

Error Estimates of Four Level Conservative Finite Difference Schemes for Multidimensional Boussinesq Equations

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Introduction

We study the Cauchy problem for

Boussinesq Equation (BE)

$$\frac{\partial^2 u}{\partial t^2} = \Delta u + \beta_1 \Delta \frac{\partial^2 u}{\partial t^2} - \beta_2 \Delta^2 u + \Delta f(u), \quad x \in \mathbb{R}^n, \quad t > 0,$$

$$u(x, 0) = u_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = u_1(x),$$

on the unbounded region \mathbb{R}^n with asymptotic boundary conditions

$$u(x, t) \rightarrow 0, \quad \Delta u(x, t) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty,$$

where Δ is the Laplace operator, β_1 and β_2 are positive constants.

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where Δ is the Laplace operator, β_1 and β_2 are positive constants.

This is a 4-th order differential equation in x and 2-nd order in t with non-linearity contained in the term $f(u)$.

f is a polynomial of u . Examples: $f(u) = \alpha u^2$; $f(u) = au^3 + bu^5$.

We assume that the functions u_0 , u_1 and $f(u)$ satisfy such regularity conditions that BE has a unique solution which is smooth enough.

Numerical methods for BE, [references](#), [1D problem](#):

- finite difference method (Ortega, Sanz Serna, 1990);
- finite element methods (Pani, 1997);
- spectral method with Christov functions (Christou, 2010);
- Godunov-type central-upwind scheme (Chertock, Christov, Kurganov, 2011);
- predictor- corrector scheme (Bratsos, 2009);
- [2D problem](#): 3 level FDS- theoretical analysis, numerical implementation (Kolkovska, Christov, Vasileva, Dimova, 2010-2012);
- vector additive schemes (multicomponent alternating direction method) (Kolkovska, Angelow, 2013).

Properties to the Boussinesq equation

Let $\|\cdot\|$ denote the standard norm in $L_2(R^n)$.

Define the energy functional

$$E(u(t)) = \left\| (-\Delta)^{-1/2} \frac{\partial u}{\partial t} \right\|^2 + \beta_1 \left\| \frac{\partial u}{\partial t} \right\|^2 + \|u\|^2 + \beta_2 \|\nabla u\|^2 + \int_{R^n} F(u) du$$

with

$$F(u) = \int_0^u f(s) ds$$

Theorem (Conservation law)

The solution u to Boussinesq problem satisfies the following energy identity

$$E(u(t)) = E(u(0)) \quad \forall t \in [0, T].$$

We obtain similar energy identities for the solutions of the FDS employed in the discretization of BE.

Notations for case $n = 2$:

- Domain $\Omega = [-L_1, L_1] \times [-L_2, L_2]$, L_1, L_2 – sufficiently large;
- a uniform mesh with steps h_1, h_2 in Ω :
 $x_i = ih_1, i = -M_1, M_1; y_j = jh_2, j = -M_2, M_2;$
- τ - the time step, $t_k = k\tau, k = 0, 1, 2, \dots;$
- mesh points $(x_i, y_j, t_k);$
- $v_{(i,j)}^{(k)}$ denotes the discrete approximation $u(x_i, y_j, t_k);$
- notations for some discrete derivatives of mesh functions:
 - $v_{t,(i,j)}^{(k)} = (v_{(i,j)}^{(k+1)} - v_{(i,j)}^{(k)})/\tau;$
 - $v_{\bar{x}x,(i,j)}^{(k)} = (v_{(i+1,j)}^{(k)} - 2v_{(i,j)}^{(k)} + v_{(i-1,j)}^{(k)})/h_1^2;$
 - $v_{\bar{t}t,(i,j)}^{(k)} = (v_{(i,j)}^{(k+1)} - 2v_{(i,j)}^{(k)} + v_{(i,j)}^{(k-1)})/\tau^2;$
 - $\Delta_h v = v_{\bar{x}x} + v_{\bar{y}y}$ – the 5-point discrete Laplacian.
 - $(\Delta_h)^2 v = v_{\bar{x}x\bar{x}x} + v_{\bar{y}y\bar{y}y} + 2v_{\bar{x}x\bar{y}y}$ – the discrete biLaplacian

The second time derivative at the time level $t^k + \tau/2$ is approximated with error $O(\tau^2)$ using four consecutive time levels $(k+2)$, $(k+1)$, (k) and $(k-1)$ as

$$v_{\hat{t}\hat{t}}^{(k)} = 0.5(v^{(k+2)} - v^{(k+1)} - v^{(k)} + v^{(k-1)})\tau^{-2}.$$

For the approximation of $\Delta_h v$ and $(\Delta_h)^2 v$ we introduce two symmetric approximations to $u(\cdot, t^k + \tau/2)$ with real parameters θ and μ :

$$v^{\theta(k)} = \theta v^{(k+2)} + (0.5 - \theta)v^{(k+1)} + (0.5 - \theta)v^{(k)} + \theta v^{(k-1)},$$
$$v^{\mu(k)} = \mu v^{(k+2)} + (0.5 - \mu)v^{(k+1)} + (0.5 - \mu)v^{(k)} + \mu v^{(k-1)}$$

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$$\begin{aligned}v^{\theta(k)} &= \theta v^{(k+2)} + (0.5 - \theta)v^{(k+1)} + (0.5 - \theta)v^{(k)} + \theta v^{(k-1)}, \\v^{\mu(k)} &= \mu v^{(k+2)} + (0.5 - \mu)v^{(k+1)} + (0.5 - \mu)v^{(k)} + \mu v^{(k-1)}\end{aligned}$$

For the approximation of non-linear term we use

$$\frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}}.$$

Note that function $f(v)$ is a polynomial of v , thus the integrals $F(v)$ could be explicitly evaluated!

Four level FDS:

$$(I - \beta_1 \Delta_h)(v^{(k+2)} - v^{(k+1)} - v^{(k)} + v^{(k-1)}) / (2\tau^2) - \Delta_h v^{\theta(k)} + \beta_2 (\Delta_h)^2 v^{\mu(k)} = \Delta_h \frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}}$$

Here I stands for the identity operator.

Initial values $v^{(0)}$, $v^{(1)}$ and $v^{(-1)}$ on time levels $t = 0$, $t = \tau$ and $t = -\tau$ are evaluated by formulas

$$v_{i,j}^{(0)} = u_0(x_i, y_j),$$

$$v_{i,j}^{(1)} = u_0(x_i, y_j) + \tau u_1(x_i, y_j)$$

$$+ 0.5\tau^2 (I - \beta_1 \Delta_h)^{-1} (\Delta_h u_0 - \beta_2 (\Delta_h)^2 u_0 + \Delta_h f(u_0)) (x_i, y_j),$$

$$v_{tt(i,j)}^{(0)} = \left(v_{(i,j)}^{(1)} - 2v_{(i,j)}^{(0)} + v_{(i,j)}^{(-1)} \right) \tau^{-2}$$

$$= (I - \beta_1 \Delta_h)^{-1} (\Delta_h u_0 - \beta_2 \Delta_h^2 u_0 + \Delta_h f(u_0)) (x_i, y_j).$$

Discrete conservation law

Consider the space of functions, which vanish on the boundary of Ω_h , with the scalar product

$$\langle v, w \rangle = \sum_{i,j} h_1 h_2 v_{(i,j)}^{(k)} w_{(i,j)}^{(k)};$$

We define operators

$$A = -\Delta_h$$

$$B = I - \beta_1 \Delta_h - 2\tau^2 \theta \Delta_h + 2\tau^2 \beta_2 \mu (\Delta_h)^2$$

A , B - self-adjoint and positive definite operators for $\theta \geq 0$ and $\mu \geq 0$

We introduce the linear functional $E_{h,L}(v^{(k)})$ as

$$E_{h,L}(v^{(k)}) = 0.5 \left\langle A^{-1} B v_t^{(k)}, v_t^{(k-1)} \right\rangle + 0.5 \left\langle v^{(k)} + \beta_2 A v^{(k)}, v^{(k)} \right\rangle$$

and the full discrete energy functional $E_h(v^{(k)})$ as

$$E_h(v^{(k)}) = E_{h,L}(v^{(k)}) + \left\langle F(v^{(k)}), 1 \right\rangle.$$

Theorem (Discrete conservation law)

The solution to the considered FDS satisfies the energy equalities

$$E_h(v^{(k)}) = E_h(v^{(0)}), \quad k = 1, 2, \dots$$

i.e. the discrete energy is conserved in time.

Our calculations confirm that the discrete energy functional $E_h(v^{(k)})$ is preserved in time with a high accuracy (for $t \in (0, 20]$ - with 10^{-8} error)

Theorem (Convergence of the method)

Assume that: f is a polynomial of u ; $u \in C^{4,4}(\mathbb{R}^2 \times [0, T])$;

(i) parameters θ and μ satisfy the operator inequality

$$A^{-1} + \beta_1 I + \tau^2(2\theta - 0.5)I + \tau^2\beta_2(2\mu - 0.5)A > \epsilon I, \quad \epsilon > 0$$

with some positive real number ϵ independent on h, τ, u ;

(ii) the discrete solution v is bounded in the maximal norm.

Let $M \geq \max_{i,j,k \leq N} (|u(x_i, y_j, t_k)|, |v_{i,j}^{(k)}|)$ and $\tau < C_1 M^{-1}$.

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Then the discrete solution v converges to the exact solution u as $|h|, \tau \rightarrow 0$ and there is a constant C (independent of h, τ and u) such that the following estimate holds for the error $z = u - v$:

$$\begin{aligned} \epsilon \|z_t^{(k)}\|^2 + 0.25 \|z^{(k)} + z^{(k+1)}\|^2 + 0.25\beta_2 \|A^{1/2}(z^{(k)} + z^{(k+1)})\|^2 \\ \leq C e^{Mt^k} (|h|^2 + \tau^2)^2. \end{aligned}$$

Restrictions on parameters θ and μ for validity of condition (i) in the convergence Theorem:

- if $\theta \geq 0.25$ and $\mu \geq 0.25$ then no restrictions on h and τ are needed;
- if $\mu = 0$ and $\beta_1 \neq 0$ then $\tau^2 < h^2 \frac{\beta_1 - \epsilon + \tau^2(2\theta - 0.5)}{2n\beta_2}$;
- if $\mu = 0$ and $\beta_1 = 0$ then $\tau^2 < h^4 \frac{1}{n(4n\beta_2 + h^2(1 - 4\theta))}$.

Here $n = 1, 2$ is the dimension.

Combining convergence Theorem with the embedding theorems we get error estimates in the uniform norm:

Corollary

Under the assumptions of the main Theorem the 4 level FDS admits the following **error estimate in the uniform norm**:

$$\max_i |z_i^{(k)} + z_i^{(k+1)}| \leq Ce^{Mt^k} (|h|^2 + \tau^2), n = 1;$$

$$\max_{i,j} |z_i^{(k)} + z_i^{(k+1)}| \leq Ce^{Mt^k} \sqrt{\ln(\max\{N_1, N_2\})} (|h|^2 + \tau^2), n = 2.$$

The above estimates are optimal for the 1D case and *almost* optimal (up to a logarithmic factor) for the 2D case.

Numerical algorithm

1. Evaluate $v^{(0)}$, $v^{(1)}$, $v^{(-1)}$ from the initial conditions;
2. For $k = 0, 1, 2, \dots$ do ($v^{(k-1)}$, $v^{(k)}$, $v^{(k+1)}$ are known):

$$\begin{aligned}
 & (I - \beta_1 \Delta_h)(v^{(k+2)}) / (2\tau^2) - \theta \Delta_h v^{(k+2)} + \mu \beta_2 \Delta_h^2 v^{(k+2)} \\
 &= \Delta_h \frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}} - (0.5 - \mu) \beta_2 (\Delta_h)^2 (v^{(k+1)} + v^{(k)}) \\
 & - \mu \beta_2 \Delta_h^2 v^{(k-1)} + (0.5 - \theta) \Delta_h (v^{(k+1)} + v^{(k)}) + \theta \Delta_h v^{(k-1)} \\
 & + (I - \beta_1 \Delta_h)(v^{(k+1)} + v^{(k)} - v^{(k-1)}) / (2\tau^2)
 \end{aligned}$$

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$$\begin{aligned} & (I - \beta_1 \Delta_h)(v^{(k+2)}) / (2\tau^2) - \theta \Delta_h v^{(k+2)} + \mu \beta_2 \Delta_h^2 v^{(k+2)} \\ &= \Delta_h \frac{F(v^{(k+1)}) - F(v^{(k)})}{v^{(k+1)} - v^{(k)}} - (0.5 - \mu) \beta_2 (\Delta_h)^2 (v^{(k+1)} + v^{(k)}) \\ & - \mu \beta_2 \Delta_h^2 v^{(k-1)} + (0.5 - \theta) \Delta_h (v^{(k+1)} + v^{(k)}) + \theta \Delta_h v^{(k-1)} \\ & + (I - \beta_1 \Delta_h)(v^{(k+1)} + v^{(k)} - v^{(k-1)}) / (2\tau^2) \end{aligned}$$

Remarks:

if $\mu \neq 0$ - 4-th order elliptic equation for $v^{(k+2)} \Rightarrow$ choose $\mu = 0!$
for $\mu = 0$ - second order elliptic equation for $v^{(k+2)}$ - the numerical method is efficient!

No inner iterations are needed for evaluation of $v^{(k+2)}$.

Despite this fact, this method is **conservative!**

Preliminaries

- An analytical solution of the 1D equation (**one solitary wave**):

$$u(x, t; x_0, c) = \frac{3}{2} \frac{c^2 - 1}{\alpha} \operatorname{sech}^2 \left(\frac{x - x_0 - ct}{2} \sqrt{\frac{c^2 - 1}{\beta_1 c^2 - \beta_2}} \right),$$

where x_0 is the initial position of the peak of the solitary wave,

- Parameters: $\alpha = 3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, c is the wave speed.
- Initial conditions for **one solitary wave** or **two solitary waves**:

$$u(x, 0) = u(x, 0; -40, 2) + u(x, 0; 50, -1.5)$$

$$\frac{du}{dt}(x, 0) = u(x, 0; -40, 2)_t + u(x, 0; 50, -1.5)_t$$

- schemes with $\mu = 0$ and several θ : $\theta = 0.25$, $\theta = 0.5$, $\theta = 0$.

One solitary wave

Errors in uniform norm and rate of convergence for
 $t \in [0, 20]$, $\theta = 0.5$

h	$c=2$		$c=0.5$	
	Error	Rate	Error	Rate
0.1	0.0011424		0.0094145	
0.05	0.00028569	1.99954544	0.0022174	2.08601543
0.025	7.1534 e-005	1.99783019	0.0005475	2.01793817
0.0125	1.9402 e-005	1.88234306	0.0001359	2.01031351

- $\tau = h\sqrt{(\beta_1/(8\beta_2))}$, $\epsilon = 0.5\beta_1$, $\tau^2 < 0.5\beta_1$
- The error is the difference between the calculated and the exact solution in uniform norm for $t = 20$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.

One solitary wave, Different parameters θ Errors in uniform norm for $t \in [0, 40]$, $c = 2$

h	$\theta = 0.5$	$\theta = 0.25$	$\theta = 0$	$Error_{0.5}/Error_0$
0.2	0.0253530	0.0128760	0.0047763	5.3080837
0.1	0.0063896	0.0032394	0.0011790	5.4195081
0.05	0.0015989	0.0008109	0.0002938	5.4423227
0.025	0.0003999	0.0002029	7.3306e-05	5.4554879
0.0125	0.0001014	5.252e-05	1.678e-05	6.0446961

- The error is the difference between the calculated and the exact solution in uniform norm for $t = 40$.
- For one solitary wave the scheme with $\theta = 0$ is 5 to 6 times more precise than the scheme with $\theta = 0.5$!

Interaction of two solitary waves with different speeds

Errors in uniform norm and rate of convergence for $t \in [0, 40]$

h	$\theta = 0.5$		$\theta = 0$	
	error	rate	error	rate
0.08				
0.04	0.00231463		0.00034355	
0.02	0.00057865	2.00002063	8.55658155e-005	2.31582697
0.01	0.00013966	2.05076806	1.71856875e-005	2.00541487

- For every h the error is calculated by Runge method as $E_1^2 / (E_1 - E_2)$ with $E_1 = \|u_{[h]} - u_{[h/2]}\|$, $E_2 = \|u_{[h/2]} - u_{[h/4]}\|$, where $u_{[h]}$ is the calculated solution with step h for $t = 40$.
- The numerical rate of convergence is $(\log E_1 - \log E_2) / \log 2$.
- The calculations confirm the schemes are of order $O(h^2 + \tau^2)$.
- For two solitary waves the scheme with $\theta = 0$ is 6 to 7 times more precise than the scheme with $\theta = 0.5$!

Comparison with a 3-level conservative scheme

Errors in uniform norm for one and two solitary waves

	1 soliton, T=40			2 solitons, T=80	
	4-level	3-level		4-layer	3-level
h	$\theta = 0.25$	Cons.FDS	h	$\theta = 0.25$	Cons.FDS
0.2	0.01288	0.14412	0.2		
0.1	0.00324	0.03753	0.1	0.04019	
0.05	0.00081	0.00948	0.05	0.01907	0.102754
0.025	0.00020	0.00238	0.025	0.009212	0.026027
0.0125	5.25e-05	0.00059	0.0125	0.004010	0.006528

- for one solitary wave: the 4-level FDS is approximately 10 times more precise than the 3-level FDS;
- for two solitary waves: the 4-level FDS is approximately 2 times more precise than the 3-level FDS.

With respect to the error magnitude the 'new' four-level scheme performs much better than the 'old' three-level schemes!

Justification: Consider both FDS. We expand all terms in Taylor series about the point $(x_i, t^{(k)} + \tau/2)$ or (x_i, t^k) and get for the leading terms

$$R_{4-lev} = \frac{1}{8} \alpha \Delta_h \frac{\partial f}{\partial u}(x_i, t^{(k)} + \tau/2) \frac{\partial^2 u}{\partial t^2}(x_i, t^{(k)} + \tau/2),$$
$$R_{3-lev} = \frac{1}{4} \alpha \Delta_h \frac{\partial f}{\partial u}(x_i, t^k) \frac{\partial^2 u}{\partial t^2}(x_i, t^k).$$

Thus, $R_{3-lev} \approx 2 * R_{4-lev}$. This has essential impact on the total error, when the solution has large derivatives!

Summary:

- We develop a four level FDS for BE.
- The 4-level FDS is conservative, i.e. the discrete energy of the numerical solution is preserved in time.
- Error estimates in the uniform norm and in the first Sobolev norm are obtained.
- For $\mu = 0$ the numerical algorithm for evaluation of the discrete solution is efficient.
- The numerical experiments show good agreement with the theoretical results for 1D problem.

Thank you
for your attention!