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ON SOME SOURCES OF COMBINATORIAL PROBLEMS

Agnis Andžāns

This paper considers some approaches to obtaining (original) contest problems. Some of the most promising areas in combinatorics are listed.

1. Introduction. The number of mathematical competitions in the world is increasing rapidly. There are several reasons for this:

a) new states become engaged in olympiad movement,

b) regional contests are emerging,

c) non-traditional forms, such as team competitions, mathematical battles etc., become popular,

d) competitions are organized also for younger pupils, sometimes even from 2^{nd}-3^{rd} grades.

This process creates a demand for more and more problems. In the age of Internet, of course, it is not hard to use problems from other competitions, and this is often done. Nevertheless, such "cheating" can solve the problem neither in general, nor in a long run for any individual contest.

In this note we describe some of the approaches used in Latvia to cope with this, mainly in the area of combinatorics.

2. The use of problem classification. The classification of contest problems and the methods of their solution is elaborated. The problems of Latvian competitions as well as of other ones are distributed into "cells" accordingly to this classification. Then the "cells" which are not very "rich" are paid special attention to.

Let's describe the main parameters of the classification.

At first, problems are divided according to the area of mathematics: algebra, geometry, number theory, combinatorics or a combination of these.

Another parameter is the algorithmic/ deductive character of possible solutions.

The accessibility for students of various grades according to Latvian school curricula is also taken into account.

A very important parameter is the possibility to use one or some of general methods in the solution. We are especially interested in the appearances of the following methods:

a) mathematical induction,

b) the method of invariants/ semi-invariants,

c) the method of extremal element,

d) the mean value method (Pigeonhole principle and its generalizations),

e) the method of interpretation, when a problem is replaced by isomorphic one (very often "translated into a language of another area of mathematics").

In our opinion, the large majority of contest problems can be solved (and even the large majority of research in areas of combinatorics and theoretical computer science is performed) by clever use of these methods, their extensions and combinations. So, the acquaintance of students with them seems to be one of the most important tasks in advanced education.

Of course, extensive subdivisions are exploited.

Let's mention some examples of "rich" and "poor" cells.

An example of a rich cell

Branch of mathematics: geometry

Topic: incidence

Subtopic: concurrence of circumferences

General methods: none

Special methods: complex numbers

Algorithmic/deductive: deductive

An example of a poor cell

Use all the previous characteristics, only "mathematical induction" mentioned as general method. Only some theorems, e.g., Clifford theorems, Simson lines of polygons etc., fit within this category. So the topics of this cell are very suitable for independent investigations of high school students and teachers.

At this moment we are trying to arrange a corresponding electronic database.

3. Introducing twists into traditional types of problems. This approach seems to be very promising. It is more convenient to illustrate it with examples of algorithmic problems from sorting/searching area.

A great number of problems are scattered over various competitions where something must be determined using "yes-no" questions. By default, usually all answers are supposed to be correct and obtained immediately, and the correctness of the result must be guaranteed.

Let's consider some variations of this topic.

Problem (Team competition "Baltic Way '03"; A. Ambainis). *It is known that n is a positive integer, $n \leq 144$. Ten questions of the type "Is n smaller than a?" are allowed. Answers are given with a delay: an answer to the i-th question is given only after the $(i+1)^{\mathrm{st}}$ question is asked, $i = 1, 2, \ldots, 9$. The answer to the 10^{th} question is given immediately after it is asked. Find a strategy for identifying n.*

Solution. Let's denote the Fibonacci numbers as F_1=1, F_2=1, F_3=2, F_4=3, $\ldots$, F_{12}=144. We will consider two types of situations: "N" denotes that we know for sure that n is one of N consecutive integers (and we know these integers); "N?M" denotes that we know for sure that n is one of N+M consecutive integers (and we know these integers), and the question mark separating these N+M numbers into intervals of length N resp. M is a set with an answer unknown so far.

Clearly, the initial situation is "F_{12}".

Theorem. *There exists a strategy which guarantees that after setting i questions and receiving answers to the first $(i-1)$ of them $(i = 1, 2, \ldots, 9)$ we get one of the following situations: "$\mathrm{F}_{12-\mathrm{i}}$"; "$\mathrm{F}_{11-\mathrm{i}}?\mathrm{F}_{12-\mathrm{i}}$"; "$\mathrm{F}_{12-\mathrm{i}}?\mathrm{F}_{11-\mathrm{i}}$".*

The proof is by a straightforward induction, the next question dividing the segment of length F_{12-i} into two segments of length F_{11-i} and F_{10-i}, the longest of them being situated at one or another end of the whole "segment of hypotheses".

So after setting 9 questions we get one of the following situations (hypothetical numbers are denoted by $\bigcirc$): "$\bigcirc\bigcirc$"; "$\bigcirc?\bigcirc\bigcirc$"; "$\bigcirc\bigcirc?\bigcirc$". It is clear that with the next, the 10^{th} question, "separating" the still unseparated hypotheses, we will find n.

Comment. It can be proved that this "Fibonacci strategy" is the optimal one.

A very useful idea in combinatorial problems is to consider a situation when the result must be obtained only with a certain probability. Let's consider an example.

Problem (Latvian summer competition 2001; A. Andžāns). *There are* 4 *equally looking coins; all of them have different masses. We can use a pan balance without counterfeits. Develop an algorithm which uses a pan balance twice and finds the heaviest coin with a probability* $\frac{3}{4}$.

Solution. At first, using any generator of random numbers (for example, throwing the fair coin twice), decide which coin will be called "red"; other coins will be called "blue". After that find the heaviest blue coin deterministically within two weightings in a standard way. Announce this coin the heaviest among all four.

Clearly there is a probability $\frac{3}{4}$ that the heaviest coin (among all four) will be blue. Then it will be announced the heaviest, QED.

Comment. This problem demonstrates the advantage of "clever" probabilistic algorithm over both deterministic algorithms and pure guessing. It can be easily proved that the task can not be completed deterministically. Of course, simple guessing gives the right answer only with a probability $\frac{1}{4}$.

The next example demonstrates the possibility of passing from deterministic algorithms to nondeterministic ones.

Problem (All-Union Olympiad 1973; R. Freivalds). *There are* 14 *equally looking coins. The experts have established that* 7 *of them are exact and* 7 *of them are false. The court knows only that all exact coins have equal masses, all false coins have equal masses and an exact coin is heavier than the false one. How can expert demonstrate to the court which coins are exact and which are false using only* 3 *weightings on a pan balance without counterfeits?*

Solution. At first the expert places one exact coin on the left pan and one false coin on the other. The court becomes aware "which is which" of these coins. The expert adds two exact coins to the false one and two false coins to the exact one – and the court again becomes aware "which is which". Then the expert gathers 3 coins "proved to be exact" on one pan and adds 4 coins "unproved to be false" to them; other 7 coins are placed on the other pan. It's not hard to understand that all should be clear to the court after this.

Comment. This problem has great educational value; it demonstrates to the student that a proof itself can be **principally** simpler than a process of establishing it. Really, an easy generalization shows that n exact coins can be separated from n false ones using $[\log_2 n] + 1$ **demonstrations**; on the other hand, information theory lover bound shows that at least $n \cdot \log_3 2$ weightings are necessary to **establish** which n coins are the exact ones.

Other possible variations are to introduce the possibility of unreliable information, to consider parallel processes, to deal with more powerful/ more restricted identifying devices than *yes/no* questions or their equivalents, etc. All these are topics of serious investigations in computer science, but have not found an adequate reflection in math contests yet.

4. Some promising problem areas. Almost each serious contest problem can be a source of eventually infinite sequence of generalizations and variations. In this section we mention some areas which have been exploited for a long time and yet seems to be extremely rich with undiscovered treasures.

4.1. Sorting and searching problems. Some examples are considered in the section 2, with different types of algorithms as main topics of interest. Along with the classical problems where something must be **found**, there are various other natural questions that can be asked. The most important of them are:

1) optimality of algorithms, especially lover bounds,
2) the analysis of performance of a given algorithm,
3) impossibility of the algorithm in the given basis of elementary operations,
4) inference of the algorithm from its history,
5) proof of correctness of the algorithm.

The classical text in this area is [1]. The contest-oriented approach can be found in many places, e.g., [2].

4.2. Desert crossing problems. The general situations is as follows. A traveller is to cross the desert. The task can not be accomplished on one load of life resources. However, storages are allowed to be established in the desert. What is the maximal breadth of the desert the traveller can cross using a given amount of life resources?

In the classical version, all fragments of desert are "homogenically difficult"- the same amount of resources must be used for equal segments; there are no assistants; the storages are safe for ever (the resources are not robbed out from them), etc. However, there is a legion of possible variations.

A very good introduction can be found in [3], pp. 204-208. For further development see, e.g., [4], pp. 208–213.

4.3. Fair division problems. The sample problem of this area can be, e.g., the following one.

Each of two persons has his/her own opinion of comparative value of various portions of cake. How can they divide the cake among them fairly (i.e., so that each of them is sure that he has got at least $\frac{1}{2}$ of the total value of cake)?

The almost obvious solution (A divides, B chooses) can not be directly applied to the case of n persons. Moreover, other concepts of fairness can be introduced (e.g., a division is fair if everybody is sure he has received not less than any other). Furthermore, the problem of minimality of the number of cuts, "nearly-fair" divisions etc. can be considered.

The best survey with many topics for undergraduate research can be found in [5].

4.4. Graph numerations. A tree with n vertices is said to be gracious if its vertices can be numbered with integers from 1 to n so that the modules of the differences of edge endpoint numbers are all different. A still unsolved hypothesis of G. Ringel (1963) that all trees are gracious has created many investigations concerning other types of numerations. For example, introducing various definitions of magic numerations provide a rich area of problems, from easy to very hard ones.

4.5. Ramsey type results. This is the broadest and, may be, along with this the deepest of all mentioned topics. Ramsey theory can be considered as the branch of mathematics investigating which properties can and which can not be eliminated by cutting a rich structure into substructures; so it can be considered as the theory of "what is deeper than other in mathematics". Van der Varden theorem, the problematics of plane colouring, theorems on graphs with coloured edges are some manifestations of it in math contests. Ramsey-type questions can be asked in any branch of mathematics.

The classical and very clear, though not complete, survey is [6]. There are many contest–oriented publications; see, e.g., [4].

5. Conclusion. The main and, we hope, everlasting source of problems for math competitions is the current scientific research. On the other side, creative approach to contest problems can stimulate the independent research of students/ teachers significantly.

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Agnis Andžāns
University of Latvia
Rainis boulevard 19
Rīga, LV-1586, Latvia
e-mail: agnis@lanet.lv

ВЪРХУ НЯКОИ ИЗТОЧНИЦИ НА КОМБИНАТОРНИ ЗАДАЧИ

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В доклада са разгледани някои подходи за съставяне на оригинални задачи за състезания. Изброени са най-обещаващите от тази гледна точка области от комбинаториката.