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DISCRETE GENERALIZATION OF GRONWALL-BELLMAN INEQUALITY WITH MAXIMA AND APPLICATION*

Snezhana Hristova, Kremena Stefanova, Liliana Vankova

Several new types of linear discrete inequalities containing the maximum of the unknown function over a past time interval are solved. Some of these inequalities are applied to difference equations with maximum and the continuous dependence of a perturbation is studied.

1. Introduction. The theory of finite difference equations has been rapidly developed in recent years and also it has been proved the fundamental importance of its applications in modeling of real world problems. At the same time there are many real world processes in which the present state depends significantly on its maximal value on a past time interval. Adequate mathematical models of these processes are the so-called difference equations with maximum. Meanwhile, this type of difference equations is not widely studied yet and there are only some isolated results ([3]).

Finite difference inequalities which exhibit explicit bounds of unknown functions provide, in generally, a very useful and important tool in the development of the theory of finite difference equations. During the past few years, motivated and inspired by their applications in various branches of difference equations, many such inequalities have been established [1, 2, 4].

The main purpose of the paper is solving of a new type of linear discrete inequalities containing the maximum over a past time interval. Also, they are applied to difference equations with maximum and the continuous dependence on a perturbation is studied.

2. Preliminary notes. Let $\mathbb{R}_+ = [0, \infty)$, $\mathbb{N}$ be the set of nonnegative integers (i.e. the numbers 0, 1, 2, 3, …) and $a,\ b \in \mathbb{N}$ be such that $a < b$. Denote by $\mathbb{N}(a, b) = \{k \in \mathbb{N} : \ a \le k \le b\}$ and $\mathbb{N}(a) = \{k \in \mathbb{N} : \ k \ge a\}$.

We assume that $\sum\limits_{l=n}^{m} = 0$ and $\prod\limits_{l=n}^{m} = 1$ for $n > m$.

In the proofs of our main results we need the following lemma:

Lemma 1 [1, Theorem 4.1.1]**.** *Let $f, q : \mathbb{N}(a) \to \mathbb{R}_+,\ p, u : \mathbb{N}(a) \to \mathbb{R}$ and let for all $k \in \mathbb{N}(a)$ $u(k) \le p(k) + q(k) \sum\limits_{l=a}^{k-1} f(l)u(l)$ be satisfied.*

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Then, for all $k \in \mathbb{N}(a)$ the inequality

$$u(k) \le p(k) + q(k) \sum_{l=a}^{k-1} p(l) f(l) \prod_{\tau=l+1}^{k-1} \Big(1 + q(\tau) f(\tau)\Big) \tag{1}$$

holds.

Remark 1. Note that Lemma 1 is true if $p(k)$ and $u(k)$ change sign on $\mathbb{N}(a)$.

3. Main results. Let $a, h \in \mathbb{N}$ be fixed so that $a \ge h$.

Theorem 1. *Let the following conditions be fulfilled:*

1. *The functions $p, g, G : \mathbb{N}(a) \to \mathbb{R}_+$ are nondecreasing.*
2. *The functions $q, Q : \mathbb{N}(a) \to \mathbb{R}_+$, the function $\varphi : \mathbb{N}(a-h, a) \to \mathbb{R}_+$, and $\max\limits_{k \in \mathbb{N}(a-h,\, a)} \varphi(k) \le p(a)$.*
3. *The function $u : \mathbb{N}(a-h) \to \mathbb{R}_+$ satisfies the inequalities*

$$u(k) \le p(k) + g(k) \sum_{l=a}^{k-1} q(l) u(l) + G(k) \sum_{l=a}^{k-1} Q(l) \max_{s \in [l-h,\, l]} u(s), \qquad k \in \mathbb{N}(a), \tag{2}$$

$$u(k) \le \varphi(k), \qquad k \in \mathbb{N}(a-h, a). \tag{3}$$

Then, for $k \in \mathbb{N}(a)$ the inequality

$$u(k) \le p(k) + S(k) \sum_{l=a}^{k-1} p(l) \big[q(l) + Q(l)\big] \prod_{\tau=l+1}^{k-1} \Big(1 + S(\tau) \big[q(\tau) + Q(\tau)\big]\Big) \tag{4}$$

holds, where $S(k) = \max\big(g(k), G(k)\big)$ for $k \in \mathbb{N}(a)$.

Proof. Define the function $z(k) : \mathbb{N}(a-h) \to \mathbb{R}_+$ by the equalities

$$z(k) = \begin{cases} p(k) + g(k) \sum\limits_{l=a}^{k-1} q(l) u(l) \\ \quad + G(k) \sum\limits_{l=a}^{k-1} Q(l) \max\limits_{s \in [l-h,\, l]} u(s), & \text{for } k \in \mathbb{N}(a), \\ p(a) & \text{for } k \in \mathbb{N}(a-h, a). \end{cases}$$

For any $k \in \mathbb{N}(a),\ k \ge a-1$ we have

$$\begin{aligned} z(k+1) &= p(k+1) + g(k+1) \sum_{l=a}^{k} q(l) u(l) + G(k+1) \sum_{l=a}^{k} Q(l) \max_{s \in [l-h,\, l]} u(s) \\ &\ge p(k+1) + g(k+1) \sum_{l=a}^{k-1} q(l) u(l) + G(k+1) \sum_{l=a}^{k-1} Q(l) \max_{s \in [l-h,\, l]} u(s) \\ &\ge p(k) + g(k) \sum_{l=a}^{k-1} q(l) u(l) + G(k) \sum_{l=a}^{k-1} Q(l) \max_{s \in [l-h,\, l]} u(s) = z(k). \end{aligned}$$

Therefore, the function $z(k)$ is nondecreasing in $\mathbb{N}(a-h)$.

From the definition of the function $z(k)$, its monotonicity, inequalities (2), (3) and condition 2 it follows that $u(k) \le z(k)$, $k \in \mathbb{N}(a-h)$. Therefore, $\max_{s\in[k-h,\,k]} u(s) \le \max_{s\in[k-h,\,k]} z(s) = z(k)$ for $k \in \mathbb{N}(a)$ and for any $k \in \mathbb{N}(a)$ we obtain

$$\begin{aligned} z(k) &\le p(k) + g(k)\sum_{l=a}^{k-1} q(l)z(l) + G(k)\sum_{l=a}^{k-1} Q(l)z(l) \\ &\le p(k) + S(k)\sum_{l=a}^{k-1}\big[q(l)+Q(l)\big]z(l). \end{aligned} \tag{5}$$

According to Lemma 1, from inequality (5) we get for $k \in \mathbb{N}(a)$

$$z(k) \le p(k) + S(k)\sum_{l=a}^{k-1} p(l)\big[q(l)+Q(l)\big]\prod_{\tau=l+1}^{k-1}\Big(1+S(\tau)\big[q(\tau)+Q(\tau)\big]\Big). \tag{6}$$

Inequality (6) implies the validity of the required inequality (4). $\square$

Corollary 1. *Let the conditions* 2, 3 *of Theorem* 1 *be satisfied where* $p(k) \equiv p$, $g(k) \equiv g$, $G(k) \equiv G$ *for* $k \in \mathbb{N}(a)$ *and* $p, g, G \ge 0$ *are constants.*

Then, $u(k) \le p\prod_{l=a}^{k-1}\Big(1+S\big[q(l)+Q(l)\big]\Big)$ *for* $k \in \mathbb{N}(a)$, *where* $S = \max\big(g, G\big)$.

Remark 2. The proof of Corollary 1 is based on the inequality

$$\sum_{l=a}^{b} A(l)\prod_{\tau=l+1}^{b}(1+A(\tau)) \le \prod_{l=a}^{b}(1+A(l)).$$

Corollary 2. *Let the conditions of Theorem* 1 *be satisfied and* $g(k) \ge 1$, $G(k) \ge 1$ *for* $k \in \mathbb{N}(a)$.

Then $u(k) \le p(k)S(k)\prod_{l=a}^{k-1}\Big(1+S(l)\big[q(l)+Q(l)\big]\Big)$ *for* $k \in \mathbb{N}(a)$.

Corollary 3. *Let the conditions* 2, 3 *of Theorem* 1 *be satisfied where* $p(k) \equiv p = const \ge 0$ *and* $g(k) = G(k) = 1$ *for* $k \in \mathbb{N}(a)$.

Then $u(k) \le p\prod_{l=a}^{k-1}\Big[1+q(l)+Q(l)\Big]$ *for* $k \in \mathbb{N}(a)$.

4. Applications. Let $a, b \in \mathbb{N}:\ a < b \le \infty$ and the function $\tau : \mathbb{N}(a,b) \to \mathbb{N}$ be given such that there exists $h \in \mathbb{N}$, $h \le a$: $k-h \le \tau(k) \le k$ for $k \in \mathbb{N}(a,b)$.

Let $u : \mathbb{N} \to \mathbb{R}$. Denote $\Delta u(k) = u(k+1) - u(k)$, $k \in \mathbb{N}$.

Consider the following difference equation with "maxima":

$$\Delta u(k) = f\Big(k,\ u(k),\ \max_{s\in\mathbb{N}(\tau(k),\,k)} u(s)\Big), \quad k \in \mathbb{N}(a,b), \tag{7}$$

and its perturbed difference equation with "maxima":

$$\Delta v(k) = f\Big(k,\ v(k),\ \max_{s\in\mathbb{N}(\tau(k),\,k)} v(s)\Big) + g\big(k, v(k)\big), \quad k \in \mathbb{N}(a,b) \tag{8}$$

with initial conditions

$$u(k) = \varphi(k), \qquad k \in \mathbb{N}(a-h, a). \tag{9}$$

Remark 3. Note that in the case $\tau(k) \equiv k$ the equation with maxima (7) reduces to a difference equation which is well-known in the literature [1].

In our further investigations we assume that the initial value problems (7), (9) and (8), (9) have solutions $u(k)$ and $v(k)$ for $k \in \mathbb{N}(a-h, b)$.

Theorem 2 (Continuous dependence on the perturbation)**.** *Let the following conditions be fulfilled:*

1. *The function $f : \mathbb{N}(a,b) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies Liptschitz's condition*

$$|f(k, x_1, y_1) - f(k, x_2, y_2)| \le \lambda(k)|x_1 - x_2| + \tilde{\lambda}(k)|y_1 - y_2|,$$

where $x_i,\, y_i \in \mathbb{R}$, $k \in \mathbb{N}(a,b)$ and the functions $\lambda, \tilde{\lambda} : \mathbb{N}(a,b) \to \mathbb{R}_+$.

2. *The function $g : \mathbb{N}(a,b) \times \mathbb{R} \to \mathbb{R}$ satisfies the condition*

$$\big|g(k,\zeta)\big| \le \mu(k), \quad k \in \mathbb{N}(a,b), \quad \zeta \in \mathbb{R}, \tag{10}$$

where the function $\mu : \mathbb{N}(a,b) \to \mathbb{R}_+$ and $\sum\limits_{k=a}^{b} \mu(k) = M < \infty$.

3. *The function $\varphi : \mathbb{N}(a-h, a) \to \mathbb{R}$.*

Then, for $k \in \mathbb{N}(a,b)$ the following inequality holds

$$\big|u(k) - v(k)\big| \le M\left(\prod_{l=a}^{k-1}\big[1 + \lambda(l) + \tilde{\lambda}(l)\big]\right). \tag{11}$$

Proof. The functions $u(k)$ and $v(k)$ satisfy the equalities $u(k) = v(k)$ for $k \in \mathbb{N}(a-h, a)$ and for $k \in \mathbb{N}(a,b)$:

$$u(k) = \varphi(a) + \sum_{l=a}^{k-1} f\Big(l,\ u(l),\ \max_{s\in\mathbb{N}(\tau(l),\,l)} u(s)\Big)$$

$$v(k) = \varphi(a) + \sum_{l=a}^{k-1}\left(f\Big(l,\ v(l),\ \max_{s\in\mathbb{N}(\tau(l),\,l)} v(s)\Big) + g\big(l, v(l)\big)\right).$$

Then, since $\mathbb{N}(\tau(k),\, k) \subset [k-h, k]$, we get for $k \in \mathbb{N}(a)$

$$\big|u(k) - v(k)\big| \le M + \sum_{l=a}^{k-1}\lambda(l)\big|u(l) - v(l)\big| + \sum_{l=a}^{k-1}\tilde{\lambda}(l) \max_{s\in[l-h,\,l]}\big|u(s) - v(k)\big|. \tag{12}$$

According to Corollary 3, from inequality (12) we obtain inequality (11). $\square$

Example. Consider the linear difference equation with "maxima"

$$u(k+1) = Au(k) + B \max_{s\in\mathbb{N}(\tau(k),\,k)} u(s), \quad k \in \mathbb{N}(a,b), \tag{13}$$

and the perturbed difference equation with "maxima"

$$v(k+1) = Av(k) + B \max_{s\in\mathbb{N}(\tau(k),\,k)} v(s) + \frac{e^{-v(k)}}{2^k}, \quad k \in \mathbb{N}(a,b), \tag{14}$$

with initial conditions $u(k) = v(k) = C, \quad k \in \mathbb{N}(a-h, a)$, where $A = \text{const} \ge 1$, $B = \text{const} \ge 0$, $C = \text{const} \ge 0$.

Then, $\lambda(k) \equiv A - 1 \ge 0$ and $\tilde{\lambda}(k) \equiv B \ge 0$ for $k \in \mathbb{N}(a,b)$.

Let $g\big(k,v\big) = \dfrac{e^{-v}}{2^k}$. Then $\mu(k) = \dfrac{1}{2^k}$ and $\sum\limits_{k=a}^{b} \dfrac{1}{2^k} \le 2 = M < \infty$. According to Theorem 2, we get $\big|u(k) - v(k)\big| \le 2\big(A + B\big)^{k-a-1}$, $k \in N(a+1, b)$.

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Snezhana Hristova
Kremena Stefanova
Liliana Vankova
Faculty of Mathematics and Informatics
Plovdiv University
236, Bulgaria Blvd
4000 Plovdiv, Bulgaria
e-mail: snehri@uni-plovdiv.bg
kstefanova@uni-plovdiv.bg
lilqna.v@gmail.com

ДИСКРЕТНО ОБОБЩЕНИЕ С МАКСИМУМИ НА НЕРАВЕНСТВОТО НА ГРОНУОЛ-БЕЛМАН И ПРИЛОЖЕНИЯ

Снежана Христова, Кремена Стефанова, Лиляна Ванкова

В работата са решени няколко нови видове линейни дискретни неравенства, които съдържат максимума на неизвестната функция в отминал интервал от време. Някои от тези неравенства са приложени за изучаване непрекъснатата зависимост от смущения при дискретни уравнения с максимуми.