

МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2026
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2026
*Proceedings of the Fifty-Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Tryavna, Bulgaria, April 5–9, 2026*

ON THE BANACH-MAZUR ELLIPSE

V. D. Babev¹, M. Ivanov², R. Nikolov

¹ Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", Sofia, Bulgaria

² Radiant Life Technologies Ltd., Nicosia, Cyprus

e-mails: ¹ babev@fmi.uni-sofia.bg, ² milen@radiant-life-technologies.com

We provide a new proof of Ader's [1] characterisation of the ellipse of minimal Banach-Mazur distance to the unit circle of a normed plane in terms of contact and extremal points. Our method reveals the relation of this problem to the Chebyshev alternance.

Keywords: Banach-Mazur distance, normed plane, distance ellipse, Chebyshev alternance

ЕЛИПСА НА БАНАХ-МАЗУР

В. Д. Бабев¹, М. Иванов², Р. Николов

¹ Факултет по математика и информатика, Софийски университет „Св. Климент Охридски“, София, България

² Radiant Life Technologies Ltd., Никозия, Кипър

emails: ¹ babev@fmi.uni-sofia.bg, ² milen@radiant-life-technologies.com

Представяме ново доказателство на характеризацията на Адер за елипсата с минимално разстояние на Банах–Мазур до единичната окръжност на нормирана равнина, която характеризация е, че контактните и екстремалните точки имат определено свойство, подобно на алтернанс. Нашият метод явно посочва връзката с чебишовия алтернанс.

Ключови думи: разстояние на Банах-Мазур, нормирана равнина, елипса на най-добро приближение, чебишов алтернанс

<https://doi.org/10.55630/мем.2026.55.009-018>

2020 Mathematics Subject Classification: Primary 46B20; Secondary 52A10, 41A50.

¹ Research partially supported by Grant No. 80-10-116 from 28.05.2025 of the Research Fund of Sofia University.

² Research partially supported by the Bulgarian National Science Fund under Grant No. KP-06-H92/6 (December 8, 2025).

1 Introduction

Recall that the Banach-Mazur distance between the isomorphic normed spaces X and Y is denoted by

$$(1) \quad d(X, Y) := \inf\{\|T\| \cdot \|T^{-1}\| : T : X \rightarrow Y \text{ isomorphism}\},$$

for details see e.g. [13, Section 6.1.1].

If X and Y are of one and the same finite dimension, then the minimum in (1) is attained.

An important characteristic of a finite dimensional space is its distance to the Euclidean space. Here we study this characteristic in the simplest case: the plane.

So, for a 2-dimensional normed space $(X, \|\cdot\|)$

$$d_2(X) := d(X, \mathbb{R}^2),$$

where \mathbb{R}^2 is considered with the standard Euclidean norm $|\cdot|$. It is easy to check that

$$(2) \quad d_2(X) = \min\{\|T\| : T : X \rightarrow \mathbb{R}^2, T(B_X) \supset B_{\mathbb{R}^2}\},$$

where B_X and S_X stand for the unit ball and the unit sphere of X respectively. Indeed, $T(B_X) \supset B_{\mathbb{R}^2}$ is equivalent to $\|T^{-1}\| \leq 1$, so we have that $d_2(X)$ is at most the right-hand side of (2). On the other hand, multiplying the operator T by a real constant does not change $\|T\| \cdot \|T^{-1}\|$, so we may assume that $\|T^{-1}\| = 1$ in (1).

Using *polar decomposition*, we can reduce (2). Note that if U is in $\mathbf{O}(2)$ – the group of isometries of \mathbb{R}^2 , that is, $U^{-1} = U^t$ if considered as a matrix – then $\|UT\| = \max_{x \in S_X} |UTx| =$

$\max_{x \in S_X} |Tx| = \|T\|$, and similarly $\|T^{-1}U^{-1}\| = \|T^{-1}\|$, so we can quotient by $\mathbf{O}(2)$. Denote by $\mathbf{PD}(X, \mathbb{R}^2)$ the operators from X to \mathbb{R}^2 that are positive definite symmetric matrices in any basis. By [4, Theorem 7.3.1, p.449] each isomorphism $T : X \rightarrow \mathbb{R}^2$ can be represented as $T = UT_1$, where $U \in \mathbf{O}(2)$ and $T_1 \in \mathbf{PD}(X, \mathbb{R}^2)$. Therefore,

$$(3) \quad d_2(X) = \min\{\|T\| : T \in \mathbf{PD}(X, \mathbb{R}^2), T(B_X) \supset B_{\mathbb{R}^2}\}.$$

We prove in a new way the following characterisation, which is essentially contained in [1, 10].

Theorem 1. Let $\dim X = 2$. There is a unique $\hat{T} \in \mathbf{PD}(X, \mathbb{R}^2)$ such that $\hat{T}(B_X) \supset B_{\mathbb{R}^2}$ and

$$(4) \quad \|\hat{T}\| = d_2(X).$$

\hat{T} is characterised by the following property:

There are $x_i \in S_X$, $i = 1, 2$, with $x_1 \neq \pm x_2$, and $y_i \in S_X$, $i = 1, 2$, with $y_1 \neq \pm y_2$, such that:

$$(5) \quad |\hat{T}x_i| = \|\hat{T}\|, \quad i = 1, 2,$$

$$(6) \quad |\hat{T}y_i| = 1, \quad i = 1, 2,$$

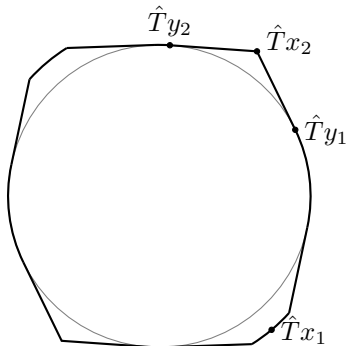
$$(7) \quad y_1 \in \text{cone}\{x_1, x_2\}, \quad y_2 \in \text{cone}\{-x_1, x_2\}.$$

Here

$$\text{cone}\{x, y\} := \{\alpha x + \beta y : \alpha, \beta \geq 0\}.$$

Note that the condition characterising the optimal operator depends only on the operator itself and not on $d_2(X)$. This is to be expected, because the problem suitably stated is convex. Not strictly convex, however, so the uniqueness is not immediate.

As an illustration, consider the following picture.



From it we can also see one important, but immediate, corollary. Since one of the angles between $\hat{T}y_1$ and $\hat{T}y_2$, and between $\hat{T}y_2$ and $-\hat{T}y_1$ (say the one between $\hat{T}y_1$ and $\hat{T}y_2$ as on the picture) will be $\leq \pi/2$; and since $\hat{T}x_2$ will be within the angle between the tangents to the unit circle at $\hat{T}y_1$ and $\hat{T}y_2$, due to convexity, the distance from $\hat{T}x_2$ to zero, that is $d_2(X)$, is at most $\sqrt{2}$. Therefore,

$$d_2(X) \leq \sqrt{2}.$$

Note that Theorem 1 easily follows from *Ader's Non-separation Principle* ([1]). For the unique \hat{T} there is no angle such that all points $x \in S_X$ with $|\hat{T}x| = \|\hat{T}\|$ are inside this angle and all points $y \in S_X$ with $|\hat{T}y| = 1$ are outside.

Here, an angle is

$$\text{cone}\{a, b\} \cup \text{cone}\{-a, -b\}$$

for some $a, b \in \mathbb{R}^2$.

The original work [1] states that there should be at most 4 points realising Ader's Non-separation Principle, and for these 4 points it is then clear that the alternance described in Theorem 1 must be in place. By Maurey's result [10], see also [3], the ellipse of minimal Banach-Mazur distance is unique, which yields the uniqueness part of Theorem 1.

Obviously, the problem can be re-stated more geometrically like: find the minimal $k \geq 1$ such that there is an ellipse inscribed in B_X whose homothetic image with ratio k is circumscribed about B_X . We will actually be using this form below, but we mention it here to point out the relationship to the celebrated Löwner–John ellipse, see e.g. [7], that is, the ellipse of maximal area inscribed in B_X . The Löwner–John ellipsoid is extensively used for estimating the Banach-Mazur distance to Euclidean space, probably because Ader's result [1] had apparently been forgotten until Grundbacher and Kobos recently unearthed it, see [3].

However, if B_X is not symmetric, the estimate through the Löwner–John ellipsoid can grow progressively worse with the dimension. Consider \mathbb{R}^n and let B_{X_n} be the convex hull of $B_{\mathbb{R}^n}$ and $\pm\sqrt{n}e$, where $e \in S_{\mathbb{R}^n}$. Then $B_{\mathbb{R}^n}$ is the Löwner–John ellipsoid for B_{X_n} , see [14] for a nice and simple proof, so it gives only the generic estimate \sqrt{n} . On the other hand, by the rotational symmetry of B_{X_n} around the axis through e , $d(X_n, \mathbb{R}^n) \leq \sqrt{3}$.

Our interest in the specifically two-dimensional case comes from our previous works [6, 5, 11, 12]. Of course, similar problems can also be considered in different metrics, like e.g. Hausdorff, see [8, 2, 9].

In conclusion, note that the convexity of B_X will not be used at all in the proofs

below, which is not surprising, as they go through Chebyshev approximation.

2 An equivalent problem in polar co-ordinates

In view of (3), Theorem 1 characterises the solution of the optimisation problem

$$\begin{cases} \|T\| \rightarrow \min \\ T \in \mathbf{PD}(X, \mathbb{R}^2), \quad \|T^{-1}\| \leq 1. \end{cases}$$

By substituting T^{-1} in place of T , this problem can be rewritten as

$$(8) \quad \begin{cases} \|T^{-1}\| \rightarrow \min \\ T \in \mathbf{PD}(\mathbb{R}^2, X), \quad \|T\| \leq 1, \end{cases}$$

which is more suited for our purposes. Theorem 1 is immediately equivalent to the following Proposition 1 with which we will be working from now on.

Proposition 1. The only solution to (8), say \tilde{T} , is characterised by the following property:

There are $u_i, v_i \in S_{\mathbb{R}^2}$, $i = 1, 2$, with $u_1 \neq \pm u_2$, $v_1 \neq \pm v_2$, $v_1 \in \text{cone}\{u_1, u_2\}$, $v_2 \in \text{cone}\{-u_1, u_2\}$, and

$$(9) \quad \|\tilde{T}u_i\| = 1/\|\tilde{T}^{-1}\|, \quad \|\tilde{T}v_i\| = 1, \quad i = 1, 2.$$

Let a co-ordinate system be fixed in X in which

$$(10) \quad S_X = \{r(\varphi)(\cos \varphi, \sin \varphi) : \varphi \in \mathbb{R}\},$$

for a strictly positive, π -periodic and continuous $r : \mathbb{R} \rightarrow \mathbb{R}^+$.

In these fixed coordinates each $T \in \mathbf{PD}(\mathbb{R}^2, X)$ is identified with a matrix

$$T = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a, c > 0, \quad \det T = ac - b^2 > 0.$$

Obviously,

$$x \in T(S_{\mathbb{R}^2}) \iff |T^{-1}x| = 1 \iff x^t T^{-2} x = 1.$$

That is, $T(S_{\mathbb{R}^2})$ is the ellipse:

$$a_1 x_1^2 + 2b_1 x_1 x_2 + c_1 x_2^2 = 1,$$

where

$$\begin{pmatrix} a_1 & b_1 \\ b_1 & c_1 \end{pmatrix} = T^{-2},$$

so $a_1, c_1 > 0$ and $b_1^2 < a_1 c_1$. In polar co-ordinates then $T(S_{\mathbb{R}^2}) = \{\rho(\varphi)(\cos \varphi, \sin \varphi) : \varphi \in \mathbb{R}\}$, where

$$(11) \quad \rho^{-2}(\varphi) = a_2 + b_2 \cos 2\varphi + c_2 \sin 2\varphi,$$

with $a_2 = (a_1 + c_1)/2$, $b_2 = (a_1 - c_1)/2$ and $c_2 = b_1$, so

$$(12) \quad a_2 > 0, \quad a_2^2 > b_2^2 + c_2^2.$$

Usually this is expressed in terms of $\cos 2(\varphi - \theta)$, where θ is the tilt angle. Let \mathcal{E} be the family of all ellipses in polar co-ordinates, that is, all curves in the form (11) for all values of the parameters a_2, b_2, c_2 which satisfy (12). It is clear that the cone (12) parametrises one-to-one the ellipses, as well as $\mathbf{PD}(\mathbb{R}^2, X)$, which we will use when establishing the uniqueness part of our main result. Let

$$(13) \quad \mathcal{C} := \{\log \rho : \rho \in \mathcal{E}\}.$$

Let also, see (10),

$$f := \log r.$$

Since $\|T\| \leq 1$ is equivalent to $T(S_{\mathbb{R}^2}) \subset B_X$, which in polar co-ordinates translates to

$$\rho \leq r \iff g := \log \rho \leq f,$$

the constraint in (8) becomes $g \in \mathcal{C}$, $g \leq f$. For the objective, since $\rho(\varphi)(\cos \varphi, \sin \varphi) \in T(S_{\mathbb{R}^2})$, we have $|T^{-1}(\rho(\varphi)(\cos \varphi, \sin \varphi))| = 1$, hence $|T^{-1}(\cos \varphi, \sin \varphi)| = 1/\rho(\varphi)$. For $x = r(\varphi)(\cos \varphi, \sin \varphi) \in S_X$,

$$|T^{-1}x| = \frac{r(\varphi)}{\rho(\varphi)} = \exp(f(\varphi) - g(\varphi)),$$

and the problem (8) is equivalent to

$$\begin{cases} \max_{\varphi} \exp(f(\varphi) - g(\varphi)) \rightarrow \min \\ g \in \mathcal{C}, \quad g \leq f. \end{cases}$$

Since the exponential is strictly monotonically increasing, Proposition 1 is equivalent to Proposition 2 below. There, for a π -periodic function h

$$\|h\|_{\infty} := \max_{\varphi} |h(\varphi)|.$$

Proposition 2. The optimisation problem

$$(14) \quad \begin{cases} \|f - g\|_{\infty} \rightarrow \min \\ g \in \mathcal{C}, \quad g \leq f \end{cases}$$

has a unique solution \bar{g} characterised by the following property:

There are points

$$(15) \quad \varphi_1 < \psi_1 < \varphi_2 < \psi_2 < \varphi_1 + \pi$$

such that for $i = 1, 2$

$$(16) \quad f(\varphi_i) - \bar{g}(\varphi_i) = \|f - \bar{g}\|_{\infty} \text{ and } \bar{g}(\psi_i) = f(\psi_i).$$

We will prove this proposition in the next section.

3 Chebyshev Alternance of second order on the circle

Let $C_{\pi}(\mathbb{R})$ be the Banach space of all continuous π -periodic functions on \mathbb{R} with the norm $\|\cdot\|_{\infty}$.

Let \mathfrak{A} be the function that maps the triple $(a, b, c) \in \mathbb{R}^3$ to $\mathfrak{A}(a, b, c) \in C_{\pi}(\mathbb{R})$ given by

$$\mathfrak{A}(a, b, c) := a + b \cos 2\varphi + c \sin 2\varphi,$$

see (11). So, if K is the cone given by (12) then

$$\mathcal{C} = \{-(1/2) \log \mathfrak{A}(a, b, c) : (a, b, c) \in K\}.$$

Obviously, \mathcal{C} is invariant under addition of a constant, that is,

$$(17) \quad \forall r \in \mathbb{R}, \forall g \in \mathcal{C} \Rightarrow g + r \in \mathcal{C}.$$

This fact allows considering the standard Chebyshev, that is, *uniform*, approximation instead of the one-sided one (14).

Proposition 3. Let \mathfrak{d} be the distance from f to \mathcal{C} , that is, the value of

$$(18) \quad \begin{cases} \|f - g\|_{\infty} \rightarrow \min \\ g \in \mathcal{C}. \end{cases}$$

The function $\bar{g} \in \mathcal{C}$ is a solution to (18) if and only if the function $(\bar{g} - \mathfrak{d}) \in \mathcal{C}$ is a solution to (14).

Proof. Let \bar{g} be a solution to (18) and let for short $g_1 := \bar{g} - \mathfrak{d}$. Since $\bar{g} - f \leq \|\bar{g} - f\|_\infty = \mathfrak{d}$, we get $g_1 \leq f$. Let $g_2 \leq f$ be a solution to (14) and let $c := \|f - g_2\|_\infty$.

Since g_1 is feasible for (14), we have $c \leq \|f - g_1\|_\infty = \max(f - g_1) = \max(f - \bar{g}) + \mathfrak{d} \leq 2\mathfrak{d}$. That is,

$$(19) \quad c \leq \|f - g_1\|_\infty \leq 2\mathfrak{d}.$$

On the other hand, if $g_3 := g_2 + c/2$ then $\mathfrak{d} \leq \|f - g_3\|_\infty$. Let $\varphi \in \mathbb{R}$ be arbitrary. If $f(\varphi) \leq g_3(\varphi)$ then $g_2(\varphi) \leq f(\varphi) \leq g_2(\varphi) + c/2$, so $|f(\varphi) - g_3(\varphi)| \leq c/2$. If $f(\varphi) > g_3(\varphi)$ then $|f(\varphi) - g_3(\varphi)| = f(\varphi) - g_3(\varphi) = (f(\varphi) - g_2(\varphi)) - c/2 \leq c/2$. So,

$$(20) \quad \mathfrak{d} \leq \|f - g_3\|_\infty \leq c/2.$$

From (19) and (20) it follows that $c = 2\mathfrak{d}$, so (19) gives that g_1 is a solution to (14), and (20) gives that g_3 is a solution to (18). \square

We will prove in the present framework the well known geometrical theorem that at most one ellipse centered at the origin passes through three distinct points.

Lemma 1. Let

$$(21) \quad \varphi_1 < \varphi_2 < \varphi_3 < \varphi_1 + \pi$$

be fixed. Let $A : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear operator

$$A(a, b, c) := (\mathfrak{R}(a, b, c)(\varphi_1), \mathfrak{R}(a, b, c)(\varphi_2), \mathfrak{R}(a, b, c)(\varphi_3)).$$

Then A is an isomorphism.

Proof. In co-ordinates

$$A = \begin{pmatrix} 1 & \cos 2\varphi_1 & \sin 2\varphi_1 \\ 1 & \cos 2\varphi_2 & \sin 2\varphi_2 \\ 1 & \cos 2\varphi_3 & \sin 2\varphi_3 \end{pmatrix},$$

so $\det A = 4 \sin(\varphi_2 - \varphi_1) \sin(\varphi_3 - \varphi_1) \sin(\varphi_3 - \varphi_2) > 0$, since all three differences are in $(0, \pi)$. \square

Proposition 4. Let $g_1, g_2 \in \mathcal{C}$.

If g_1 and g_2 coincide at three points as in (21) then $g_1 = g_2$.

If there are φ_1, φ_2 such that $0 < |\varphi_1 - \varphi_2| < \pi$, and

$$(22) \quad g_1(\varphi_1) = g_2(\varphi_1), \quad g_1'(\varphi_1) = g_2'(\varphi_1) \quad \text{and} \quad g_1(\varphi_2) = g_2(\varphi_2),$$

then $g_1 = g_2$.

Proof. Let $g_i = -(1/2) \log \mathfrak{R}(a_i, b_i, c_i)$ for $i = 1, 2$. If g_1 and g_2 coincide at three points as in (21) then, since \log is injective, $\mathfrak{R}(a_1, b_1, c_1)$ and $\mathfrak{R}(a_2, b_2, c_2)$ also coincide at those three points, so by Lemma 1 we get $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$.

Let now (22) hold. Since

$$g_i'(\varphi) = -\frac{1}{2} \left(\frac{d}{d\varphi} \mathfrak{R}(a_i, b_i, c_i)(\varphi) \right) / \mathfrak{R}(a_i, b_i, c_i)(\varphi),$$

the three conditions translate to $\mathfrak{R}(a_1, b_1, c_1)$ and $\mathfrak{R}(a_2, b_2, c_2)$ agreeing at φ_1 in value and derivative, and at φ_2 in value. This implies

$$\begin{pmatrix} 1 & \cos 2\varphi_1 & \sin 2\varphi_1 \\ 0 & -2 \sin 2\varphi_1 & 2 \cos 2\varphi_1 \\ 1 & \cos 2\varphi_2 & \sin 2\varphi_2 \end{pmatrix} \begin{pmatrix} a_2 - a_1 \\ b_2 - b_1 \\ c_2 - c_1 \end{pmatrix} = 0.$$

The determinant of the above matrix is $4\sin^2(\varphi_2 - \varphi_1) > 0$, so $a_1 = a_2$, $b_1 = b_2$ and $c_1 = c_2$. \square

We are ready to prove the uniqueness part of our main result.

Proposition 5. Let $\bar{g} \in \mathcal{C}$. Assume that there exist points φ_i, ψ_i , $i = 1, 2$, satisfying (15) and such that

$$(23) \quad f(\varphi_i) - \bar{g}(\varphi_i) = \bar{g}(\psi_i) - f(\psi_i) = \|f - \bar{g}\|_\infty, \quad i = 1, 2.$$

Then \bar{g} is the unique solution to (18).

Proof. Let $\hat{g} \in \mathcal{C}$ be such that

$$(24) \quad \|f - \hat{g}\|_\infty \leq \|f - \bar{g}\|_\infty.$$

If we can prove that from this follows $\hat{g} = \bar{g}$ then obviously \bar{g} is a solution to (18) and, moreover, unique.

Assume that

$$(25) \quad \bar{g} \neq \hat{g}.$$

Let $h := \bar{g} - \hat{g}$. We can write $h(\varphi_i) = (f - \hat{g})(\varphi_i) - (f - \bar{g})(\varphi_i)$. By (23) we have $(f - \bar{g})(\varphi_i) = \|f - \bar{g}\|_\infty$, so $h(\varphi_i) = (f - \hat{g})(\varphi_i) - \|f - \bar{g}\|_\infty \leq \|f - \hat{g}\|_\infty - \|f - \bar{g}\|_\infty \leq 0$ by (24). Thus $h(\varphi_i) \leq 0$. Similarly, $h(\psi_i) = (f - \hat{g})(\psi_i) + (\bar{g} - f)(\psi_i) = (f - \hat{g})(\psi_i) + \|f - \bar{g}\|_\infty \geq \|f - \bar{g}\|_\infty - \|f - \hat{g}\|_\infty \geq 0$. We have shown that

$$(26) \quad h(\varphi_i) \leq 0 \leq h(\psi_i), \quad i = 1, 2.$$

Consequently, there exist $\xi_i \in [\varphi_i, \psi_i]$, $i = 1, 2$ (in particular, $\xi_1 < \xi_2$) such that $h(\xi_i) = 0$ for $i = 1, 2$. If there were a third point in $[\varphi_1, \varphi_1 + \pi)$ at which $h = 0$, then Proposition 4 would have implied $\bar{g} = \hat{g}$, contradicting (25), so

$$(27) \quad h(\varphi) \neq 0, \quad \forall \varphi \in [\varphi_1, \varphi_1 + \pi) \setminus \{\xi_1, \xi_2\}.$$

Also, if $h'(\xi_i) = 0$ for some $i = 1, 2$, then again Proposition 4 implies $\bar{g} = \hat{g}$, so

$$(28) \quad h'(\xi_i) \neq 0, \quad i = 1, 2.$$

From (27) and $\xi_1 \leq \psi_1 < \varphi_2 \leq \xi_2$ it follows that $h(\varphi) \neq 0$ for all $\varphi \in (\psi_1, \varphi_2)$. Since h is continuous, it has a constant sign on (ψ_1, φ_2) , so there are two possibilities:

CASE 1. $h > 0$ on (ψ_1, φ_2) . Since $h(\varphi_2) \leq 0$, see (26), we have $h(\varphi_2) = 0$, that is, $\xi_2 = \varphi_2$. From (28) it follows that $h'(\varphi_2) < 0$, so $h(\varphi_2 + \delta) < 0$ for $\delta > 0$ small enough. Since $h \neq 0$ on $(\xi_2, \psi_2]$ by (27), we get $h(\psi_2) < 0$, contradiction to (26).

CASE 2. $h < 0$ on (ψ_1, φ_2) . Since $h(\psi_1) \geq 0$, we have $\xi_1 = \psi_1$, and $h'(\xi_1) < 0$, so h is strictly positive to the left of ξ_1 , implying $h(\varphi_1) > 0$, contradiction. \square

Proposition 6. Let $g \in \mathcal{C}$ be arbitrary. Let $\alpha < \beta < \alpha + \pi$. Let $\varepsilon > 0$.

There exists $g_\varepsilon \in \mathcal{C}$ such that $\|g_\varepsilon - g\|_\infty < \varepsilon$ and

$$g_\varepsilon(\varphi) < g(\varphi), \quad \forall \varphi \in (\alpha, \beta), \quad g_\varepsilon(\varphi) > g(\varphi), \quad \forall \varphi \in (\beta, \alpha + \pi).$$

Proof. Let $g = -(1/2) \log \mathfrak{R}(a, b, c)$ for some $(a, b, c) \in K$. Let $\gamma := (\alpha + \beta)/2$. Let A be the operator from Lemma 1 for α, γ, β in place of the φ s in (21).

Let $(a_t, b_t, c_t) = A^{-1}(\mathfrak{R}(a, b, c)(\alpha), \exp(2t)\mathfrak{R}(a, b, c)(\gamma), \mathfrak{R}(a, b, c)(\beta))$. It is clear that for small enough $t > 0$ the point $(a_t, b_t, c_t) \in K$ and for $g_t := -(1/2) \log \mathfrak{R}(a_t, b_t, c_t)$

$$g_t(\alpha) = g(\alpha), \quad g_t(\beta) = g(\beta) \quad \text{and} \quad g_t(\gamma) = g(\gamma) - t < g(\gamma).$$

Since $(a_t, b_t, c_t) \rightarrow (a, b, c)$, as $t \rightarrow 0$, and \mathfrak{R} is continuous, we have $\|g - g_t\|_\infty < \varepsilon$ for t small enough. Take $g_\varepsilon = g_t$ for some small enough t .

If we assume that $g_\varepsilon(\xi) \geq g(\xi)$ for some $\xi \in (\alpha, \beta)$, then, since $g_\varepsilon(\gamma) < g(\gamma)$ there will be three different points in the interval $[\alpha, \beta]$ where g_ε and g coincide, which contradicts Proposition 4. So,

$$g_\varepsilon(\varphi) < g(\varphi), \quad \forall \varphi \in (\alpha, \beta).$$

From this it follows in particular that $g'_\varepsilon(\beta) \geq g'(\beta)$. Since $g'_\varepsilon(\beta) = g'(\beta)$ would contradict Proposition 4, we have that $g'_\varepsilon(\beta) > g'(\beta)$. In particular $g_\varepsilon(\beta + \delta) > g(\beta + \delta)$ for all $\delta > 0$ small enough. If there were $\xi \in (\beta, \alpha + \pi)$ such that $g_\varepsilon(\xi) \leq g(\xi)$ this would contradict Proposition 4. So,

$$g_\varepsilon(\varphi) > g(\varphi), \quad \forall \varphi \in (\beta, \alpha + \pi).$$

□

Of course, Proposition 6 can be proved by simply adding an appropriate trigonometric polynomial to \mathfrak{R} , but we chose to demonstrate a way that would work just as well for other families as long as they satisfy the interpolation Proposition 4.

Proposition 7. Let \bar{g} be a solution to (18), that is,

$$(29) \quad \|f - \bar{g}\|_\infty = \mathfrak{d} = \min\{\|f - g\|_\infty : g \in \mathcal{C}\}.$$

Then there exist points $\varphi_i, \psi_i, i = 1, 2$, satisfying (15) and (23).

Proof. If $\mathfrak{d} = 0$ then $f = \bar{g}$ and any points satisfying (15) will do.

Therefore, let $\mathfrak{d} > 0$ and

$$\Phi := \{\varphi \in \mathbb{R} : f(\varphi) - \bar{g}(\varphi) = \mathfrak{d}\}, \quad \Psi := \{\psi \in \mathbb{R} : \bar{g}(\psi) - f(\psi) = \mathfrak{d}\}.$$

Because of the assumption $\mathfrak{d} > 0$ the sets Φ and Ψ do not intersect.

If $\Phi = \emptyset$ then $\max(f - \bar{g}) < \mathfrak{d}$. Consider $g_\varepsilon := \bar{g} - \varepsilon$. Because of (17) the functions $g_\varepsilon \in \mathcal{C}$. For $\varepsilon > 0$ small enough we still have $\max(f - g_\varepsilon) = \max(f - \bar{g}) + \varepsilon < \mathfrak{d}$, while $\min(f - g_\varepsilon) = \min(f - \bar{g}) + \varepsilon > -\mathfrak{d}$. So, $\|f - g_\varepsilon\|_\infty < \mathfrak{d}$, contradiction to (29). In a similar way, $\Psi \neq \emptyset$.

Fix $\varphi_1 \in \Phi$. Since $\Psi + \pi = \Psi$, because f and \bar{g} are π -periodic, and $\Psi \neq \emptyset$, we have $\Psi \cap [\varphi_1, \varphi_1 + \pi] \neq \emptyset$. Let

$$\psi_1 := \min \Psi \cap [\varphi_1, \varphi_1 + \pi], \quad \psi_2 := \max \Psi \cap [\varphi_1, \varphi_1 + \pi].$$

Since $\varphi_1, \varphi_1 + \pi \in \Phi$ and $\Phi \cap \Psi = \emptyset$, it follows that $\psi_i \in (\varphi_1, \varphi_1 + \pi)$ for $i = 1, 2$.

We claim that

$$(30) \quad \Phi \cap [\psi_1, \psi_2] \neq \emptyset,$$

(note Ader's Non-separation Principle).

Assume (30) were false. Since $(f - \bar{g})$ is a continuous function, which is \mathfrak{d} at φ_1 and $-\mathfrak{d}$ at ψ_1 , it vanishes somewhere in (φ_1, ψ_1) . Let $\alpha := \max\{\xi < \psi_1 : f(\xi) = \bar{g}(\xi)\}$. In a similar way, let $\beta := \min\{\xi > \psi_2 : f(\xi) = \bar{g}(\xi)\}$. We have then

$$[\psi_1, \psi_2] \subset (\alpha, \beta), \quad [\alpha, \beta] \subset (\varphi_1, \varphi_1 + \pi).$$

Moreover, it is easy to see that

$$\Phi \cap [\alpha, \beta] = \emptyset,$$

because (30) was assumed false, and $f \leq \bar{g}$ on $[\alpha, \psi_1]$ and $[\psi_2, \beta]$.

By construction, $\Psi \cap ([\varphi_1, \alpha] \cup [\beta, \varphi_1 + \pi]) = \emptyset$. Since $\Psi + \pi = \Psi$, the latter translates to

$$\Psi \cap [\beta, \alpha + \pi] = \emptyset.$$

So, there is $\varepsilon > 0$ such that

$$\max_{[\alpha, \beta]} (f - \bar{g}) + \varepsilon < \mathfrak{d}, \quad \max_{[\beta, \alpha + \pi]} (\bar{g} - f) + \varepsilon < \mathfrak{d}.$$

From Proposition 6 there is $g_\varepsilon \in \mathcal{C}$ such that $\|g_\varepsilon - \bar{g}\|_\infty < \varepsilon$ and

$$g_\varepsilon < \bar{g} \text{ on } (\alpha, \beta), \quad g_\varepsilon > \bar{g} \text{ on } (\beta, \alpha + \pi).$$

We will show that

$$(31) \quad \|f - g_\varepsilon\|_\infty = \max_{[\alpha, \alpha + \pi]} |f - g_\varepsilon| < \mathfrak{d}.$$

Indeed, $\max_{[\alpha, \beta]} (f - g_\varepsilon) \leq \max_{[\alpha, \beta]} (f - \bar{g}) + \|\bar{g} - g_\varepsilon\|_\infty < \mathfrak{d}$, and $\max_{[\alpha, \beta]} (g_\varepsilon - f) = g_\varepsilon(\xi) - f(\xi)$ for some $\xi \in [\alpha, \beta]$, but since $f = \bar{g}$ at α and β , and $\max_{[\alpha, \beta]} (g_\varepsilon - f) \geq g_\varepsilon(\psi_1) - f(\psi_1) = \mathfrak{d} + (g_\varepsilon(\psi_1) - \bar{g}(\psi_1)) \geq \mathfrak{d} - \|\bar{g} - g_\varepsilon\|_\infty > 0$; we have that $\xi \in (\alpha, \beta)$, thus $\max_{[\alpha, \beta]} (g_\varepsilon - f) = g_\varepsilon(\xi) - f(\xi) < \bar{g}(\xi) - f(\xi) \leq \mathfrak{d}$.

In a similar way, $\max_{[\beta, \alpha + \pi]} (f - g_\varepsilon) = f(\xi) - g_\varepsilon(\xi)$ for some $\xi \in (\beta, \alpha + \pi)$, so $< f(\xi) - \bar{g}(\xi) \leq \mathfrak{d}$, and $\max_{[\beta, \alpha + \pi]} (g_\varepsilon - f) \leq \max_{[\beta, \alpha + \pi]} (\bar{g} - f) + \|\bar{g} - g_\varepsilon\|_\infty < \mathfrak{d}$.

Thus (31) is verified, but it contradicts (29). Therefore, there is $\varphi_2 \in \Phi \cap (\psi_1, \psi_2)$ and φ_i, ψ_i satisfy (15) and (23). \square

In view of the work so far the following completes our study.

Proposition 8. The problem (18) has a solution.

Proof. Here it is easier to work with the standard form of the ellipses in polar co-ordinates.

Take $(a, b, c) \in K$, that is, $a > \sqrt{b^2 + c^2}$, and set

$$b' := \sqrt{b^2 + c^2}, \quad b = b' \cos 2\theta, \quad c = b' \sin 2\theta.$$

Then

$$\mathfrak{R}(a, b, c) = a + b' \cos 2(\varphi - \theta).$$

K translates to

$$K' := \{(a, b', \theta) : a > b' \geq 0, \theta \in [0, \pi]\},$$

and

$$\mathcal{C} = \{-(1/2) \log(a + b' \cos 2(\varphi - \theta)) : (a, b', \theta) \in K'\}.$$

Let $(a_n, b'_n, \theta_n) \in K'$ be such that for

$$g_n(\varphi) := -\frac{1}{2} \log(a_n + b'_n \cos 2(\varphi - \theta_n))$$

we have

$$\lim_{n \rightarrow \infty} \|f - g_n\|_\infty = \mathfrak{d}.$$

By taking a subsequence we may assume that $\theta_n \rightarrow \theta$, as $n \rightarrow \infty$. Since

$$g_n(\theta_n + \pi/4) = -\frac{1}{2} \log a_n,$$

and $\|g_n\|_\infty < \|f\|_\infty + \mathfrak{d} + 1$ for large enough ns , the sequence $\{\log a_n\}_{n=1}^\infty$ is bounded, so there is $r > 0$ such that

$$r \leq a_n \leq 1/r, \quad \forall n \in \mathbb{N}.$$

Therefore, we may assume that $a_n \rightarrow a > 0$, as $n \rightarrow \infty$. We also assume that $b'_n \rightarrow b' \leq a$, as $n \rightarrow \infty$. If $b' = a$ then

$$g_n(\theta_n + \pi/2) = -\frac{1}{2} \log(a_n - b'_n) \rightarrow \infty, \text{ as } n \rightarrow \infty,$$

which is a contradiction. So, $(a, b', \theta) \in K'$ and by continuity it follows that the corresponding $g \in \mathcal{C}$ solves (18). \square

References

- [1] O. B. ADER, An affine invariant of convex regions, *Duke Math. J.* 4 (1938), no. 2, 291–299.
- [2] P. M. GRUBER, P. KENDEROV, Approximation of convex bodies by polytopes, *Rend. Circ. Mat. Palermo*, Ser. II, 31 (1982), 195–225.
- [3] F. GRUNDBACHER, T. KOBOS, On certain extremal Banach–Mazur distances and Ader’s characterization of distance ellipsoids, *Mathematika* 72 (2026), e70062. <https://doi.org/10.1112/mtk.70062>
- [4] R. A. HORN, C. R. JOHNSON, *Matrix Analysis*, 2nd ed., Cambridge University Press, Cambridge, 2013. ISBN 978-0-521-83940-2.
- [5] M. IVANOV, A. J. PALLARÉS, S. TROYANSKI, On the geometry of Banach spaces with modulus of convexity of power type 2, *Studia Math.* 197 (2010), no. 1, 81–91.
- [6] M. IVANOV, S. TROYANSKI, Uniformly smooth renorming of Banach spaces with modulus of convexity of power type 2, *J. Funct. Anal.* 237 (2006), 373–390.
- [7] F. JOHN, Extremum problems with inequalities as subsidiary conditions, in: *Studies and Essays Presented to R. Courant on his 60th Birthday*, January 8, 1948, Interscience Publishers, New York, 1948, pp. 187–204.
- [8] P. S. KENDEROV, Polygonal approximation of plane convex compacta, *J. Approx. Theory* 38 (1983), 221–239.
- [9] P. S. KENDEROV, N. K. KIROV, A dynamical systems approach to the polygonal approximation of plane convex compacta, *J. Approx. Theory* 74 (1993), no. 1, 1–15.
- [10] B. MAUREY, unpublished.
- [11] R. NIKOLOV, On a geometric characteristic of Banach spaces with modulus of convexity of power type 2, *C. R. Acad. Bulg. Sci.* 70 (2017), no. 9, 1189–1194.
- [12] R. NIKOLOV, On a differential inequality, *Ann. Sofia Univ., Fac. Math. and Inf.* 104 (2017).
- [13] A. PIETSCH, *History of Banach Spaces and Linear Operators*, Birkhäuser, Boston, 2007. ISBN 978-0-8176-4367-6.
- [14] S. RAKOV, On uniformly smooth renormings of uniformly convex Banach spaces, *J. Soviet Math.* 31 (1985), 2713–2721.