МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2017
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2017
Proceedings of the Forty-sixth Spring Conference
of the Union of Bulgarian Mathematicians
Borovets, April 9–13, 2017

GLOBAL ASYMPTOTIC STABILIZATION OF A MODEL OF ANAEROBIC BIOREACTOR WITH METHANE PRODUCTION*

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In this paper we consider a four-dimensional bioreactor model, describing an anaerobic digestion process with methane (biogas) production. Different control strategies for global asymptotic stabilization of the model towards a previously chosen operating point are proposed and discussed. A model-based extremum seeking algorithm is applied to optimize the biogas production. Numerical results are presented to confirm the theoretical studies.

1. Introduction. The anaerobic digestion is a complex biotechnological process, where organic wastes are decomposed by microorganisms inside a continuously stirred tank bioreactor. The goal is to reduce the pollutant concentration in the outlet stream below a specified value, usually fixed by environmental and safety rules. At the same time this process can produce valuable energy (methane).

Mathematical modeling of anaerobic digestion has recently become an active research area. Dynamic mathematical models are used as a powerful tool to simulate different operating, control and optimization strategies in designing and operating the bioreactor in order to predict its behavior long before the physical prototype is built and tested in real life.

One of the main drawbacks in the modeling and control of anaerobic digestion lies in the difficulty to monitor online the key biological variables of the process and to obtain explicit analytic expressions for the specific growth rates of the microorganisms. Thus developing control systems based on simple measurements and general assumptions on the specific growth rates, that guarantee stability of the process, is of primary importance.

The present paper summarizes the authors' results presented in the papers [7]–[10]. We consider here three approaches for asymptotic stabilization of a known four-dimensional nonlinear control system (cf. [2], [4], [13], [14], [17]), that models anaerobic biological digestion of a wastewater treatment process in a continuously stirred tank bioreactor. The first approach is based on an adaptive feedback stabilization, the second, on a state (nonadaptive) feedback; the third approach is not based on feedback laws, but uses directly the control (manipulated) variable of the model. In all cases it is shown

*__2010 Mathematics Subject Classification:__ 34C60, 34D20, 34D23, 37N25, 92D40.
Key words: continuous bioreactor, dynamical nonlinear model, global asymptotic stabilization, feedback control, model uncertainty, numerical extremum seeking algorithm.

that the control system is globally stabilizable towards a previously chosen operating (also called *reference* or *set*) point. For practical applications it is worth mentioning that the feedback control laws depend on online measurements only.

The paper is organized as follows. Section 2 presents shortly the dynamic model of the digestion process. The adaptive asymptotic stabilization of the dynamic system towards a previously chosen operating point is studied in Section 3. Section 4 considers the non-adaptive stabilization problem for the same model. Section 5 is devoted to the global stabilization of the model without using any feedback law. In order to prove that the closed-loop system is asymptotically stable, suitable Lyapunov-like functions are constructed explicitly in the three cases. An extremum seeking algorithm is applied to optimize the biogas flow rate in real time. The algorithm is shortly described in Section 6. Computer simulations are reported in Section 7 to illustrate the theoretical results of the three approaches. The robustness of the extremum seeking algorithm under model uncertainties is also demonstrated.

2. Model description. We consider a model of an anaerobic digestion process, based on two main reactions (cf. [2], [4], [13], [14], [17]):

(a) acidogenesis, where the organic substrate is degraded into volatile fatty acids (VFA) by acidogenic bacteria;

(b) methanogenesis, where VFA are degraded into methane CH_4 (and carbon dioxide CO_2) by methanogenic bacteria.

The mass balance model of the process in a continuously stirred tank bioreactor is described by the following nonlinear system of ordinary differential equations

$$\frac{ds_1}{dt} = u(s_1^i - s_1) - k_1\ \mu_1(s_1)\ x_1 \tag{1}$$

$$\frac{dx_1}{dt} = (\mu_1(s_1) - \alpha u)x_1 \tag{2}$$

$$\frac{ds_2}{dt} = u(s_2^i - s_2) + k_2\ \mu_1(s_1)x_1 - k_3\ \mu_2(s_2)\ x_2 \tag{3}$$

$$\frac{dx_2}{dt} = (\mu_2(s_2) - \alpha u)x_2 \tag{4}$$

with gaseous output

$$Q = k_4\mu_2(s_2)x_2. \tag{5}$$

The state variables s_1, s_2 and x_1, x_2 denote substrate and biomass concentrations, respectively: s_1 represents the organic substrate, characterized by its chemical oxygen demand (COD), s_2 denotes the volatile fatty acids (VFA), x_1 and x_2 are the acidogenic and methanogenic bacteria respectively. The parameter $\alpha \in (0, 1)$ represents the proportion of bacteria that are affected by the dilution; $\alpha = 0$ and $\alpha = 1$ correspond to an ideal fixed bed reactor and to an ideal continuous stirred tank reactor, respectively (cf. [1], [2], [4], [13], [20]). The functions $\mu_1(s_1)$ and $\mu_2(s_2)$ model the specific growth rates of the microorganisms.

The input substrate concentrations s_1^i and s_2^i are assumed to be constant. The dilution rate u is considered as a control (manipulated) input.

It is known that the methane flow rate Q is a measurable output.

The definition of the model parameters is given in Table 1. There the constants m_1, m_2, k_{s_1}, k_{s_2} and k_I are related to the particular expressions of the specific growth rate

Table 1. Definition of the model parameters

s_1	concentration of chemical oxygen demand (COD) [g/l]
s_2	concentration of volatile fatty acids (VFA) [mmol/l]
x_1	concentration of acidogenic bacteria [g/l]
x_2	concentration of methanogenic bacteria [g/l]
u	dilution rate [day^{-1}]
s_1^i	influent concentration s_1 [g/l]
s_2^i	influent concentration s_2 [mmol/l]
k_1	yield coefficient for COD degradation [g COD/(g x_1)]
k_2	yield coefficient for VFA production [mmol VFA/(g x_1)]
k_3	yield coefficient for VFA consumption [mmol VFA/(g x_2)]
k_4	coefficient [l^2/g]
m_1	maximum acidogenic biomass growth rate [day^{-1}]
m_2	maximum methanogenic biomass growth rate [day^{-1}]
k_{s_1}	saturation parameter associated with s_1 [g COD/l]
k_{s_2}	saturation parameter associated with s_2 [mmol VFA/l]
k_I	inhibition constant associated with s_2 [(mmol VFA/l)$^{1/2}$]
α	proportion of dilution rate reflecting process heterogeneity
Q	methane gas flow rate

functions $\mu_1(s_1)$ and $\mu_2(s_2)$, which are used later in Section 7.

In the theoretical studies we do not assume to know explicit expressions for μ_1 and μ_2, we only impose the following general assumptions on the latter:

Assumption A1. *$\mu_j(s_j)$ is defined for $s_j \in [0, +\infty)$, $\mu_j(0) = 0$, $\mu_j(s_j) > 0$ for $s_j > 0$; $\mu_j(s_j)$ is continuously differentiable and bounded for all $s_j \in [0, +\infty)$, $j = 1, 2$.*

3. Adaptive asymptotic stabilization. In this section we propose an adaptive stabilizing controller of (1)–(4) and show that the dynamics can be globally asymptotically stabilized towards a previously chosen operating point.

Denote

$$s(t) = \frac{k_2}{k_1} s_1(t) + s_2(t), \quad s^i = \frac{k_2}{k_1} s_1^i + s_2^i. \tag{6}$$

The quantity $s(t)$ is called *biological oxygen demand* (BOD). For the practical application it is important to note that BOD is online measurable. This fact is discussed in details in [4], see also [1], [2], [5] and the references therein.

Let us fix an operating (reference) point $\bar{s}$,

$$\bar{s} \in (0, s^i). \tag{7}$$

Assumption A2. *There exists a point $\bar{s}_1$ such that*

$$\mu_1(\bar{s}_1) = \mu_2\left(\bar{s} - \frac{k_2}{k_1}\bar{s}_1\right) > 0, \quad \bar{s}_1 \in \left(0, s_1^i\right).$$

Assumption A2 is called *regulability of the system* [13]: it means that there exists (at least one) nontrivial equilibrium of the system (1)–(4), corresponding to some value of the dilution rate $u > 0$.

Define further

$$\bar{s}_2 = \bar{s} - \frac{k_2}{k_1}\bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad \bar{x}_2 = \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1}{\alpha k_3} = \frac{s^i - \bar{s}}{\alpha k_3}. \tag{8}$$

It is straightforward to see that the point

$$\bar{\zeta} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is an equilibrium point for the system (1)–(4). Our goal is to construct an adaptive feedback law to stabilize asymptotically the system (1)–(4) to $\bar{\zeta}$.

Using the definitions of s and s^i from (6), we define the sets

$$\begin{aligned}
\Omega_0 &= \{(s_1, x_1, s_2, x_2) |\ s_1 > 0,\ x_1 > 0,\ s_2 > 0,\ x_2 > 0\}, \\
\Omega_1 &= \left\{(s_1, x_1, s_2, x_2) |\ s_1 + k_1 x_1 \le \frac{s_1^i}{\alpha},\ s + k_3 x_2 \le \frac{s^i}{\alpha}\right\}, \\
\Omega_2 &= \left\{\left(s_1, x_1, \bar{s} - \frac{k_2}{k_1}s_1, \bar{x}_2\right) |\ 0 < s_1 < \frac{k_1}{k_2}\bar{s},\ x_1 > 0\right\}, \\
\Omega &= \Omega_0 \cap \Omega_1.
\end{aligned} \tag{9}$$

Assumption A3. *Let the inequality* $\mu_1'(s_1) + \frac{k_2}{k_1} \cdot \mu_2'\left(\bar{s} - \frac{k_2}{k_1}s_1\right) > 0$ *be satisfied on the set* $\Omega \cap \Omega_2$.

Assumption A3 is technical and not restrictive. It has been discussed in details in [7], [8], so we skip here the comments about it.

Denote

$$\bar{\beta} = \frac{1}{\alpha k_4 \bar{x}_2} = \frac{k_3}{k_4(s^i - \bar{s})} \tag{10}$$

and make the next technical

Assumption A4. *Let* $\beta^- > 0$ *and* $\beta^+ > 0$ *be arbitrary real numbers such that* $\bar{\beta} \in (\beta^-, \beta^+)$ *and the following inequality hold true* $s^i > \frac{k_3}{\alpha k_4 \beta^-}$.

Following [2], [18], we extend the system (1)–(4) by adding the differential equation

$$\frac{d\beta}{dt} = -C(\beta - \beta^-)(\beta^+ - \beta)k_4\, \mu_2(s_2)\, x_2\, (s - \bar{s}), \tag{11}$$

where $C > 0$ is a constant.

We consider the extended system (1)–(4), (11) in the augmented state space (ζ, β) with $\zeta = (s_1, x_1, s_2, x_2)$ and define the following continuous feedback control law

$$\kappa_1(\zeta, \beta) = k_4 \beta\, \mu_2(s_2)\, x_2. \tag{12}$$

Then the following theorem holds true (a similar assertion can be found in [7] and [8]):

Theorem 1. *Let us fix an arbitrary reference point* $\bar{s} \in (0, s^i)$. *Let Assumptions A1, A2, A3 and A4 be satisfied. Then the feedback* (12) *stabilizes asymptotically the control system* (1)*–*(4)*,* (11) *to the point* $(\bar{\zeta}, \bar{\beta})$ *for each starting point* $\zeta^0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ *and* $\beta^0 \in (\beta^-, \beta^+)$.

Proof. Let us fix an arbitrary point $\zeta^0 \in \Omega_0$ and a positive value $u_0 > 0$ for the control. According to Lemma 1 from [13] there exists time $T > 0$ such that the value of the corresponding trajectory of (1)–(4) for $t = T$ belongs to the set Ω, i. e. the trajectory of (1)–(4) starting from the point ζ^0 enters the set Ω after a finite time. For that reason we are studying the control system (1)–(4) after this moment of time, i. e. we assume that the starting point belongs to the set Ω.

Let Σ_1 denote the closed-loop system obtained from (1)–(4) and (11) by substituting the control variable u by the feedback $\kappa_1(\zeta, \beta)$ and let

$$\widetilde{\Omega} = \overline{\Omega} \times [\beta^-, \beta^+],$$

where $\overline{\Omega}$ is the closure of the set Ω. One can directly check that $\widetilde{\Omega}$ is positively invariant (cf. [6]) with respect to the trajectories of Σ_1.

Let $(\zeta(t), \beta(t)) = (s_1(t), x_1(t), s_2(t), x_2(t), \beta(t))$ be the trajectory of the closed loop system Σ_1 starting from an arbitrary point $(\zeta^0, \beta^0) = (s_1^0, x_1^0, s_2^0, x_2^0, \beta_0)$ of $\Omega_0 \times (\beta^-, \beta^+)$. Using the definitions of $s(t)$ and s^i from (6), one can check that $s(t)$ and $x_2(t)$ satisfy the following system:

$$\begin{aligned}
\frac{ds}{dt} &= \kappa_1(\zeta, \beta)(s^i - s) - k_3\ \mu_2(s_2)\ x_2 = -\kappa_1(\zeta, \beta)\left(s - s^i + \frac{k_3}{k_4 \beta}\right) \\
\frac{dx_2}{dt} &= (\mu_2(s_2) - \alpha\kappa_1(\zeta, \beta))x_2 = -\alpha\kappa_1(\zeta, \beta)\left(x_2 - \frac{1}{\alpha k_4 \beta}\right).
\end{aligned}$$

From here we obtain the estimates:

$$-\kappa_1(\zeta, \beta)\left(s - s^i + \frac{k_3}{k_4\beta^-}\right) \le \frac{ds}{dt} \le -\kappa_1(\zeta, \beta)\left(s - s^i + \frac{k_3}{k_4\beta^+}\right),$$

$$-\alpha\kappa_1(\zeta, \beta)\left(x_2 - \frac{1}{\alpha k_4\beta^+}\right) \le \frac{dx_2}{dt} \le -\alpha\kappa_1(\zeta, \beta)\left(x_2 - \frac{1}{\alpha k_4\beta^-}\right).$$

These estimates and Assumption A4 imply that

$$\begin{aligned}
&0 < \delta := \min\left\{s^0,\ s^i - \frac{k_3}{\alpha k_4\beta^-}\right\} \le s(t) \le \max\left\{s^0,\ s^i - \frac{k_3}{\alpha k_4 \beta^+}\right\}, \\
&0 < \min\left\{x_2^0,\ \frac{1}{\alpha k_4 \beta^+}\right\} \le x_2(t) \le \max\left\{x_2^0,\ \frac{1}{\alpha k_4\beta^-}\right\}.
\end{aligned} \tag{13}$$

Define the following function

$$V_1(\zeta, \beta) = (s - \bar{s} + k_3(x_2 - \bar{x}_2))^2 + \Gamma \int_{\bar{s}}^{s} \frac{v - \bar{s}}{s^i - v} dv + \frac{1}{C} \int_{\bar{\beta}}^{\beta} \frac{w - \bar{\beta}}{(w - \beta^-)(\beta^+ - w)}\, dw,$$

where the parameter $\Gamma > 0$ will be determined later. Clearly, the values of this function are nonnegative.

Denote by $\dot{V}_1(\zeta, \beta)$ the Lie derivative of the function V_1 with respect to the right-hand side of the closed-loop system Σ_1 at the point (ζ, β). It is easy to see that for each point $(\zeta, \beta) \in \widetilde{\Omega}$ the following presentation holds true:

$$\begin{aligned}
\dot{V}_1(\zeta, \beta) &= -\kappa_1(\zeta, \beta)\left(2 + \Gamma\, \frac{k_3}{k_4\beta(s^i - s)(s^i - \bar{s}))}\right)(s - \bar{s})^2 \\
&\quad - 2(1 + \alpha)k_3 \cdot \kappa_1(\zeta, \beta)(s - \bar{s})(x_2 - \bar{x}_2) \\
&\quad - 2\alpha k_3^2 \cdot \kappa_1(\zeta, \beta)(x_2 - \bar{x}_2)^2.
\end{aligned}$$

The boundedness of the set $\widetilde{\Omega}$ implies the existence of a sufficiently large constant $\Gamma > 0$

so that

$$\dot V_1(\zeta,\beta) \le 0 \ \text{ for each point } \ (\zeta,\beta) \in \widetilde{\Omega}. \tag{14}$$

Let us denote by $\phi(t,\zeta,\beta)$ the value of the trajectory of Σ_1 calculated at time t starting from the point $(\zeta,\beta) \in \widetilde{\Omega}$. The positive limit set (or ω-limit set) of the solution $\phi(t,\zeta,\beta)$ is defined as (cf. [15])

$$L^+(\zeta,\beta) = \Big\{(\tilde\zeta,\tilde\beta) : \text{ there exists a sequence } \\ \{t_n\} \to +\infty \text{ with } (\tilde\zeta,\tilde\beta) = \lim_{t_n\to+\infty} \phi(t_n,\zeta,\beta)\Big\}.$$

In what follows we shall show that the ω-limit set $L^+(\zeta^0,\beta^0)$ belongs to the set

$$\widetilde{\Omega}_2 := \left\{(s_1,x_1,s_2,\bar x_2,\bar\beta) \in \widetilde{\Omega} : \ \frac{k_2}{k_1}s_1 + s_2 = \bar s\right\}.$$

According to a refinement of the LaSalle invariance principle (cf. [3], see also [11] and [19], where similar stabilizability problems are studied) every solution of Σ_1 starting from a point of $\widetilde{\Omega}$ is defined in the interval $[0,+\infty)$ and approaches the largest invariant set $\mathcal{M}$ (with respect to Σ_1) which is contained in one connected component of the closure of the set Z, where the Lie derivative $\dot V_1$ of V_1 with respect to the right-hand side of Σ_1 is equal to zero, i. e.

$$Z := \{(s_1,x_1,s_2,x_2,\beta) \in \widetilde{\Omega} : \ \dot V_1(s_1,x_1,s_2,x_2,\beta) = 0\}.$$

Let $(\tilde\zeta,\tilde\beta)$ be an arbitrary point from the set $L^+(\zeta^0,\beta^0)$, where $\tilde\zeta = (\tilde s_1,\tilde x_1,\tilde s_2,\tilde x_2)$. We set $\tilde s = \dfrac{k_2}{k_1}\tilde s_1 + \tilde s_2$.

If we assume that $\kappa_1(\tilde\zeta,\tilde\beta) = 0$, then the inequalities (13) imply that $\tilde x_2 > 0$ and that $\tilde s \ge \delta > 0$ for some $\delta > 0$. Then the equality $\kappa_1(\tilde\zeta,\tilde\beta) = 0$ implies that $\tilde s_2 = 0$, and hence $\tilde s_1 \ge \delta k_1/k_2$. If $\tilde x_1 > 0$, then we have that $k_2\mu_1(\tilde s_1)\tilde x_1 > 0$; the latter inequality follows from the right-hand side of equation (3) of the closed loop system Σ_1 at the point $\tilde\zeta$. Then the continuity of the right-hand side of Σ_1 implies the existence of a neighborhood U_1 of the point $\tilde\zeta$ so that for each point $\zeta \in U_1$ the right-hand side of (3) in Σ_1 is greater than $k_2\mu_1(\tilde s_1)\tilde x_1/2$. But this means that if a trajectory of Σ_1 enters the set U_1, then $s_2(t)$ will increase, so the trajectory could not tend to the point $\tilde\zeta$ for which $\tilde s_2 = 0$.

Next we consider the case when $\tilde x_1 = 0$. In this case we have that $\mu_1(\tilde s_1) > 0 = \tilde x_1 = \alpha\kappa_1(\tilde\zeta,\tilde\beta)$. The continuity of the feedback $\kappa_1(\cdot,\cdot)$ and of the function $\mu_1(\cdot)$ implies the existence of a neighborhood U_2 of $\tilde\zeta$ so that for each point $\zeta \in U_2$ the following inequality holds true: $\mu_1(s_1) > \alpha\kappa_1(\zeta,\beta)$. Then for each point $\zeta \in U_2$ the right-hand side of equation (2) in Σ_1 is $x_1(\mu_1(s_1) - \alpha\kappa_1(\zeta,\beta)) > 0$. But this means that if a trajectory of Σ_1 enters the set U_2, then $x_1(t)$ will strictly increase and the trajectory could not tend to the point $\tilde\zeta$ for which $\tilde x_1 = 0$.

Hence, our assumption that the point $\tilde\zeta$ belongs to the set $L^+(\zeta^0,\beta^0)$ and $\tilde s_2 = 0$ is wrong. The obtained contradiction shows that $\kappa_1(\tilde\zeta,\tilde\beta) \ne 0$. This and the equality $\dot V_1(\tilde\zeta,\tilde\beta) = 0$ imply that $\tilde x_2 = \bar x_2$ and $\tilde s = \bar s$. Remind that we have denoted by $\mathcal{M}$ the largest positive invariant (with respect to the trajectories of the closed-loop system Σ_1) subset of the set Z. One can directly check that the positive invariance of the set $\mathcal{M}$ with respect to Σ_1 implies that $\mathcal{M}$ coincides with the set $\widetilde{\Omega}_2$. Hence $L^+(\zeta^0,\beta^0)$ is a subset of

$\widetilde{\Omega}_2$, and $\widetilde{\Omega}_2$ can be presented in the equivalent form

$$\widetilde{\Omega}_2 = \left\{ \left(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta} \right) \in \widetilde{\Omega} \right\}.$$

Next we shall prove that the ω-limit set $L^+(\zeta^0, \beta^0)$ coincides with the equilibrium point $(\bar{\zeta}, \bar{\beta})$ of Σ_1.

One can directly check that the set $\widetilde{\Omega}_2$ is invariant with respect to the trajectories of Σ_1. Using (8), (10) and (12), the dynamics of Σ_1 on the set $\widetilde{\Omega}_2$ can be described by the following system

$$\begin{aligned} \frac{ds_1}{dt} &= \frac{1}{\alpha} \chi_1(s_1)(s_1^i - s_1) - k_1 \mu_1(s_1) x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi_1(s_1)) x_1, \end{aligned} \tag{15}$$

where $\chi_1(s_1) = \mu_2 \left(\bar{s} - \frac{k_2}{k_1} s_1 \right)$. Taking into account that $\bar{s} = \frac{k_2}{k_1} \bar{s}_1 + \bar{s}_2$ and $s_1^i = \bar{s}_1 + \alpha k_1 \bar{x}_1$, (15) can be rewritten as follows:

$$\begin{aligned} \frac{ds_1}{dt} &= -\frac{1}{\alpha} \chi_1(s_1) \cdot (s_1 - \bar{s}_1 + \alpha k_1 (x_1 - \bar{x}_1)) - k_1 \left(\mu_1(s_1) - \chi_1(s_1) \right) \cdot x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi_1(s_1)) \cdot x_1. \end{aligned}$$

Consider the function

$$W_1(\zeta, \beta) = (s_1 - \bar{s}_1 + \alpha k_1 (x_1 - \bar{x}_1))^2 + \alpha(1-\alpha) k_1^2 (x_1 - \bar{x}_1)^2.$$

Clearly, this function depends only on the variables s_1 and x_1, i. e. $W_1(\zeta, \beta) = W_1(s_1, x_1)$, and takes only nonnegative values; moreover, for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}_2$, the following presentation holds true:

$$\begin{aligned} \dot{W}_1(s_1, x_1) &= -\frac{2}{\alpha} \chi_1(s_1)(s_1 - \bar{s}_1 + \alpha k_1 (x_1 - \bar{x}_1))^2 \\ &\quad - 2(1-\alpha) k_1 x_1 (s_1 - \bar{s}_1)(\mu_1(s_1) - \chi_1(s_1)). \end{aligned} \tag{16}$$

We have

$$\begin{aligned} &\mu_1(s_1) - \chi_1(s_1) = \mu_1(s_1) - \mu_2 \left(\bar{s} - \frac{k_2}{k_1} s_1 \right) = \mu_1(s_1) - \mu_2 \left(\bar{s}_2 - (s_1 - \bar{s}_1) \frac{k_2}{k_1} \right) \\ &= \mu_1(\bar{s}_1) + \int_{\bar{s}_1}^{s_1} \mu_1'(\theta)\, d\theta - \mu_2(\bar{s}_2) + \frac{k_2}{k_1} \int_{\bar{s}_1}^{s_1} \mu_2' \left(\bar{s}_2 - (\theta - \bar{s}_1) \frac{k_2}{k_1} \right) d\theta \\ &= \int_{\bar{s}_1}^{s_1} \left(\mu_1'(\theta) + \frac{k_2}{k_1} \mu_2' \left(\bar{s}_2 - (\theta - \bar{s}_1) \frac{k_2}{k_1} \right) \right) d\theta, \end{aligned}$$

and by means of Assumption A3 it follows that

$$(s_1 - \bar{s}_1) \int_{\bar{s}_1}^{s_1} \left(\mu_1'(\theta) + \frac{k_2}{k_1} \mu_2' \left(\bar{s}_2 - (\theta - \bar{s}_1) \frac{k_2}{k_1} \right) \right) d\theta \geq 0.$$

From this inequality and from (16) we obtain that

$$\dot{W}_1(s_1, x_1) \leq 0 \tag{17}$$

for each point $(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}_2$.

Let us fix an arbitrary point (ζ^0, β^0) from $\widetilde{\Omega}$. The invariance of the bounded set $\widetilde{\Omega}$ with respect to the trajectories of Σ_1 implies that the ω-limit set $L^+(\zeta^0, \beta^0)$ is a nonempty compact connected invariant set. It follows from the LaSalle invariance principle that $L^+(\zeta^0, \beta^0)$ is a subset of $\widetilde{\Omega}_2$. Using (17) it is possible to obtain a better estimate of the location of $L^+(\zeta^0, \beta^0)$. Namely, Theorem 6 from [3] implies that $L^+(\zeta^0, \beta^0)$ is contained in one connected component of the set

$$L^\infty := \left\{ (s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) \in \widetilde{\Omega}_2 : \ \dot{W}_1(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) = 0 \right\}.$$

Let $(\tilde{\zeta}, \tilde{\beta})$ be an arbitrary point from $L^+(\zeta^0, \beta^0)$. Then the following two cases are possible: (i) $\tilde{s}_2 = 0$ and $\tilde{x}_1 = 0$; (ii) $\tilde{s}_1 = \bar{s}_1$ and $\tilde{x}_1 = \bar{x}_1$.

Assume that case (i) holds true. Then $\tilde{s}_1 = \dfrac{k_1}{k_2} \bar{s} > 0$, and hence $\mu_1(\tilde{s}_1) > 0 = \chi_1(\tilde{s}_1) = \dfrac{1}{\alpha} \mu_2(0)$. The continuity of $\mu_1(\cdot)$ and $\chi_1(\cdot)$ implies the existence of a neighborhood U_3 of $\tilde{\zeta}$ so that for each point $\zeta \in U_3$ the inequality $\mu_1(s_1) > \chi_1(s_1)$ holds true. Hence, if a trajectory $(s_1(t), x_1(t), s_2(t), x_2(t), \beta(t))$ of Σ_1 enters the set U_3, then the right-hand side of equation (2) in Σ_1 is $x_1(t)(\mu_1(s_1(t)) - \chi_1(s_1(t))) > 0$ (because $x_1(t) > 0$). This means that $x_1(t)$ will strictly increase and the trajectory could not tend to the point $\tilde{\zeta}$ for which $\tilde{x}_1 = 0$. So we obtain that the case (i) is impossible. Hence the case (ii) is fulfilled. From here it follows that $L^\infty = \{(\bar{\zeta}, \bar{\beta})\}$, and therefore $L^+(\zeta^0, \beta^0) = \{(\bar{\zeta}, \bar{\beta})\}$.

Finally, we shall show that the point $(\bar{\zeta}, \bar{\beta})$ is a Lyapunov stable equilibrium of Σ_1.

Let $\varepsilon > 0$ be an arbitrary real number and $(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta})$ be an arbitrary point from the set $\widetilde{\Omega}_2$. Assume that $W_1(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) = W_1(s_1, x_1) = \varepsilon$. Then the inequalities (14) and (17), the continuity of the function W_1 and of the right-hand side of the closed-loop system Σ_1 imply the existence of $\tilde{\varepsilon}$ with $0 < \tilde{\varepsilon} < \varepsilon$, such that the Lie derivative $\dot{W}_1(s_1, x_1, s_2, x_2, \beta)$ of W_1 with respect to Σ_1 at the point (ζ, β) is negative whenever $V_1(\zeta, \beta) \le \tilde{\varepsilon}$. Let us choose an arbitrary sufficiently small positive real number $\varepsilon > 0$, determine as above the number $\tilde{\varepsilon}$ with $0 < \tilde{\varepsilon} < \varepsilon$ and define the set

$$\begin{aligned} K(\varepsilon) &= \left\{ (s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega} \ : \ W_1(s_1, x_1, \bar{s} - \frac{k_2}{k_1} s_1, \bar{x}_2, \bar{\beta}) \le \varepsilon \right\} \\ & \qquad \bigcap \left\{ (s_1, x_1, s_2, x_2, \beta) \in \widetilde{\Omega} : V_1(s_1, x_1, s_2, x_2, \beta) \le \tilde{\varepsilon} \right\}. \end{aligned}$$

Using the definitions of $V_1(\zeta, \beta)$ and $W_1(s_1, x_1)$ one can directly check that there exists a constant $\varrho > 0$ such that the ball $B_{\varrho\sqrt{\varepsilon}}(\bar{\zeta}, \bar{\beta})$ centered at the point $(\bar{\zeta}, \bar{\beta})$ with radius $\varrho\sqrt{\varepsilon}$ contains the set $K(\varepsilon)$. Clearly, the inequalities (14) and (17) imply that the set $K(\varepsilon)$ is invariant with respect to the trajectories of Σ_1. Moreover, the equalities $V_1(\bar{\zeta}, \bar{\beta}) = W_1(\bar{\zeta}, \bar{\beta}) = 0$ and the continuity of these functions imply the existence of $\delta > 0$ so that for each point $(\zeta, \beta) \in B_\delta(\bar{\zeta}, \bar{\beta})$ the following inequalities hold true: $V_1(\zeta, \beta) \le \tilde{\varepsilon}$ and $W_1(\zeta, \beta) \le \varepsilon$. This shows that each trajectory of the closed-loop system Σ_1 starting from a point of $B_\delta(\bar{\zeta}, \bar{\beta})$ will remain forever in the ball $B_{\varrho\sqrt{\varepsilon}}(\bar{\zeta}, \bar{\beta})$, i. e. the point $(\bar{\zeta}, \bar{\beta})$ is a Lyapunov stable equilibrium point for the closed-loop system Σ_1. This completes the proof of Theorem 1. $\square$

Remark 1. The external equation (11) can be rewritten in the form

$$\frac{d\beta}{dt}(t) = -C \cdot (\beta(t) - \beta^-) \cdot (\beta^+ - \beta(t)) \cdot Q(t) \cdot (s(t) - \bar{s}), \tag{18}$$

where $Q(t)$ denotes the methane flow rate measured at time t. As mentioned before, $Q(t)$ and $s(t)$ in (18) are online measurable, therefore the values of the solution $\beta(t)$ can be determined online as well. Since the solution of (18) depends on $\bar{s}$, we denote it by $\beta_{\bar{s}}(t)$, $t \in [0, +\infty)$. Then the feedback control law $\kappa_1(\cdot, \cdot)$ can be presented in the form

$$(s, Q, \beta_{\bar{s}}) \longmapsto \kappa_1(s, Q, \beta_{\bar{s}}) = \beta_{\bar{s}}\, Q. \tag{19}$$

According to Theorem 1, this feedback will asymptotically stabilize the control system (1)–(4), (18) to the point $(\bar{\zeta}, \bar{\beta}_{\bar{s}})$ with $\bar{\beta}_{\bar{s}} = \dfrac{k_3}{k_4(s^i - \bar{s})}$. The feedback (19) will be used in the computer simulations in Section 7.

4. Nonadaptive asymptotic stabilization. In the previous section we have constructed an adaptive feedback law for stabilizing the dynamics (1)–(4). The application of this feedback is however not a trivial task and different technical difficulties have to be overcome, mainly caused by the lack of interpretation of the external equation (11) in terms of the real process.

In this section we propose a state (nonadaptive) feedback of the form $u \equiv \kappa_2(s_2, x_2) = \beta k_4 \mu_2(s_2) x_2$, where β does not depend on time, but is a properly chosen positive parameter.

Consider the control system (1)–(4) in the state space $\zeta = (s_1, x_1, s_2, x_2)$. Using the definition of s^i from (6) we make the following assumption:

Assumption A5. *Lower bounds s^{i-} and k_4^- for the values of s^i and k_4 respectively, as well as an upper bound k_3^+ for the value of k_3 are known.*

Define the following feedback control law:

$$\kappa_2(\zeta) = \beta\, k_4\, \mu_2(s_2)\, x_2 \quad \text{with} \quad \beta \in \left(\frac{k_3^+}{s^{i-} \cdot k_4^-},\ +\infty \right). \tag{20}$$

Denote by Σ_2 the closed-loop system obtained from (1)–(4) by substituting the control variable u by the feedback $\kappa_2(\zeta)$ from (20).

Choose some $\beta \in \left(\dfrac{k_3^+}{s^{i-} \cdot k_4^-},\ +\infty \right)$ and let $\bar{\xi} = s^i - \dfrac{k_3}{\beta k_4}$; obviously, $\bar{\xi}$ belongs to the interval $(0, s^i)$. The next assumption is similar to the regulability Assumption A2 with $\bar{\xi}$ instead of $\bar{s}$:

Assumption A2′. *There exists a point $\bar{s}_1$ such that*

$$\mu_1(\bar{s}_1) = \mu_2\left(\bar{\xi} - \frac{k_2}{k_1} \bar{s}_1 \right) > 0, \quad \bar{s}_1 \in \left(0, s_1^i\right).$$

Find $\bar{s}_1$ according to Assumption A2′ and define

$$\bar{s}_2 = \bar{\xi} - \frac{k_2}{k_1}\bar{s}_1, \quad \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad \bar{x}_2 = \frac{1}{\alpha \beta k_4}. \tag{21}$$

It is straightforward to see that the point

$$\bar{p}_\beta = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is an equilibrium point for Σ_2. We shall prove below that the feedback law (20) stabilizes asymptotically the closed-loop system to $\bar{p}_\beta$ (cf. [9]).

Let the sets Ω_0, Ω_1, Ω be defined according to (9), and let

$$\Omega_3 = \left\{ \left(s_1, x_1, \bar{\xi} - \frac{k_2}{k_1} s_1, \bar{x}_2 \right) \mid 0 < s_1 < \frac{k_1}{k_2}\, \bar{\xi},\ x_1 > 0 \right\}.$$

Theorem 2. *Let Assumptions A1, A2′, A3 and A5 be satisfied. Let us fix an arbitrary number* $\beta \in \left(\frac{k_3^+}{s^{i-} \cdot k_4^-}, +\infty \right)$ *and let* $\bar{p}_\beta = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$ *be the corresponding equilibrium point. Then the feedback control law* $\kappa_2(\cdot)$ *defined by* (20) *stabilizes asymptotically the control system* (1)–(4) *to the point* $\bar{p}_\beta$ *for each starting point* $\zeta^0 = (s_1^0, x_1^0, s_2^0, x_2^0)$ *from* Ω_0.

Proof. The proof follows the main steps in the proof of Theorem 1.

Similar arguments as in the beginning of the proof of Theorem 1 show that we can consider the closed-loop system Σ_2 only in the set Ω.

Let s be defined according to (6), i. e. $s = \frac{k_2}{k_1} s_1 + s_2$. Then it is not difficult to see that the following ordinary differential equations are satisfied:

$$\begin{aligned} \frac{ds}{dt} &= -\beta k_4 \mu_2(s_2) x_2 (s - \bar{\xi}) \\ \frac{dx_2}{dt} &= -\alpha\beta k_4 \mu_2(s_2) x_2 (x_2 - \bar{x}_2). \end{aligned} \tag{22}$$

Integrating the equations (22), one obtains

$$\begin{aligned} s(t) &= \bar{\xi} + (s(0) - \bar{\xi}) e^{-\int_0^t \beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau} \\ x_2(t) &= \bar{x}_2 + (x_2(0) - \bar{x}_2) e^{-\int_0^t \alpha\beta k_4 \mu_2(s_2(\tau)) x_2(\tau) d\tau}. \end{aligned}$$

Since the integrands are strictly positive, we have that for each $t > 0$ the following inequalities hold true

$$\begin{aligned} &\max\{s(0), \bar{\xi}\} \geq s(t) \geq \min\{s(0), \bar{\xi}\} > 0 \\ &\max\{x_2(0), \bar{x}_2\} \geq x_2(t) \geq \min\{x_2(0), \bar{x}_2\} > 0. \end{aligned} \tag{23}$$

Define the function

$$V_2(\zeta) = (s - \bar{\xi})^2 + (x_2 - \bar{x}_2)^2.$$

Clearly, the values of $V_2(\zeta)$ are nonnegative. If we denote by $\dot{V}_2(\zeta)$ the Lie derivative of V_2 with respect to the right-hand side of (22) then for each point $\zeta \in \Omega$ we obtain

$$\dot{V}_2(\zeta) = -\beta k_4 \mu_2(s_2) x_2 (s - \bar{\xi})^2 - \alpha\beta k_4 \mu_2(s_2) x_2 (x_2 - \bar{x}_2)^2 \leq 0.$$

Applying LaSalle's invariance principle (cf. [15]), we get that every solution of Σ_2 starting from a point of Ω is defined in the interval $[0, +\infty)$ and approaches the largest invariant set with respect to Σ_2, which is contained in the set Ω_∞, where Ω_∞ is the closure of the set $\Omega \cap \Omega_3$. Taking into account the estimates (23), the dynamics of Σ_2 on Ω_∞ can be described in the following way

$$(24)\qquad \begin{aligned} \frac{ds_1}{dt} &= \frac{1}{\alpha}\,\chi_2(s_1)(s_1^i - s_1) - k_1\mu_1(s_1)x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi_2(s_1))x_1, \end{aligned}$$

where $\chi_2(s_1) = \mu_2\left(\bar{\xi} - \frac{k_2}{k_1}s_1\right)$ (remind that $\bar{\xi} = s^i - \frac{k_3}{\beta k_4}$). According to (21) we have that $\bar{\xi} = \frac{k_2}{k_1}\bar{s}_1 + \bar{s}_2$ and $s_1^i = \bar{s}_1 + \alpha k_1\bar{x}_1$. Then (24) can be rewritten as follows:

$$\begin{aligned} \frac{ds_1}{dt} &= -\frac{1}{\alpha}\,\chi_2(s_1)\cdot(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1)) - k_1\left(\mu_1(s_1) - \chi_2(s_1)\right)\cdot x_1 \\ \frac{dx_1}{dt} &= (\mu_1(s_1) - \chi_2(s_1))\cdot x_1. \end{aligned}$$

Consider the function

$$(25)\qquad W_2(s_1,x_1) = (s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 + \alpha(1-\alpha)k_1^2(x_1 - \bar{x}_1)^2.$$

This function takes nonnegative values. It is straightforward to see that for each point $\left(s_1, x_1, \bar{\xi} - \frac{k_2}{k_1}s_1, \bar{x}_2\right) \in \Omega_\infty$,

$$\begin{aligned} \dot{W}_2(s_1,x_1) &= -\frac{2}{\alpha}\chi_2(s_1)(s_1 - \bar{s}_1 + \alpha k_1(x_1 - \bar{x}_1))^2 \\ &\quad -2(1-\alpha)k_1x_1(s_1 - \bar{s}_1)(\mu_1(s_1) - \chi_2(s_1)). \end{aligned}$$

Assumptions A3 and A5 imply

$$\mu_1(s_1)-\chi_2(s_1) = \mu_1(s_1)-\mu_2\left(\bar{\xi} - \frac{k_2}{k_1}s_1\right) = \int_{\bar{s}_1}^{s_1}\left(\mu_1'(\theta) + \frac{k_2}{k_1}\mu_2'\left(\bar{s}_2 - (\theta - \bar{s}_1)\frac{k_2}{k_1}\right)\right)d\theta,$$

and therefore

$$\dot{W}_2(s_1,x_1) \le 0$$

for each point $\left(s_1, x_1, \bar{\xi} - \frac{k_2}{k_1}s_1, \bar{x}_2\right)$ from Ω_∞.

Finally, to prove the Lyapunov stability of $\bar{p}_\beta$, we can use the same arguments as at the end of the proof of Theorem 1. This completes the proof of Theorem 2. □

Remark 2. The feedback control law $\kappa_2(\cdot)$ can be written in the form

$$(26)\qquad (Q,\beta) \longmapsto \kappa_2(Q,\beta) = \beta\cdot Q,$$

and according to Theorem 2, the latter will asymptotically stabilize the control system (1)–(4) to the point $\bar{p}_\beta$. This feedback (26) will be used in the computer simulations in Section 7.

5. Asymptotic stabilization using constant dilution rate. In the previous two sections the global stabilizability of the dynamics (1)–(4) depends on quantities external for the model – the function $\beta(t)$ in $\kappa_1(\cdot)$ and the parameter (scalar) β in $\kappa_2(\cdot)$, which makes the practical implementation of the feedback laws not easy.

This section is devoted to investigating the global stability of the model (1)–(4) without using any feedback control strategy.

In what follows we shall consider u varying in the interval

(27) $$u \in (0,\ u_b)\,,\ \text{where } u_b \le \frac{1}{\alpha}\ \min\left\{\mu_1(s_1^i),\ \mu_2(s_2^i)\right\}.$$

Let for any $\bar{u} \in (0, u_b)$ the following regulability assumption hold true:

Assumption A6. *There exist points* $s_1(\bar{u}) = \bar{s}_1 \in \left(0, s_1^i\right)$ *and* $s_2(\bar{u}) = \bar{s}_2 \in \left(0, s_2^i\right)$, *such that the following equalities hold true*

$$\bar{u} = \frac{1}{\alpha}\mu_1(\bar{s}_1) = \frac{1}{\alpha}\mu_2\left(\bar{s}_2\right).$$

Define the points $\bar{s}_1$ and $\bar{s}_2$ according to Assumption A6 and further

(28) $$x_1(\bar{u}) = \bar{x}_1 = \frac{s_1^i - \bar{s}_1}{\alpha k_1}, \quad x_2(\bar{u}) = \bar{x}_2 = \frac{s_2^i - \bar{s}_2 + \alpha k_2 \bar{x}_1}{\alpha k_3}.$$

Then the point

$$p(\bar{u}) = \bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$$

is an equilibrium point for the system (1)–(4).

The last assumption is

Assumption A7. *The following inequalities hold true*

$$\mu_1(s_1^-) < \mu_1(\bar{s}_1) < \mu_1(s_1^+), \quad \mu_2(s_2^-) < \mu_2(\bar{s}_2) < \mu_2(s_2^+)$$

for each $s_1^- \in (0, \bar{s}_1)$, $s_1^+ \in (\bar{s}_1, s_1^i)$, $s_2^- \in (0, \bar{s}_2)$ *and* $s_2^+ \in (\bar{s}_2, s_2^i + \alpha k_2 \bar{x}_1)$.

This assumption is technical. It is always fulfilled when the functions $\mu_j(\cdot)$, $j = 1, 2$, are monotone increasing (like the Monod specific growth rate, see Section 7). If the function $\mu_j(\cdot)$ is not monotone increasing (like the Haldane law, see Section 7) then the points $\bar{s}_j$ (i. e. $\bar{u}$) have to be chosen sufficiently small in order to satisfy Assumption A7.

The following Lemma will be used further.

Barbălat's Lemma (cf. [12])**.** *If* $f : (0, \infty) \to R$ *is Riemann integrable and uniformly continuous, then* $\lim_{t\to\infty} f(t) = 0$.

Let $\bar{u} \in (0,\ u_b)$ be chosen according to Assumptions A6 and A7. Denote by Σ_3 the system obtained from (1)–(4) by substituting the control variable u by $\bar{u}$. We shall prove below that the equilibrium point $\bar{p}$ is globally asymptotically stable for Σ_3.

Theorem 3. *Let the Assumptions A1, A6, A7 be fulfilled and let* $p_0 = (s_1^0, x_1^0, s_2^0, x_2^0)$ *be an arbitrary point of the set* $\Omega_0 = \{(s_1, x_1, s_2, x_2) : s_1 > 0, x_1 > 0, s_2 > 0, x_2 > 0\}$. *Then the solution of* Σ_3 *starting from the point* p_0 *converges asymptotically towards* $\bar{p}$.

Proof. We fix an arbitrary point $p_0 = (s_1^0, x_1^0, s_2^0, x_2^0) \in \Omega_0$ and denote by $\phi(\cdot, p_0) = (s_1(\cdot), x_1(\cdot), s_2(\cdot), x_2(\cdot))$ the corresponding solution of Σ_3 starting from p_0. Clearly, $s_1(t) > 0$, $x_1(t) > 0$, $s_2(t) > 0$ and $x_2(t) > 0$ for each $t > 0$, where the solution is defined. Denote

(29) $$\sigma(t) = k_2 s_1(t) + k_1 s_2(t) \quad \text{and} \quad \sigma^i = k_2 s_1^i + k_1 s_2^i.$$

Then $\sigma(t)$ satisfies the differential equation

$$\dot{\sigma}(t) = \bar{u}(\sigma^i - \sigma(t)) - k_1 k_3 \mu_2(s_2(t)) x_2(t).$$

We set

$$q_1(t) = \sigma(t) + k_1 k_3 x_2(t) - \sigma^i/\alpha \ \text{ and } \ q_2(t) = \sigma(t) + k_1 k_3 x_2(t) - \sigma^i.$$

One can directly check that

$$\dot{q}_1(t) = \bar{u}(\sigma^i - \sigma(t)) - \alpha\bar{u}k_1k_3x_2(t) \le \bar{u}\sigma^i - \alpha\bar{u}\,(\sigma(t) + k_1k_3x_2(t)) = -\alpha\bar{u}q_1(t),$$

and hence

$$q_1(t) \le q_1(0) \cdot e^{-\alpha t\,\bar{u}}. \tag{30}$$

The latter inequality shows that $q_1(t)$ is bounded. Using the fact that the values of $s_1(t)$, $s_2(t)$ and $x_2(t)$ are positive, we get that $s_1(t)$, $s_2(t)$ and $x_2(t)$ are bounded as well. Analogously one can obtain that

$$q_2(t) \ge q_2(0) \cdot e^{-t\,\bar{u}}. \tag{31}$$

The estimates (30), (31) and the definition of $\sigma(\cdot)$ from (29) imply that for each ε there exists $T_\varepsilon > 0$ such that for each $t \ge T_\varepsilon$ the following inequalities hold true

$$\sigma^i - \varepsilon < k_2s_1(t) + k_1s_2(t) + k_1k_3x_2(t) < \frac{\sigma^i}{\alpha} + \varepsilon. \tag{32}$$

It is easy to see (in the same way as the estimates (32)) that for each $\varepsilon > 0$ there exists a finite time $T_\varepsilon > 0$ such that for all $t \ge T_\varepsilon$ the following inequalities are valid (cf. [13])

$$s_1^i - \varepsilon < s_1(t) + k_1x_1(t) < \frac{s_1^i}{\alpha} + \varepsilon. \tag{33}$$

The inequalities (33) imply that $x_1(t)$ is also bounded. Thus the trajectory $\phi(t, p_0) = (s_1(t), x_1(t), s_2(t), x_2(t))$ is well defined and bounded for all $t \ge 0$.

Denote by $L^+(p)$ the ω-limit set of the solution $\phi(t, p)$ of Σ_3 starting from some point p, i. e.

$$L^+(p) = \big\{\tilde{p} \,:\, \text{there exists a sequence } \{t_n\} \to +\infty \\ \text{such that } \tilde{p} = \lim_{t_n\to+\infty} \phi(t_n, p)\big\}.$$

It is well known that $L^+(p_0)$ is invariant with respect to the trajectories of Σ_3, i. e. $\phi(t, p) \in L^+(p_0)$ for each point $p \in L^+(p_0)$ and for each real number t. Also (because the trajectory $\phi(\cdot, p_0)$ is contained in a compact set), $L^+(p_0)$ is nonempty, compact and connected.

Suppose now that $s_1(t) \ge s_1^i$ for each $t \ge 0$. Then we have

$$\dot{s}_1(t) = \bar{u}(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t) < 0.$$

According to Barbălat's Lemma we obtain

$$0 = \lim_{t\to\infty} \dot{s}_1(t) = \lim_{t\to\infty} [\bar{u}(s_1^i - s_1(t)) - k_1\mu_1(s_1(t))x_1(t)].$$

The last equalities imply that $s_1(t) \to s_1^i$ and $x_1(t) \to 0$ as $t \to \infty$.

According to Assumption A7 we have that $\mu_1(\bar{s}_1) < \mu_1(s_1^i)$. The continuity of $\mu_1(\cdot)$ and the relation $s_1(t) \to s_1^i$ as $t \to \infty$ imply that there exists $\nu > 0$ such that

$$\mu_1(s_1(t)) - \alpha\bar{u} = \mu_1(s_1(t)) - \mu_1(\bar{s}_1) \ge \nu$$

for all sufficiently large t. It follows then that $\dot{x}_1(t) = (\mu_1(s_1(t)) - \alpha\bar{u})x_1(t) \ge \nu x_1(t)$ for all sufficiently large t, which contradicts the boundedness of $x_1(t)$. Hence, there exists a sufficiently large $T_1 > 0$ with $s_1(T_1) \le s_1^i$. Moreover, if the equality $s_1(T_1) = s_1^i$ holds true, then we have

$$\dot{s}_1(T_1) = \bar{u}(s_1^i - s_1(T_1)) - k_1\mu_1(s_1(T_1))x_1(T_1) = -k_1\mu_1(s_1(T_1))x_1(T_1) < 0.$$

The last inequality shows that $s_1(t) < s_1^i$ for each $t > T_1$. Hence,

$$L^+(p_0) \subset \{(s_1, x_1, s_2, x_2) :\ 0 \le s_1(t) \le s_1^i, 0 \le x_1, 0 \le s_2, 0 \le x_2\}. \tag{34}$$

Let us fix an arbitrary $\bar{\eta} \in (0, (\mu_1(s_1^i) - \mu_1(\bar{s}_1))/2)$ (Assumption A7 implies that $\mu_1(\bar{s}_1) < \mu_1(s_1^i)$). Then we obtain (from the continuity of $\mu_1(\cdot)$) the existence of a small $\bar{\varepsilon} > 0$ such that $\mu_1(\bar{s}_1) + \bar{\eta} < \mu_1(s_1)$ for each $s_1 \in [s_1^i - (1 + k_1)\bar{\varepsilon}, s_1^i)$. If we assume that $x_1(\bar{t}) \le \bar{\varepsilon}$ for some $\bar{t} \ge T_2 := \max\{T_1, T_{\bar{\varepsilon}}\}$, then according to the inequalities (33) and the choice of $\bar{\varepsilon} > 0$, we obtain

$$s_1^i > s_1(\bar{t}) \ge s_1^i - k_1 x_1(\bar{t}) - \bar{\varepsilon} \ge s_1^i - (1 + k_1)\bar{\varepsilon},$$

i. e. $s_1(\bar{t}) \in [s_1^i - (1 + k_1)\bar{\varepsilon}, s_1^i)$. Therefore

$$\dot{x}_1(\bar{t}) = (\mu_1(s_1(\bar{t})) - \alpha\bar{u})x_1(\bar{t}) = (\mu_1(s_1(\bar{t})) - \mu_1(\bar{s}_1))x_1(\bar{t}) \ge \bar{\eta}\, x_1(\bar{t}) > 0.$$

The latter inequality implies that $x_1(t) \ge e^{(t-\bar{t})\,\bar{\eta}} x_1(\bar{t})$. Hence there exists a moment of time $\bar{t}_1 \ge \bar{t}$ such that $x_1(\bar{t}_1) = \bar{\varepsilon}$. Clearly, at every time $\bar{t}_2 \ge \bar{t}_1$ with $x_1(\bar{t}_2) = \bar{\varepsilon}$ the following inequality holds true

$$\dot{x}_1(\bar{t}_2) = (\mu_1(s_1(\bar{t}_2)) - \alpha\bar{u})x_1(\bar{t}_2) \ge \bar{\eta}\,\bar{\varepsilon} > 0.$$

Thus there exists $T_3 > T_2$ such that $x_1(t) \ge \bar{\varepsilon}$ for each $t \ge T_3$. This and (34) imply that

$$L^+(p_0) \subset \{(s_1, x_1, s_2, x_2) :\ 0 \le s_1 \le s_1^i, \bar{\varepsilon} \le x_1, 0 \le s_2, 0 \le x_2\}. \tag{35}$$

Following [16] we define the following weak Lyapunov-type function:

$$V_3(s_1, x_1) = \int_{\bar{s}_1}^{s_1} \frac{\bar{x}_1(\mu_1(\xi) - \alpha\bar{u})}{\bar{u}(s_1^i - \xi)}\, d\xi + \int_{\bar{x}_1}^{x_1} \frac{\eta - \bar{x}_1}{\eta}\, d\eta$$

for each $s_1 \in (0, s_1^i)$ and $x_1 > 0$. Clearly

$$\begin{aligned}
\dot{V}_3(s_1, x_1) &= \frac{\bar{x}_1(\mu_1(s_1) - \alpha\bar{u})}{\bar{u}(s_1^i - s_1)}\big(\bar{u}(s_1^i - s_1) - k_1\mu_1(s_1)x_1\big) + \frac{x_1 - \bar{x}_1}{x_1}(\mu_1(s_1) - \alpha\bar{u})x_1 \\
&= x_1(\mu_1(s_1) - \alpha\bar{u})\left(1 - \frac{\bar{x}_1 k_1 \mu_1(s_1)}{\bar{u}(s_1^i - s_1)}\right) \\
&= x_1(\mu_1(s_1) - \alpha\bar{u})\left(1 - \frac{s_1^i - \bar{s}_1}{s_1^i - s_1} \cdot \frac{\mu_1(s_1)}{\alpha\bar{u}}\right) \le 0
\end{aligned}$$

for each $s_1 \in (0, s_1^i)$ and $x_1 > 0$. Moreover, according to Assumption A6 we have that $\dot{V}_3(s_1, x_1) = 0$ iff $s_1 = \bar{s}_1$ or $x_1 = 0$.

Applying the LaSalle invariance principle [15] and taking into account (35), it follows that the set $L^+(p_0)$ is a subset of the largest invariant set with respect to Σ_3, which is contained in the set

$$\begin{aligned}
L_1^\infty &:= \left\{(s_1, x_1, s_2, x_2) : \dot{V}_3(s_1, x_1) = 0, 0 \le s_1 \le s_1^i, \bar{\varepsilon} \le x_1, 0 \le s_2, 0 \le x_2\right\} \\
&= \{(\bar{s}_1, \bar{x}_1, s_2, x_2) :\ 0 \le s_2, 0 \le x_2\},
\end{aligned}$$

i.e. $L^+(p_0) \subset L_1^\infty$.

The dynamics of Σ_3 on L_1^∞ can be described by the following system of ordinary differential equations

$$\frac{ds_2}{dt} = \bar{u}(s_2^i - s_2) + k_2\mu_1(\bar{s}_1)\bar{x}_1 - k_3\mu_2(s_2)x_2$$
$$\frac{dx_2}{dt} = (\mu_2(s_2) - \alpha\bar{u})x_2,$$

or equivalently by

$$(36) \qquad \begin{aligned} \frac{ds_2}{dt} &= \bar{u}(\bar{s}_2^i - s_2) - k_3\mu_2(s_2)x_2 \\ \frac{dx_2}{dt} &= (\mu_2(s_2) - \alpha\bar{u})x_2 \end{aligned}$$

with $\bar{s}_2^i = s_2^i + \alpha k_2 \bar{x}_1$. Proceeding in the same way as above we obtain that there exists $T_4 > T_3$ and a small number $\bar{\varepsilon}_2 > 0$ such that

$$(37) \qquad s_2(t) < \bar{s}_2^i \text{ and } x_2(t) \geq \bar{\varepsilon}_2 \text{ for each } t \geq T_4.$$

Consider the function

$$W_3(s_2, x_2) = \int_{\bar{s}_2}^{s_2} \frac{\bar{x}_2(\mu_2(\xi) - \alpha\bar{u})}{\bar{u}(\bar{s}_2^i - \xi)}\, d\xi + \int_{\bar{x}_2}^{x_2} \frac{\eta - \bar{x}_2}{\eta}\, d\eta.$$

Straightforward calculations deliver that $\dot{W}_3(s_2, x_2) \leq 0$ for each $s_2 \in (0, \bar{s}_2^i]$ and $x_2 \geq 0$. Applying again the LaSalle invariance principle and taking into account the estimates (37), we obtain that each trajectory of (36) starting from a point of L_1^∞ is defined in the interval $[0, +\infty)$ and approaches the point $\bar{p} = (\bar{s}_1, \bar{x}_1, \bar{s}_2, \bar{x}_2)$. Applying again Theorem 6 from [3], we obtain that $L^+(p_0) = \{\bar{p}\}$. To prove the Lyapunov stability of system Σ_3 at the point $\bar{p}$, we can use the same arguments as at the end of the proof of Theorem 1. This completes the proof of Theorem 3. □

6. Optimizing the biogas production via extremum seeking. Consider the equation (5) describing the process output, i.e. the biogas production. In this section we shall apply a numerical extremum seeking algorithm (cf. [7]–[10]) to steer and stabilize the dynamics (1)–(4) towards a steady state, where maximum methane flow rate $Q_{\max}$ is achieved. For that purpose we compute Q on the set of all equilibrium points, parameterized with respect to: (i) $\bar{s}$ in the case of the adaptive feedback (see (7) and Theorem 1), (ii) β in the case of the nonadaptive control (see (20) and Theorem 2), and (iii) u in the case of constant dilution rate (see (27) and Theorem 3). Denote the so obtained function by $Q(q)$, where $q \in (0, q_b)$ denotes one of $\bar{s}$, β or u, and $q_b > 0$ is the corresponding upper bound according to Theorem 1, Theorem 2 or Theorem 3. The function $Q(q)$ is called *input-output static characteristic of the model.* Assume that the function $q \to Q(q)$, $q \in (0, q_b)$, is strongly unimodal, i. e. there exists a unique point $q_{\max} \in (0, q_b)$ where $Q(q)$ takes a maximum, i. e. $Q_{\max} = Q(q_{\max})$, the function strongly increases in the interval $(0, q_{\max})$ and strongly decreases in $(q_{\max}, q_b)$.

Denote by $E(q)$ the equilibrium point parameterized on q and let

$$E(q_{\max}) = (s_1^{\max}, x_1^{\max}, s_2^{\max}, x_2^{\max})$$

be the steady state where $Q_{\max}$ is achieved. Our goal is to stabilize in real time the system (1)–(4) towards this (unknown) equilibrium point $E(q_{\max})$ and therefore to the maximum methane flow rate $Q_{\max}$. This is realized by applying a numerical model-based extremum seeking algorithm. The algorithm is developed in three modes: for the adaptive feedback control, for the nonadaptive feedback, and in the case of constant dilution rate u. In all cases the control laws have been used for global stabilization of the dynamics (1)–(4) towards the desired steady state.

The main idea of the algorithm is the following: we construct a sequence of points $q^1, q^2, \ldots, q^n, \ldots$ from the interval $(0, q_b)$, each q^j being in the form $q^j = q^{j-1} \pm h$, $h > 0$, and such that the sequence $\{q^j\}$ tends to $q_{\max}$; Theorems 1 to 3 guarantee that the dynamics is globally asymptotically stabilizable towards the equilibrium $E(q^j)$, $j = 1, 2, \ldots$. Then by computing and comparing the values $Q(q^1), Q(q^2), \ldots, Q(q^n), \ldots$, we achieve the desired equilibrium point $E(q_{\max})$ and thus $Q_{\max}$.

The extremum seeking algorithm is implemented in the computer algebra system *Maple*. The algorithm is carried out in two stages. In the first stage, "rough" intervals $[q]$ and $[Q]$ are found which enclose $q_{\max}$ and $Q_{\max}$ respectively; in the second stage, the interval $[q]$ is refined using an elimination procedure based on the golden mean value (or Fibonacci search) strategy. The second stage produces the final intervals $[q_{\max}] = [q_{\max}^-, q_{\max}^+]$ and $[Q_{\max}]$ such that $q_{\max} \in [q_{\max}]$, $Q_{\max} \in [Q_{\max}]$ and $q_{\max}^+ - q_{\max}^- \leq \epsilon$, where the tolerance $\epsilon > 0$ is specified by the user.

7. Numerical simulation. Let $\mu_1(s_1)$ and $\mu_2(s_2)$ be the Monod and the Haldane specific growth rates respectively (cf. [1], [4], [13], [14])

$$\mu_1(s_1) = \frac{m_1 s_1}{k_{s_1} + s_1}, \qquad \mu_2(s_2) = \frac{m_2 s_2}{k_{s_2} + s_2 + \left(\dfrac{s_2}{k_I}\right)^2}, \tag{38}$$

where m_1, k_{s_1}, m_2, k_{s_2} and k_I are positive coefficients (see Table 1).

Usually the formulation of the specific growth rates is based on experimental results, and therefore it is not possible to have an exact analytic form of these functions, but only some quantitative bounds, see Figure 1.

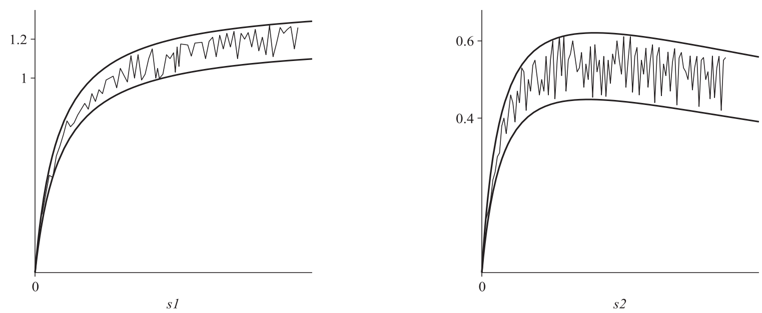

Fig. 1. Uncertain specific growth rates

This uncertainty can be simulated by assuming in (38) that instead of exact values for the coefficients m_1, k_{s_1}, m_2, k_{s_2} and k_I we have compact intervals for them.

To demonstrate the robustness of the extremum seeking algorithm (and thus of the three control strategies) with respect to model uncertainties, we assume that all model coefficients are unknown but bounded by given compact intervals: $k_j \in [k_j] = [k_j^-, k_j^+]$, $j = 1, 2, 3, 4$; $m_j \in [m_j]$, $k_{s_j} \in [k_{s_j}]$, $j = 1, 2$, $k_I \in [k_I]$.

In the simulation process we consider the following intervals for the coefficients:

$[m_1] = [1.2,\ 1.4],\ \ [k_{s_1}] = [6.5,\ 7.2],$

$[m_2] = [0.64,\ 0.84],\ \ [k_{s_2}] = [9,\ 10.28],\ \ [k_I] = [15,\ 17],$

$[k_1] = [9.5, 11.5],\ \ [k_2] = [27.6, 29.6],\ \ [k_3] = [1064, 1084],\ \ [k_4] = [650, 700].$

The above intervals enclose numerical coefficients values derived by experimental measurements [1]; we also use the values $\alpha = 0.5$, $s_1^i = 7.5$, $s_2^i = 75$ taken from [1]. The next Figure 2 shows the graphs of the input-output static characteristic $Q(q)$ in the three different cases using the specific growth rates μ_1 and μ_2 from (38); obviously, $Q(q)$ is strongly unimodal with respect to the parameter q, where q is one of $\bar{s}$, β and u.

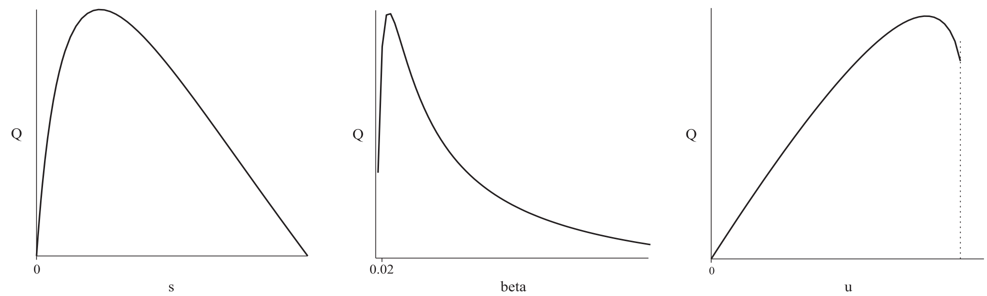

Fig. 2. The input-output static characteristic Q, parameterized on: $\bar{s}$ (left), β (middle), and u (right)

The computer simulations are carried out in two passes. First, at the initial time $t_0 = 0$, we choose random values for the coefficients in the corresponding intervals and apply the extremum seeking algorithm to stabilize the system towards $Q_{\max}$. Second, at some time $t_1 > t_0$, another set of random coefficient values is chosen and the process is repeated; thereby the last computed values for the phase variables $(s_1(t_1), x_1(t_1), s_2(t_1), x_2(t_1))$ are taken as initial conditions for the second run. The numerical results are visualized in Figures 3 to 7. The "jumps" in the solutions correspond to the choices of the points q^j, $j = 1, 2, \ldots$.

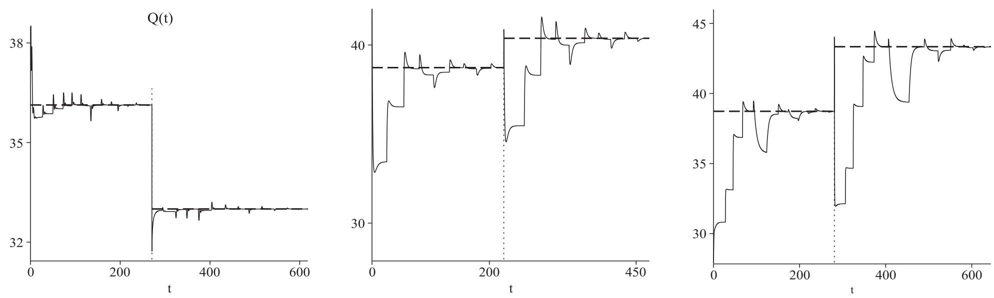

Fig. 3. Time profile of $Q(t)$ in the case of adaptive feedback (left), nonadaptive feedback (middle), and constant u (right). The horizontal dash lines pass through $Q_{\max}$

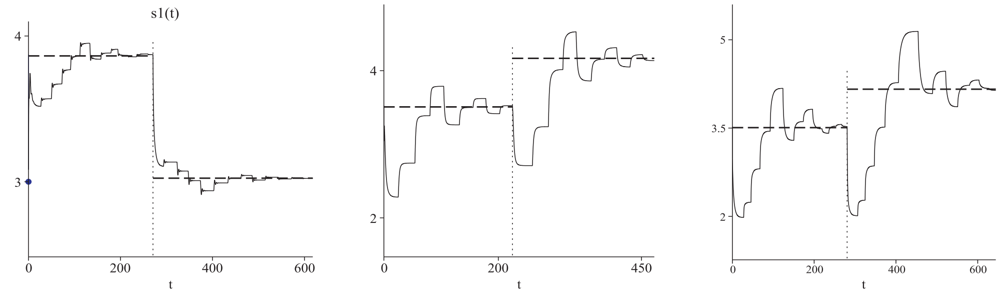

Fig. 4. Time profile of $s_1(t)$ in the case of adaptive feedback (left), nonadaptive feedback (middle), and constant u (right). The horizontal dash lines pass through $s_1^{\max}$

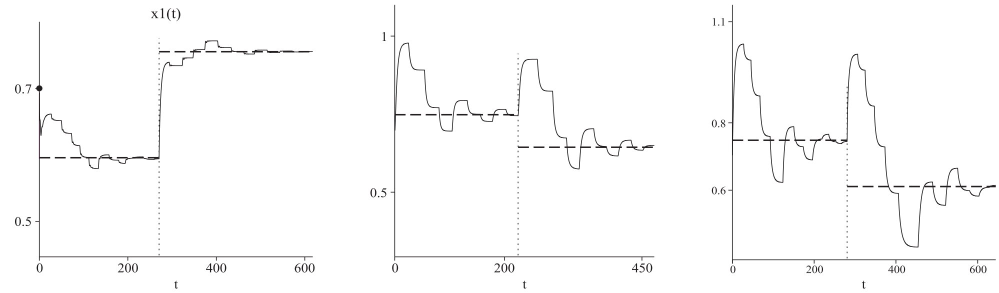

Fig. 5. Time profile of $x_1(t)$ in the case of adaptive feedback (left), nonadaptive feedback (middle), and constant u (right). The horizontal dash lines pass through $x_1^{\max}$

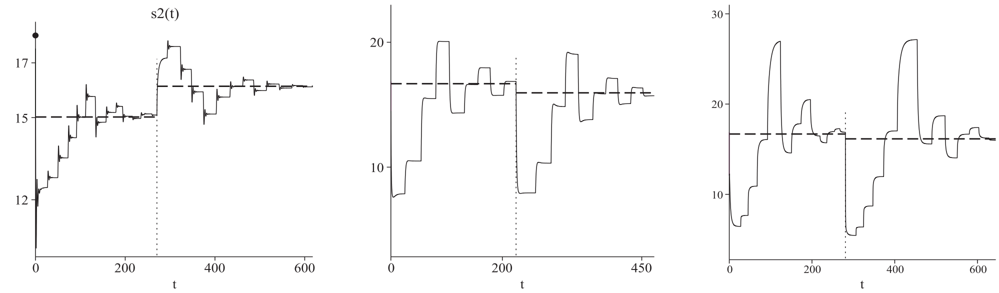

Fig. 6. Time profile of $s_2(t)$ in the case of adaptive feedback (left), nonadaptive feedback (middle), and constant u (right). The horizontal dash lines pass through $s_2^{\max}$

8. Conclusion. The paper presents an overview on different strategies for asymptotic stabilization of a four-dimensional nonlinear dynamic system, which models anaerobic degradation of organic wastes in a bioreactor with methane production. We propose two feedback laws – an adaptive, and a nonadaptive ones – and show that under some natural assumptions the model solutions are globally stabilizable towards an equilibrium point, say $E(q)$, chosen and fixed from the point of view of the practical applications. Since the two feedbacks depend on external for the model quantities, we further propose a simple solution of the global stabilizability problem, namely by using directly the input variable

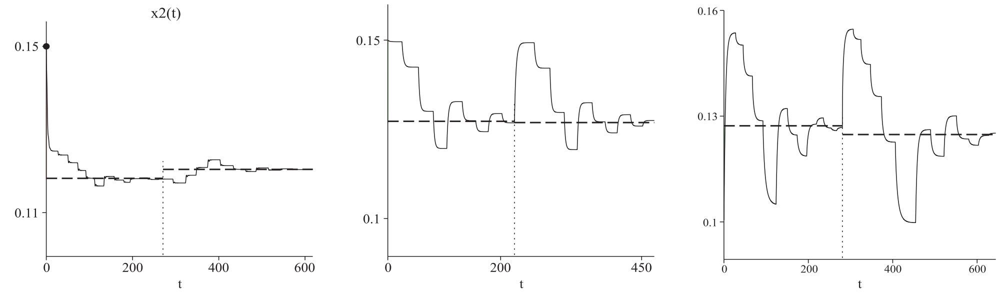

Fig. 7. Time profile of $x_2(t)$ in the case of adaptive feedback (left), nonadaptive feedback (middle), and constant u (right). The horizontal dash lines pass through $x_2^{\max}$

u. In practice, the dilution rate u can be manipulated directly by the experimenter, because u is proportional to the speed of the input pump, which feeds the bioreactor with substrate.

Using the fact that the equilibrium point $E(q)$ is globally asymptotically stable for any admissible value of the parameter $q \in (0, q_b)$, an iterative numerical extremum seeking algorithm is applied to deliver bounds (interval) for the maximum methane flow rate $[Q_{\max}]$. The interval $[Q_{\max}]$ can be made as tight as desired depending on a user specified tolerance. The theoretical results are illustrated numerically and the robustness of the extremum seeking algorithm is shown by assuming that the model coefficients are unknown but bounded within given intervals.

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ГЛОБАЛНА АСИМПТОТИЧНА СТАБИЛИЗАЦИЯ НА МОДЕЛ НА АНАЕРОБЕН БИОРЕАКТОР С ДОБИВ НА МЕТАН

Нели Стоянова Димитрова, Михаил Иванов Кръстанов

В настоящата статия е разгледан четиримерен динамичен модел на биореактор, описващ биотехнологичен процес на анаеробно разграждане на органични отпадъци, придружен с производство на метан (биогаз). Предложени са различни стратегии за управление на модела, които водят до глобална асимптотична стабилизация на динамичната система към предварително избрана работна точка. Приложен е числен алгоритъм за търсене на екстремум, който оптимизира добива на метан в реално време. Представени са резултати от компютърни симулации като илюстрация на теоретичните изследвания.