backslash\varphi_2) \wedge \overline{R}^*\big) \wedge \overline{R}^* \Rightarrow (R\backslash\backslash(\varphi_1; \varphi_2))$	2-4, PC

This concludes the proof. $\dashv$

The role of the following proposition is analogous to that of Proposition 32, yet with respect to $(.\backslash\backslash.)$ instead of $(.;.)$.

Proposition 33 *Let $\sigma, \sigma' \in \mathbf{I}_{\backslash\backslash}(T)$, $\sigma' = \langle [c_0], \ldots, [c_n] \rangle$ and $\sigma' \subseteq \sigma \subseteq \langle [c_0], [c_n] \rangle$. Let $R \in C$ be such that $\overline{R}^*, (\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)^R \in \mu(\sigma)$. Then $\{\varphi : (R\backslash\backslash\varphi) \in \mu(\sigma)\} = \mu(\sigma)$.*

Proof: We do the cases $\sigma \in \mathbf{I}(T)$ and $\sigma \in \mathbf{P}_{fin}(T)$ separately.

In case $\sigma \in \mathbf{I}(T)$, $\sigma = \langle [c_0], [c_n] \rangle$, because $\min \sigma' = \min \langle [c_0], [c_n] \rangle$, $\max \sigma' = \max \langle [c_0], [c_n] \rangle$ and $\sigma' \subseteq \sigma \subseteq \langle [c_0], [c_n] \rangle$. In this case the proposition follows immediately from the definitions of $\mu^R(\sigma')$ and $\mu(\sigma)$.

Now let $\sigma \in \mathbf{P}_{fin}(T)$. Let P denote the formula $(\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)$ for the sake of brevity. Let $\sigma = \langle [d_0], \ldots, [d_m] \rangle$. Then $c_0 \in [d_0]$ and $c_n \in [d_m]$, and $\mu(\sigma) = \mu^{R'}(\sigma)$ for some $R' \in C$ such that

$$(\ell = d_0; \overline{R'}^* \wedge (\ell = d_1 \wedge \delta; \ldots; \ell = d_{m-1} \wedge \delta)^{R'}; \ell = d_m) \in \Gamma.$$

Similarly, let $\mu(\sigma') = \mu^{R''}(\sigma')$ for some $R'' \in C$ such that

$$(\ell = d_0; \overline{R''}^* \wedge P^{R''}; \ell = d_m) \in \Gamma.$$

Let Δ denote the theory $\mu(\langle [c_0], [c_n] \rangle)$ of the ordinary interval $\langle [c_0], [c_n] \rangle$, which underlies σ, for the sake of brevity. Now $P^R, \overline{R}^* \in \mu(\sigma)$ entails that $(R'\backslash\backslash P^R \wedge \overline{R}^*) \in \Delta$, whence, using that $\overline{R'}^* \in \Delta$, we obtain $(R'\backslash\backslash P^R), (R'\backslash\backslash\overline{R}^*) \in \Delta$ by $T4$ of Theorem 8. Similarly, using that

$$\vdash_{ITL_{\backslash\backslash}} P^R \wedge \overline{R}^* \Rightarrow (R\backslash\backslash P)$$

follows from $P3$, whence

$$\vdash_{ITL_{\backslash\backslash}} \overline{R'}^* \Rightarrow (R'\backslash\backslash P^R \wedge \overline{R}^* \Rightarrow (R\backslash\backslash P))$$

by PN and the definition of $\overline{R'}^*$, we obtain $(R'\backslash\backslash(R\backslash\backslash P)) \in \Delta$. This implies $((R'\backslash\backslash R)\backslash\backslash P) \in \Delta$ by $P4$. Furthermore, $(R'\backslash\backslash\overline{R}^*) \in \Delta$ implies $\overline{(R'\backslash\backslash R)}^* \in \Delta$ by $T7$ of Lemma 18, whence $P^{(R'\backslash\backslash R)} \in \Delta$ by $P3$ again. Now notice that

$$\overline{(R'\backslash\backslash R)}^* \wedge \overline{R''}^* \wedge P^{(R'\backslash\backslash R)} \wedge P^{R''} \Rightarrow \Box((R'\backslash\backslash R) \Leftrightarrow R'')$$

is valid in ITL^* and therefore an $ITL_{\backslash\backslash}$ theorem. Hence $\Box((R'\backslash\backslash R) \Leftrightarrow R'') \in \Delta$. This entails that $(R'\backslash\backslash(R\backslash\backslash\varphi)) \in \Delta$ and $(R''\backslash\backslash\varphi) \in \Delta$ are equivalent for all $\varphi \in \mathbf{L}(C)$ by $T5$ of Theorem 8, $P3$ and $P4$. This concludes the proof, because $\varphi \in \mu(\sigma')$ is equivalent to $(R''\backslash\backslash\varphi) \in \Delta$ by the choice of R''. $\dashv$

Proposition 34 *Let $\sigma \in \mathbf{I}_{\backslash\backslash}(T)$ and Γ_1, Γ_2 be complete Henkin theories in $\mathbf{L}(C)$ with witnesses in C and $\Gamma_1; \Gamma_2 \subset \mu(\sigma)$. Then there exists a $\tau \in \sigma$ such that $\Gamma_1 = \mu(\sigma \cap [\min \sigma, \tau])$ and $\Gamma_2 = \mu(\sigma \cap [\tau, \max \sigma])$.*

Proof: Let $\sigma = \langle [c_0], \ldots, [c_n] \rangle$ for some $c_0, \ldots, c_n \in C$, if $\sigma \in \mathbf{P}_{fin}(T)$. Let $\sigma = \langle [c_0], [c_n] \rangle$ otherwise. Let $c', c'', d_1, d_2 \in C$ be such that $\ell = d_i \in \Gamma_i$, $i = 1, 2$, and $c' = c_0 + d_1, c' = d_2 + c''$. Then a direct check shows that τ can be chosen to be $\langle [c'], [c''] \rangle$. $\dashv$

Theorem 35 (Truth lemma) *Let $t, \varphi \in \mathbf{L}(C)$, $\sigma \in \mathbf{I}_{\backslash\backslash}(T)$ and $\sigma \subseteq \sigma_0$. Then $I_\sigma(t) = [c]$ iff $t = c \in \mu(\sigma)$, and $M, \sigma \models \varphi$ iff $\varphi \in \mu(\sigma)$.*

Proof: Induction on the construction of terms t and formulas φ. We omit the details about terms, because they are standard. The equivalence for atomic formulas φ is an immediate consequence of the definition of $\mu(\sigma)$. Inductive steps which correspond to propositional connectives and $\exists$ are trivial.

Let φ be $(\varphi_1; \varphi_2)$. Then $\varphi \in \mu(\sigma)$ implies $(\varphi_1 \wedge \ell = d_1; \varphi_2 \wedge \ell = d_2) \in \mu(\sigma)$ for some $d_1, d_2 \in C$, because $\vdash_{ITL_{\backslash\backslash}} (\varphi_1; \varphi_2) \Rightarrow \exists x \exists y (\varphi_1 \wedge \ell = x; \varphi_2 \wedge \ell = y)$, and $\mu(\sigma)$ is a Henkin theory with witnesses in C. Now Corollary 22 implies that there are two complete theories Henkin theories Γ_1 and Γ_2 with witnesses in C such that $\varphi_i, \ell = d_i \in \Gamma_i$, $i = 1, 2$, and $\Gamma_1; \Gamma_2 \subset \mu(\sigma)$. Hence there exist a $\tau \in \sigma$ and $\sigma_1, \sigma_2 \subseteq \sigma$ such that $\sigma_1 = [\min T, \tau] \cap \sigma$, $\sigma_2 = [\tau, \max T] \cap \sigma$, $\Gamma_1 = \mu(\sigma_1)$ and $\Gamma_2 = \mu(\sigma_2)$ by Proposition 34. This is equivalent to $M, \sigma_i \models \varphi_i$, $i = 1, 2$, by the induction hypothesis, whence $M, \sigma \models \varphi$. For the converse implication, note that $M, \sigma \models \varphi$ implies $M, \sigma_i \models \varphi_i$, $i = 1, 2$, for some σ_1, σ_2 such that $\sigma_1; \sigma_2 = \sigma$, whence $\varphi_i \in \mu(\sigma_i)$, $i = 1, 2$, by the induction hypothesis. This implies $\varphi \in \mu(\sigma)$, because $\mu(\sigma_1); \mu(\sigma_2) \subset \mu(\sigma)$ by Proposition 32.

The inductive step is similar about φ being φ_1^*, because $\varphi \in \mu(\sigma)$ is equivalent to the existence of a $k < \omega$ such that $\varphi_1^k \in \mu(\sigma)$ by Lemma 16, and $M, \sigma \models \varphi$ is equivalent to the existence of a $k < \omega$ such that $M, \sigma \models \varphi^k$ by the definition of $\models$.

Let φ be $(\varphi_1 \backslash\backslash \varphi_2)$. We consider the cases $\ell = 0 \in \mu(\sigma)$ and $\ell \neq 0 \in \mu(\sigma)$ separately.

Let $\ell = 0 \in \mu(\sigma)$. Then $m(\sigma) = 0$ by the definition of m, and $\varphi_2 \in \mu(\sigma)$ by $T3$ of Theorem 8, whence, $M, \sigma \models \varphi_2$ by the induction hypothesis. This entails $M, \sigma \models \varphi$. The converse implication follows from $T3$ in a similarly straightforward way.

Let $\ell \neq 0 \in \mu(\sigma)$. Since $\mu(\sigma)$ is a Henkin theory, $\varphi \in \mu(\sigma)$ implies that there exists an $R \in C$ such that $\overline{R}^*, \Box(R \Rightarrow \varphi_1), \varphi_2^R \in \mu(\sigma)$. This entails that $(R \backslash\backslash \varphi_2) \in \mu(\sigma)$ by $P3$. $\overline{R}^* \in \mu(\sigma)$ implies that there is a unique $k < \omega$, $k \neq 0$ such that $R^k \in \mu(\sigma)$. Let $c_1, \ldots, c_k \in C$ be such that $(\ell = c_1 \wedge R; \ldots; \ell = c_k \wedge R) \in \mu(\sigma)$. Such constants exist, because $\mu(\sigma)$ is a Henkin theory with witnesses in C. Hence $(\ell = c_1; \ldots; \ell = c_k) \in \mu(\sigma)$ too. The definition of $\overline{R}^*$ implies that $c_1 \neq 0, \ldots, c_{k-1} \neq 0 \in \Gamma$.

We can choose the constants $c_0, c_{k+1} \in C$ so that $\sigma = \langle [c_0], [c_{k+1}] \rangle$, in case $\sigma \in \mathbf{I}(T)$, and $\sigma = \langle [c_0], [c_1'], \ldots, [c_l'], [c_{k+1}] \rangle$, in case $\sigma \in \mathbf{P}_{fin}(T)$ for some $c_1', \ldots, c_l' \in C$. Let $\sigma' = \langle [c_0], [c_1], \ldots, [c_k], [c_{k+1}] \rangle$. Then $(\ell = c_1; \ldots; \ell = c_k) \in \mu(\sigma)$ implies

$\sigma \supseteq \sigma'$ and $\min\sigma = \min\sigma'$, $\max\sigma = \max\sigma'$. Let us prove that $\varphi_2 \in \mu(\sigma')$ by finding an $R' \in C$ such that σ, σ' and R' satisfy the conditions of Proposition 33.

$DI2$ and $(\ell = c_1; \ldots; \ell = c_k) \in \mu(\sigma)$ imply that $(\top\backslash\backslash(\ell = c_1 \wedge \delta; \ldots; \ell = c_k \wedge \delta)) \in \mu(\sigma)$. Then there exists an $R' \in C$ such that $\overline{R'}^*, (\ell = c_1 \wedge \delta; \ldots; \ell = c_k \wedge \delta)^{R'} \in \mu(\sigma)$. Since $(\ell = c_i \wedge \delta)^{R'}$ stands for

$$\ell = c_i \wedge (R')^* \wedge \neg((R')^* \wedge \ell \neq 0; (R')^* \wedge \ell \neq 0)$$

by the definitions of δ, $(.)^{R'}$ and $P\ell$, it can be established by a purely ITL^* deduction that

$$\vdash_{ITL_{\backslash\backslash}} c_i \neq 0 \Rightarrow ((\ell = c_i \wedge \delta)^{R'} \Rightarrow \ell = c_i \wedge R')$$

where $i = 1, \ldots, k$. This implies that $(\ell = c_1 \wedge R'; \ldots; \ell = c_k \wedge R') \in \mu(\sigma)$. Now, using that

$$\vdash_{ITL_{\backslash\backslash}} \bigwedge_{i=1}^{k} c_i \neq 0 \wedge \overline{R}^* \wedge \overline{R'}^* \wedge \begin{pmatrix} (\ell = c_1 \wedge R; \ldots; \ell = c_k \wedge R) \wedge \\ (\ell = c_1 \wedge R'; \ldots; \ell = c_k \wedge R') \end{pmatrix} \Rightarrow \Box(R \Leftrightarrow R'),$$

which can be established by a purely ITL^* deduction too, we obtain $\Box(R \Leftrightarrow R') \in \mu(\sigma)$, which implies $\Box(R' \Rightarrow \varphi_1) \in \mu(\sigma)$, and, furthermore, $(R'\backslash\backslash\varphi_2) \in \mu(\sigma)$ by $T5$ of Theorem 8. Hence, $\varphi_2 \in \mu(\sigma')$ by Proposition 33.

This implies $M, \sigma' \models \varphi_2$ by the inductive hypothesis. Similarly, since

$$(\ell = c_1 \wedge R'; \ldots; \ell = c_k \wedge R') \in \mu(\sigma),$$

the subintervals $\sigma_i = \langle [c_0 + \ldots + c_{i-1}], [c_{i+1} + \ldots + c_{k+1}] \rangle \cap \sigma$ of σ satisfy $M, \sigma_i \models R'$, and, consequently, $M, \sigma_i \models \varphi_1$, because $\Box(R' \Rightarrow \varphi_1) \in \mu(\sigma)$, for $i = 1, \ldots, k$. This can be demonstrated in detail by repeating the inductive step about $(.;.)$-formulas which appears in this proof. Hence $M, \sigma \models (\varphi_1\backslash\backslash\varphi_2)$.

Now let us prove that $M, \sigma \models \varphi$ implies $\varphi \in \mu(\sigma)$. $M, \sigma \models \varphi$ implies that there exist $c_0, \ldots, c_n \in C$ such that $M, \sigma \models (\varphi_1 \wedge \ell = c_1; \ldots; \varphi_1 \wedge \ell = c_{n-1})$ and $\sigma' = \langle [c_0], \ldots, [c_n] \rangle$ satisfies $\sigma' \subseteq \sigma$, $\min\sigma' = \min\sigma$, $\max\sigma' = \max\sigma$, and $M, \sigma' \models \varphi_2$. By the induction hypothesis, $\varphi_2 \in \mu(\sigma')$. Since $(\ell = c_1; \ldots; \ell = c_{n-1}) \in \mu(\sigma)$, we have $(\top\backslash\backslash(\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)) \in \mu(\sigma)$ by $DI2$. Hence there exists an $R \in C$ such that

$$\overline{R}^*, (\ell = c_1 \wedge \delta; \ldots; \ell = c_{n-1} \wedge \delta)^R \in \mu(\sigma).$$

This, together with $(\varphi_1 \wedge \ell = c_1; \ldots; \varphi_1 \wedge \ell = c_{n-1}) \in \mu(\sigma)$, which follows from $M, \sigma \models (\varphi_1 \wedge \ell = c_1; \ldots; \varphi_1 \wedge \ell = c_{n-1})$ by a repetition of the inductive step about $(.;.)$-formulas which appears in this proof, entails that $\Box(R \Rightarrow \varphi_1) \in \mu(\sigma)$ by a purely-ITL^* deduction. Proposition 33 implies that $(R\backslash\backslash\varphi_2) \in \mu(\sigma)$. Hence $(\varphi_1\backslash\backslash\varphi_2) \in \mu(\sigma)$ follows from $\Box(R \Rightarrow \varphi_1) \in \mu(\sigma)$ by $T5$ of Theorem 8. $\dashv$

3.2.5 The completeness theorem

Now we are ready to prove our completeness theorem about $ITL_{\backslash\backslash}$.

Theorem 36 (ω-completeness of $ITL_{\backslash\backslash}$) *Let Γ_0 be a consistent set of formulas in the $ITL_{\backslash\backslash}$ language* **L**. *Then there exists a model M for* **L** *and an interval σ in its time domain such that $M, \sigma \models \varphi$ for all $\varphi \in \Gamma_0$.*

Proof: Let the complete Henkin theory Γ considered above be an extension of Γ_0. This choice is possible due to Theorem 15. Then M can be chosen to be the canonical model for Γ built above. We have $M, \sigma_0 \models \Gamma_0$ by Theorem 35. $\dashv$

4 Related work

4.1 Moszkowski-style ITL

The original projection operator $\triangle$ introduced in [16, 17] is a special case of the operator $(.\backslash\backslash.)$ studied here. The system of ITL presented in these works is based on discrete time. That is, only the frame $F_{\mathbf{Z}}$ is considered. To distinguish the original form of ITL, as introduced in Moszkowski's works [16, 17], from the abstract time variant studied here, we call it *Moszkowski-style* ITL in this section.

In the majority of the works on Moszkowski-style ITL, flexible non-logical symbols' interpretations are assumed to depend on the beginning of the reference interval only. An exception to this is the early work [12]. This assumption means that

$(At)\ \varphi \Leftrightarrow \neg(\neg\varphi; \top)$

is valid for atomic φ in Moszkowski-style ITL. This restriction has a crucial effect on the complexity of the system. For example, propositional ITL (only 0-ary flexible relation symbols and no function symbols, nor constant symbols, not even ℓ) is not decidable in the form adopted in this paper. Yet, under the assumption that At is valid about atomic formulas, propositional discrete time ITL is decidable. Another consequence of At is that $\models \varphi$ does not imply $\models [\psi/P]\varphi$, where P stands for a 0-ary flexible relation symbol, in Moszkowski-style ITL.

Furthermore, projection is definable under the assumption At in Moszkowski-style propositional ITL: every propositional ITL formula with projection is equivalent to one without projection (in a normal form.) This enabled the demonstration of the completeness of a proof system for Moszkowski-style propositional ITL with projection and the establishment of its decidability by a tableau-based procedure in [1].

Moszkowski-style ITL with projection can be embedded into abstract time ITL with projection in the following way. Let 1 be a rigid constant and $\#$ be a flexible constant in the considered $ITL_{\backslash\backslash}$ language **L**. Consider the following axioms about $\#$ and 1:

$(1)\ 0 \neq 1$

$(\#_1)\ \ell = 0 \Rightarrow \# = 1$

$(\#_2)\ (\# = x; \ell \neq 0 \wedge \delta) \Rightarrow \# = x + 1$

$(\#_3)\ \delta^* \vee \# = 0$

The validity of these axioms in a model for **L** is equivalent to the semantical condition on $I(\#)(\sigma)$ to be equal to the number of *isolated* points of the reference interval σ for all intervals σ in the model. The flexible constant $\#$ was introduced in [11] by this semantical condition. In Moszkowski-style ITL, $\#$ is always equal to $\ell + 1$. Yet in the more general situation introduced in [11] and studied here $\#$ cannot be defined using ℓ

only. Using $\#$, we can establish the following correspondence between Moszkowski-style ITL validity and provability in our system for $ITL_{\backslash\backslash}$:

> Let φ be an $ITL_{\backslash\backslash}$ formula. Then φ is valid in Moszkowski-style ITL iff
>
> $$(\ell = 1)^* \Rightarrow (\ell = 1 \vee \ell = 0 \backslash\backslash [\# - 1/\ell]\varphi)$$
>
> is provable in the extension of our proof system by the axioms At about atomic formulas, the axiom (1) about the constant 1 and the axioms $\#_1$-$\#_3$ about $\#$.

4.2 Translating $ITL_{\backslash\backslash}$ into ITL with discrete propositional variables and quantification over them

An earlier result on the axiomatisation of projection in ITL was established in [7]. In that work ITL was extended by so-called *discrete* 0-ary relation symbols to represent discrete $ITL_{\backslash\backslash}$ intervals and quantification over them. The system thus obtained is called ITL^D.

ITL^D languages contain a countable set of distinguished flexible 0-ary relation symbols p, q, $\ldots$, which are called *discrete propositional variables*. The following restriction is imposed on the interpretations of discrete propositional variables:

> Every interval may contain at most finitely many subintervals which satisfy a given discrete propositional variable, and these subintervals should be 0-length ones.

This condition enables the use of discrete propositions to represent $ITL_{\backslash\backslash}$ discrete intervals in a straightforward way. In order to represent the dependency of the interpretations of flexible symbols on the internal points of discrete intervals, ITL^D also allows flexible symbols to take one formula argument. In the translation of $ITL_{\backslash\backslash}$ into ITL^D below this argument is always the discrete propositional variable which represents the reference interval.

The BNFs for terms and formulas in ITL^D are extended to allow the new kind of arguments of flexible symbols. Furthermore, the BNF for formulas in ITL^D allows $\exists$ to bind discrete propositional variables:

$t ::= c \mid f(t, \ldots, t) \mid f(\varphi, t, \ldots, t)$

$\varphi ::= \bot \mid R(t, \ldots, t) \mid R(\varphi, t, \ldots, t) \mid \varphi \Rightarrow \varphi \mid (\varphi; \varphi) \mid (\varphi \backslash\backslash \varphi) \mid \exists x\varphi \mid \exists p\varphi$

The interpretations of flexible symbols which take a formula argument in ITL^D models have the following types:

- $I(f) : \mathbf{I}(T) \times 2^{\mathbf{I}(T)} \times D^n \to D$, for function symbols f that take n term arguments;
- $I(R) : \mathbf{I}(T) \times 2^{\mathbf{I}(T)} \times D^n \to \{0, 1\}$, for relation symbols R that take n term arguments.

The interpretations of other symbols are as in ITL. Let $M = \langle F, I\rangle$ be an ITL^D model. Let $\tilde{I}(\varphi)$ stand for $\{\sigma \in \mathbf{I}(T_F) : M, \sigma \models \varphi\}$. The clauses for the inductive definitions of term values and $\models$ for the new kind of terms and atomic formulas are as follows:

$I_\sigma(f(\varphi, t_1, \ldots, t_n)) = I(f)(\sigma, \tilde{I}(\varphi) \cap 2^\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_n))$

$M, \sigma \models R(\varphi, t_1, \ldots, t_n)$ iff $I(R)(\sigma, \tilde{I}(\varphi) \cap 2^\sigma, I_\sigma(t_1), \ldots, I_\sigma(t_n)) = 1$

The intersections with 2^σ here guarantee that only the truth values of the formula argument at subintervals of the reference interval σ can influence the (truth) values of terms and formulas where this formula argument occurs. Quantification over discrete propositional variables is defined in the ordinary way:

$\langle F, I\rangle, \sigma \models \exists p\varphi$ iff $\langle F, J\rangle, \sigma \models \varphi$ for some J which p-agrees with I.

Given an $ITL_{\backslash\backslash}$ language $\mathbf{L}$, the corresponding ITL^D language $\mathbf{L}'$ is defined as follows: $\mathbf{L}'$ has the same rigid symbols as $\mathbf{L}$ and, of course, ℓ. For every flexible constant c in $\mathbf{L}$, except ℓ, there is a unary flexible function symbol c in $\mathbf{L}'$. Similarly, for every n-ary function (relation) symbol f (R) in $\mathbf{L}$ there is an $n+1$-ary function (relation) symbol f (R) in $\mathbf{L}'$, which takes one formula argument and n term arguments. Furthermore, $\mathbf{L}'$ contains a countable set $\{p_i : 1 \leq i < \omega\}$ of discrete propositional variables, none of which occurs in $\mathbf{L}$.

Let $\mathbf{L}$ be an $ITL_{\backslash\backslash}$ language and $\mathbf{L}'$ be its corresponding ITL^D language. Let

$\Diamond_i\varphi \rightleftharpoons \Diamond(\varphi \wedge \ell \neq 0) \vee (\ell \neq 0; \varphi; \ell \neq 0)$, $\Box_i\varphi \rightleftharpoons \neg\Diamond_i\neg\varphi$

An interval σ satisfies $\Diamond_i\varphi$ iff φ holds at some subinterval of σ which is different from the 0-length intervals $[\min\sigma, \min\sigma]$ and $[\min\sigma, \min\sigma]$ that are at the beginning and at the end of σ. In the translation from $\mathbf{L}$ to $\mathbf{L}'$ below we abbreviate $\ell = 0$ by p_0:

$$
\begin{array}{lcl}
c^{p_i} & \rightleftharpoons & c \text{ for rigid constants } c \\
x^{p_i} & \rightleftharpoons & x \text{ for individual variables } x \\
(s(t_1, \ldots, t_n))^{p_i} & \rightleftharpoons & s(t_1^{p_i}, \ldots, t_n^{p_i}) \text{ for rigid } n\text{-ary symbols } s \\
\ell^{p_i} & \rightleftharpoons & \ell \\
c^{p_i} & \rightleftharpoons & c(p_i) \text{ for other flexible constants } c \\
(s(t_1, \ldots, t_n))^{p_i} & \rightleftharpoons & s(p_i, t_1^{p_i}, \ldots, t_n^{p_i}) \text{ for flexible } n\text{-ary symbols } s \\
\bot^{p_i} & \rightleftharpoons & \bot \\
(\varphi \Rightarrow \psi)^{p_i} & \rightleftharpoons & \varphi^{p_i} \Rightarrow \psi^{p_i} \\
(\varphi; \psi)^{p_i} & \rightleftharpoons & (\varphi^{p_i}; p_i; \psi^{p_i}) \\
(\psi\backslash\backslash\varphi)^{p_i} & \rightleftharpoons & \exists p_{i+1} \left(\begin{array}{l} \Box(p_{i+1} \Rightarrow p_i) \wedge \\ \Box((p_{i+1}; \ell \neq 0 \wedge \Box_i \neg p_{i+1}; p_{i+1}) \Rightarrow \psi^{p_i}) \wedge \\ (p_{i+1}; \varphi^{p_{i+1}}; p_{i+1}) \end{array} \right) \\
(\exists x\varphi)^{p_i} & \rightleftharpoons & \exists x\varphi^{p_i}
\end{array}
$$

No special clause about iteration is needed in this translation, because it is expressible by projection:

$(\varphi^*)^{p_i} \rightleftharpoons (\varphi\backslash\backslash\top)^{p_i}$.

The intended meaning of φ^{p_i} is to express the truth value of φ at the (discrete) interval consisting of the time points in the ordinary reference interval which, if regarded as 0-length intervals, satisfy p_i. Since p_0 stands for $\ell = 0$, and this formula holds at every 0-length interval, p_0 always represents the reference interval itself. That is why if φ is projection-free, then φ^{p_0} is equivalent to φ.

Let p_i represent some possibly discrete interval σ in the above way. Then the formula for $(\varphi; \psi)^{p_i}$ states that chopping can be done at the points of σ only. The formula for $(\psi\backslash\backslash\varphi)^{p_i}$ states that there exists a discrete interval σ' represented by p_{i+1} which is required to have the properties expressed by the conjunction in the scope of $\exists p_{i+1}$. The members of this conjunction state that σ' is a subset of σ, every two adjacent points of σ' define a subinterval of σ' which satisfies ψ, and σ' has the same end points as σ and satisfies φ, respectively. The following proposition gives the precise formulation of the intended meaning of the translation:

Proposition 37 *Let F be an $ITL_{\backslash\backslash}$ frame (which is the same as an ITL^D and an ITL frame). Let $M = \langle F, I\rangle$ and $M' = \langle F, I'\rangle$ be models for $\mathbf{L}$ and $\mathbf{L}'$, respectively. Let I and I' coincide on rigid symbols from $\mathbf{L}$ and $\mathbf{L}'$. Let*

$$I'(s)([\min\sigma, \max\sigma], \{[\tau,\tau] : \tau \in \sigma\}, d_1, \ldots, d_n) = I(s)(\sigma, d_1, \ldots, d_n)$$

where $\sigma \in \mathbf{I}_{\backslash\backslash}(T_F)$ and $d_1, \ldots, d_n \in D_F$ for other flexible symbols s. Let t be a term and φ be a formula from $\mathbf{L}$.

- *Let $\sigma \in \mathbf{I}(T_F)$. Then $I_\sigma(t) = I'_\sigma(t^{\ell=0})$, and $M, \sigma \models \varphi$ iff $M', \sigma \models \varphi^{\ell=0}$.*
- *Let $\sigma \in \mathbf{P}_{fin}(T_F)$. Let $\{\tau \in [\min\sigma, \max\sigma] : I(p_1)([\tau,\tau]) = 1\} = \sigma$. Then $I_\sigma(t) = I'_\sigma(t^{p_1})$, and $M, \sigma \models \varphi$ iff $M', [\min\sigma, \max\sigma] \models \varphi^{p_1}$.*

Proof: Induction on the construction of t and φ. $\dashv$

This proposition entails that a formula φ is valid at the ordinary intervals of all $ITL_{\backslash\backslash}$ models for the language of φ iff $\varphi^{\ell=0}$ is valid in ITL^D and φ is valid at all the discrete intervals of $ITL_{\backslash\backslash}$ models iff $\forall p_1((p_1; \top; p_1) \Rightarrow \varphi^{p_1})$ is valid in ITL^D.

The proof system for ITL^D is obtained by adding the following axioms and rules to the proof system for ITL:

Extensionality axioms about flexible symbols with formula arguments

$(D_=)$ $\quad \Box(\varphi \Leftrightarrow \psi) \wedge x_1 = y_1 \wedge \ldots \wedge x_n = y_n \Rightarrow f(\varphi, x_1, \ldots, x_n) = f(\psi, y_1, \ldots, y_n)$
$(D_\Leftrightarrow)$ $\quad \Box(\varphi \Leftrightarrow \psi) \wedge x_1 = y_1 \wedge \ldots \wedge x_n = y_n \Rightarrow (R(\varphi, x_1, \ldots, x_n) \Leftrightarrow R(\psi, y_1, \ldots, y_n))$

Axioms and rules about discrete propositional variables

(D_0) $\quad p \Rightarrow \ell = 0$
(S_0) $\quad \exists p \Box \neg p$
(S_1) $\quad x \leq \ell \Rightarrow \exists p \forall y((\ell = y; p; \top) \Rightarrow y = x)$
$(S\vee)$ $\quad \exists r \Box(r \Leftrightarrow p \vee q)$
$(\exists^D)$ $\quad [q/p]\varphi \Rightarrow \exists p \varphi$

(G^D) $\quad \dfrac{[q/p]\varphi}{\forall p \varphi}$

(ω^D) $\quad \dfrac{\forall k < \omega\ [(\Box_i \neg p)^k / P]\varphi}{[\top/P]\varphi}$

This proof system is ω-complete for ITL^D [7].

There is a straightforward connection between the ITL^D discrete propositional variables p and the 0-ary flexible relation symbols R involved in our direct axiomatisation of $ITL_{\backslash\backslash}$. Namely, if p is interpreted as a discrete propositional variable, then $(p; \top; p) \Rightarrow \overline{(p; \ell \neq 0 \wedge \Box_i \neg p; p)}^*$ is valid in the corresponding model.

Although the way valid $ITL_{\backslash\backslash}$ formulas can be "proven" using the above translation and the proof system for ITL^D is indirect, this approach has some advantages over the direct axiomatisation. Given that ITL semantics has been enriched with discrete intervals, projection is only one of the many discrete interval-related modal operators which may happen to have meaning in applications. It is clear that other related operators can be handled by extending the above translation with clauses which encode their definitions.

In particular, note that $ITL_{\backslash\backslash}$ does not provide the possibility to access the underlying ordinary interval from a discrete reference interval. An operator $[.]$ to enable this can be defined as follows:

$M, \sigma \models [\varphi]$ iff $M, [\min \sigma, \max \sigma] \models \varphi$

That is, $[\varphi]$ holds at some, possibly discrete, interval, if φ holds at the ordinary interval which has the same endpoints as the given one. For example,

$(\varphi \backslash\backslash [\psi]) \Leftrightarrow \varphi^* \wedge [\psi]$ and $[\varphi] \Rightarrow \varphi \vee \delta^*$

are valid formulas, according to the proposed definition of $[.]$.

The clause about the new operator in our translation is

$[\varphi]^{p_i} \rightleftharpoons \varphi^{p_0}$,

i.e., from the (possibly discrete) interval, which is represented by p_i, "return" to the underlying ordinary interval, which is represented by $p_0 \rightleftharpoons \ell = 0$.

5 Projection in DC

The Duration Calculus (DC) [24] is probably the most interesting extension of ITL. Its language extends that of ITL by allowing *state expressions* which are boolean formulas built using a distinguished set of non-logical symbols called *state variables*. State variables and, consequently, state expressions, are interpreted as $\{0, 1\}$-valued functions of time. These functions are required to be piecewise constant. This restriction is known as *finite variability* of state in DC. State expressions S participate in DC formulas by forming *duration terms* $\int S$. Given an ITL model $M = \langle F, I \rangle$, we denote the value of the function represented by state expression S at time τ by $I_\tau(S)$. Given an interval σ in the time domain of F and a partition $\sigma_1, \ldots, \sigma_n$ of σ such that $I_\tau(S)$ is constant in every interval of the kind $[\min \sigma_i, \max \sigma_i)$, the value $I_\sigma(\int S)$ of the term $\int S$ at σ is defined by the equality

$$I_\sigma(\textstyle\int S) = \sum_{i=1,\ldots,n \;\; I_{\min \sigma_i}(S)=1} m(\sigma_i),$$

where $m(\sigma)$ stands for the measure, or the *duration* of σ, as in Definition 1. Clearly, this definition does not depend on the choice of $\sigma_1, \ldots, \sigma_n$, provided that $\sigma = \sigma_1; \ldots; \sigma_n$, and $I_\tau(S)$ is constant on $[\min \sigma_i, \max \sigma_i)$, $i = 1, \ldots, n$.

The following abbreviations are frequently used in DC:

$\mathbf{0}$	$\rightleftharpoons$	$P \wedge \neg P$,	for some arbitrarily chosen state variable P
$\mathbf{1}$	$\rightleftharpoons$	$\neg \mathbf{0}$	
$\lceil S \rceil$	$\rightleftharpoons$	$\int S = \ell \wedge \ell \neq 0$	

DC has been primarily studied with respect to its real-time frame $F_{\mathbf{R}}$. A relatively complete proof system for DC with respect to this frame was presented in [13]. A comprehensive introduction to DC is given in [14]. An ω-complete proof system for DC on the class of all ITL frames can be found in [6].

There is no established way of interpreting duration terms at $ITL_{\backslash\backslash}$ discrete intervals. One reasonable way is to refer to the value of the duration term at the underlying ordinary interval, by putting $I_\sigma(\int S) = I_{[\min \sigma, \max \sigma]}(\int S)$. This complies with the possibility to define ℓ in DC by putting $\ell \rightleftharpoons \int (P \vee \neg P)$. The following axiom can be used to characterise this property:

$(P\int)$ $\varphi^* \wedge \int S = x \Leftrightarrow (\varphi \backslash\backslash \int S = x)$

Furthermore, since iteration is available in our proof system, finite variability of state can be straightforwardly characterised by the axiom

(FV) $(\int S = \ell \vee \int(\neg S) = \ell)^* \vee \delta^*$

The subformula δ^* of this axiom accounts for the possibility of a discrete interval not to contain all the time points where the interpretation of S changes its value.

The extension of the proof system for $ITL_{\backslash\backslash}$ by the axioms $P\int$, FV and $DC0$-$DC7$ below is ω-complete for DC with projection, as introduced here.

$(DC0)$ $\ell = 0 \Rightarrow \int S = 0$

$(DC1)$ $\int \mathbf{0} = 0$

$(DC2)$ $\lceil \mathbf{1} \rceil \vee \ell = 0$

$(DC3)$ $(\int S = x; \lceil S \rceil \wedge \ell = y) \Rightarrow \int S = x + y$

$(DC4)$ $(\int S = x; \lceil \neg S \rceil) \Rightarrow \int S = x$

$(DC5)$ $\lceil S_1 \rceil \wedge \lceil S_2 \rceil \Leftrightarrow \lceil S_1 \wedge S_2 \rceil$

$(DC6)$ $\lceil S_1 \rceil \Leftrightarrow \lceil S_2 \rceil$ if $\models S_1 \Leftrightarrow S_2$ in propositional calculus.

$(DC7)$ $\lceil S \rceil \Rightarrow \Box(\lceil S \rceil \vee \ell = 0)$

A proof can be obtained easily by following the example about $ITL_{\backslash\backslash}$ given here and the ω-completeness argument about DC in [6].

Conclusion

We have presented an ω-complete proof system for the extension of first order ITL by projection and carried out the completeness argument about this system in the well-known framework provided by Henkin constructions. We have also briefly presented an alternative approach to the axiomatisation of $ITL_{\backslash\backslash}$, which employs a truth-preserving translation of $ITL_{\backslash\backslash}$ formulas into formulas of a (simpler) completely axiomatised extension of ITL. This approach gives an indirect solution to the axiomatisation of ITL, but provides a convenient way to handle other discrete interval related operators along with projection. Our approach to ω-axiomatisation applies to DC, which the best known and most widely applied extension of ITL.

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