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QUASIDOUBLES OF HADAMARD 2-(15,7,3) DESIGNS WITH AUTOMORPHISMS OF ORDER 3

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The study of quasidoubles of Hadamard designs is of interest for setting better or exact lower bounds on the number of designs that can be obtained from several designs of smaller parameters and, in particular, on the number of Hadamard designs of greater parameters. We construct all the nonisomorphic 2-(15,7,6) designs with automorphisms of order 3, and determine the orders of their groups of automorphisms. There are 92 115 nonisomorphic 2-(15,7,6) designs with automorphisms of order 3. Among them 12 716 designs are reducible into two 2-(15,7,3) designs. Most computer results were obtained by the two authors independently, and by different programmes.

1. Introduction. A 2-(v, k, λ) design is a collection of k-element subsets (*blocks*) of a set of v elements (*points*), such that each pair of points is contained in exactly λ blocks.

For the basic concepts and notations concerning combinatorial designs refer for instance to [1, 2, 3, 9].

Let b denote the number of the blocks of the design, and r – the number of blocks in which a given point is contained. An incidence matrix of the design is a matrix of v rows and b columns which contains 1 in the i-th row and j-th column iff the i-th point is contained in the j-th block, and 0 if not.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence, i.e. if the incidence matrix of the first design can be obtained from the incidence matrix of the second one by permuting rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks. The set of all automorphisms of a design form a group called its *full group of automorphisms*. Each subgroup of this group is a group of automorphisms of the design.

Each 2-(v,k,λ) design implies the existence of 2-(v,k,$m\lambda$) designs for any integer $m > 1$. These 2-(v,k,$m\lambda$) designs are called *quasimultiples* of a 2-(v,k,λ) design. A quasimultiple 2-(v,k,$m\lambda$) is *reducible* into m 2-(v,k,λ) designs if there is a partition of

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its blocks into m subcollections each of which forms a 2-(v,k,λ) design. For $m = 2$ quasimultiple designs are called *quasidoubles*, and the reducible quasidouble designs are called *doubles*.

A *Hadamard matrix of order* n is an $n \times n$ (± 1)-matrix satisfying $HH^t = nI$ (its rows are pairwise orthogonal).

Two Hadamard matrices are *equivalent* if one can be transformed into the other by a series of row or column permutations and negations. An *automorphism* of a Hadamard matrix is an equivalence with itself.

Each Hadamard matrix can be normalized, i.e. replaced by an equivalent Hadamard matrix whose first row and column entries are 1-s. Deleting the first row and column of a normalized Hadamard matrix of order $4m$, and replacing -1-s by 0-s, one obtains a Hadamard 2-($4m-1$,$2m-1$,$m-1$) design.

The 2-(15,7,6) design is quasidouble of the Hadamard 2-(15,7,3) design. According to computer estimations [8] there exist 5 nonisomorphic 2-(15,7,3) designs and at least 57 810 nonisomorphic 2-(15,7,6) designs. This lower bound is improved in the present work. The result is of interest for the classification of the double 2-(15,7,6) designs, and for setting higher lower bounds on the number of Hadamard designs of greater parameters [5, 6]. The first step towards the classification of the doubles of 2-(15,7,3) implies the construction and study of 2-(15,7,6) designs with nontrivial automorphims.

The five 2-(15,7,3) designs have automorphism groups of orders 20 160, 576, 96, 168, 168. They possess automorphisms of prime orders 7, 5, 3, and 2. A double design can have automorphisms of order 2 and automorphisms which preserve the two constituent designs (see for instance [4]). Thus, to study the reducible 2-(15,7,6) designs, we have to start with the classification of all (or only all reducible) 2-(15,7,6) with automorphisms of orders 7, 5, 3, and 2. Orders 7 and 5 are considered in [7]. Order 3 is subject of the present work. We construct all 2-(15,7,6) designs with automorphisms of order 3.

2. Automorphisms of order 3 of 2-(15,7,6) designs.

Theorem 2.1. *An automorphism of order* 3 *of a* 2-(15,7,6) *design has either no fixed points and blocks, or* 3 *fixed points and* 6 *fixed blocks.*

Proof. Let D be a 2-(15,7,6) design with an automorphism φ of order 3 fixing f points and h blocks. The number of points (15) and of blocks (30) is divisible by 3 and, thus, f and h are divisible by 3 too.

A. Let $f = 0$ or $h = 0$. The number of blocks, incident with any point of D is 14, i.e. not divisible by 3 and, thus, if $h = 0$, then $f = 0$ too. The number of points in a block of D is 7, i.e. not divisible by 3 and, thus, if $f = 0$, then $h = 0$ too.

B. Let $f > 0$ and $h > 0$. Without loss of generality we can assume that φ acts as $(1,2,3)(4,5,6)\dots(15-f+1)\dots(15)$ on the points, and as $(1,2,3)(4,5,6)\dots(30-h+1)\dots(30)$ on the blocks. Denote by u the number of nonfixed point orbits, and by w the number of nonfixed block orbits.

Then the nontrivial orbit part of the incidence matrix A of D is:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,w} \\ A_{2,1} & A_{2,2} & \dots & A_{2,w} \\ \dots & \dots & \dots & \dots \\ A_{u,1} & A_{u,2} & \dots & A_{u,w} \end{pmatrix}$$

where $A_{i,j}$, $i = 1, 2, \dots u$, $j = 1, 2, \dots, w$, are circulant matrices of order 3.

Let $m_{i,j}$, $i = 1, 2, \dots u$, $j = 1, 2, \dots, w$, be equal to the number of 1's in a row of $A_{i,j}$, and s_i to the number of fixed blocks incident with a points of the i-th nontrivial point orbit.The following equations hold for the matrix $M = (m_{i,j})_{u\times w}$

$$\sum_{j=1}^{w} m_{i,j} = 14 - s_i, \qquad i = 1, 2, \dots u, \tag{1}$$

$$\sum_{j=1}^{w} m_{i,j}^2 = 26 - 3s_i, \qquad i = 1, 2, \dots u. \tag{2}$$

1. Let $h > 6$, then $w \le 7$. Since $14 - s_i \le 26 - 3s_i$, it follows that $s_i \le 6$, $i = 1, 2, \dots u$. A direct check shows that the above equation system has no solution for $w \le 7$ and $s_i = 1, 2, \dots, 6$. Thus, $h \le 6$.

2. Let $h = 3$ or $f > 3$. Then some pairs of fixed points are contained in fixed blocks, and the λ parameter of the design cannot be kept.

3. Automorphisms of order 3 with no fixed points. Let D be a 2-(15,7,6) design with an automorphism φ of order 3, fixing no points or blocks. Without loss of generality we can assume that φ acts on the points as $(1, 2, 3)(4, 5, 6)\dots(13, 14, 15)$, and on the blocks as $(1, 2, 3)(4, 5, 6)\dots(28, 29, 30)$. The incidence matrix A of D is:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,10} \\ A_{2,1} & A_{2,2} & \dots & A_{2,10} \\ \dots & \dots & \dots & \dots \\ A_{5,1} & A_{5,2} & \dots & A_{5,10} \end{pmatrix},$$

where $A_{i,j}$, $i = 1, 2, \dots 5$, $j = 1, 2, \dots, 10$, are circulant matrices of order 3.

Let $m_{i,j}$, $i = 1, 2, \dots 5$, $j = 1, 2, \dots, 10$, be equal to the number of 1's in a row of $A_{i,j}$.The following equations hold for the matrix $M = (m_{i,j})_{5\times 10}$:

$$\sum_{j=1}^{10} m_{i,j} = 14, \qquad i = 1, 2, \dots 5 \tag{3}$$

$$\sum_{j=1}^{10} m_{i,j}^2 = 26, \qquad i = 1, 2, \dots 5, \tag{4}$$

$$\sum_{j=1}^{10} m_{i_1,j} m_{i_2,j} = 18, \qquad i_1, i_2 = 1, 2, \dots 5,\ i_1 < i_2. \tag{5}$$

There are 62 inequivalent matrix solutions for M. After replacement by circulants of their elements, 5 532 nonisomorphic 2-(15,7,6) designs were obtained, and the orders of their automorphism groups were computed. The number of reducible designs among them is 58. The results are illustrated in Table 1.

4. Automorphisms of order 3 with 3 fixed points. Let D be a 2-(15,7,6) design with an automorphism α of order 3 with 3 fixed points and 6 fixed blocks. We assume that α acts on the points as (1,2,3)(4,5,6)…(13)(14)(15) and on the blocks

Table 1. Order of the automorphism group of designs with automorphisms of order 3 with no fixed points and blocks

Aut.gr.order	3	6	9	12	15	18	24	30	36	48	72	96	120	288	576	2304	20160	All
Designs	5276	158	34	23	6	12	7	1	2	3	2	1	1	1	3	1	1	5532

as(1,2,3)(4,5,6)... (25)(26)... (30). The nontrivial orbit part of the incidence matrix A of D is:

$$\begin{pmatrix} A_{1,1} & A_{1,2} & \dots & A_{1,8} \\ A_{2,1} & A_{2,2} & \dots & A_{2,8} \\ \dots & \dots & \dots & \dots \\ A_{4,1} & A_{4,2} & \dots & A_{4,8} \end{pmatrix}$$

where $A_{i,j}$, $i = 1,2,3,4$, $j = 1,2,\dots,8$, are circulant matrices of order 3.

Let $m_{i,j}$, $i = 1,2,\dots 4$, $j = 1,2,\dots,8$, be equal to the number of 1's in a row of $A_{i,j}$, and s_i to the number of fixed blocks incident with a points of the i-th nontrivial point orbit. The following equations hold for the matrix $M = (m_{i,j})_{4\times 8}$:

$$\sum_{j=1}^{8} m_{i,j} = 14 - s_i, \qquad i = 1,2,\dots 4 \tag{6}$$

$$\sum_{j=1}^{8} m_{i,j}^2 = 26 - 3s_i, \qquad i = 1,2,\dots 4, \tag{7}$$

$$\sum_{i=1}^{4} m_{i,j} \le 7, \qquad j = 1,2,\dots 8. \tag{8}$$

The incidence matrix of D can be presented as

$$\begin{pmatrix} A & H \\ F & C \end{pmatrix}$$

The matrix H implies the incidence between nontrivial point orbits and fixed blocks, F – between fixed points and nontrivial block orbits, and C – between fixed points and fixed blocks. We generated by computer all possible matrix solutions for H, and found out that the above system has solutions for only 5 of them. Below we denote by p the transpose of $(1,1,1)$, and by o the transpose of $(0,0,0)$:

$$H = \begin{pmatrix} p & p & p & p & p & p \\ p & p & o & o & o & o \\ o & o & p & p & o & o \\ o & o & o & o & p & p \end{pmatrix}, \quad \begin{pmatrix} p & p & p & p & p & o \\ p & p & o & o & o & p \\ o & o & p & p & o & p \\ o & o & o & o & p & o \end{pmatrix},$$

$$\begin{pmatrix} p & p & p & p & o & o \\ p & p & o & o & p & p \\ o & o & p & p & p & p \\ o & o & o & o & o & o \end{pmatrix}, \begin{pmatrix} p & p & p & p & o & o \\ p & p & o & o & p & p \\ o & o & p & o & p & o \\ o & o & o & p & o & p \end{pmatrix}, \begin{pmatrix} p & p & p & o & o & o \\ p & o & o & p & p & o \\ o & p & o & p & o & p \\ o & o & p & o & p & p \end{pmatrix}.$$

There are 11 solutions for M with the above solutions for H. After replacement by circulants of the elements of these 11 solutions and construction of F and C in all possible ways, we obtain 86 639 nonisomorphic 2-(15,7,6) designs. Among them there are 56 designs which possess an automorphism of order 3 without fixed points too and, thus, are obtained from the constructions in the previous section too. There are 12 672 reducible designs, 14 of which also possess an automorphism of order 3 without fixed points. The results are presented in Table 2.

Table 2. Order of the automorphism group and reducibility of designs with automorphisms of order 3 with 3 fixed points

gr.order	3	6	9	12	18	21	24	36	42	48	72	96	168	192	288	336	384	576	2304	2688	20160	All
Designs	82066	3854	67	272	15	9	244	2	5	58	2	18	4	7	1	2	7	3	1	1	1	86639

5. Classification results. In total we construct 92 115 2-(15,7,6) designs and improve this way the existing lower bound on their number. Among them 12 716 are doubles of Hadamard 2-(15,7,3) designs , and the data about them is of interest as a step towards the full classification of all doubles and its usage for improving the lower bounds on the number of Hadamard designs of greater parameters.

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КВАЗИДВОЙНИ НА АДАМАРОВИТЕ 2-(15,7,3) ДИЗАЙНИ С АВТОМОРФИЗМИ ОТ РЕД 3

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Изследванията на квазидвойните на Адамарови дизайни представляват интерес заради приложенията при определяне на по-добри долни граници за броя на дизайни, които могат да бъдат конструирани от няколко дизайна с по-малки параметри, каквито са Адамаровите дизайни. Конструираме всички неизоморфни 2-(15,7,6) дизайни с автоморфизми от ред 3 и определяме реда на техните групи от автоморфизми. Установяваме, че броят на неизоморфните 2-(15,7,6) дизайни с автоморфизми от ред 3 е 92115, което е повече от известната досега долна граница. Сред тях има 12716 дизайна, които са разложими до два Адамарови 2-(15,7,3) дизайна. Повечето компютърни резултати са получени независимо и с различни програми от двата автора.