МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2026
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2026
Proceedings of the Fifty-Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Tryavna, Bulgaria, April 5–9, 2026

FOCAL SURFACES OF DEVELOPABLE RULED SURFACES AND THEIR INVARIANTS

Cvetelina Dinkova[1], Radostina Encheva[2]
Faculty of Mathematics and Informatics,
Konstantin Preslavsky University of Shumen, Shumen, Bulgaria
e-mails: [1]c.dinkova@shu.bg, [2]r.encheva@shu.bg

In this work the focal surfaces of developable ruled surfaces generated by a unit speed Frenet curve and its focal curve in the Euclidean three-dimensional space are investigated. For each curve, we construct the associated tangential surface and derive a parametrization of its focal surface. Working with respect to the natural parameter of the base curve, we obtain relations between the coefficients of the two fundamental forms of the original developable surfaces and those of their focal surfaces. The focal surfaces obtained in both cases are themselves developable. Formulas relating to their mean curvatures are established.

Keywords: Frenet curves, Focal curves, Ruled surfaces, Developed surfaces, Focal surfaces, Gauss curvature, Mean curvature

ФОКАЛНИ ПОВЪРХНИНИ НА РАЗВИВАЕМИ ПРАВОЛИНЕЙНИ ПОВЪРХНИНИ И ТЕХНИТЕ ИНВАРИАНТИ

Цветелина Динкова[1], Радостина Енчева[2]
Факултет по математика и информатика,
Шуменски университет „Епископ Константин Преславски“, Шумен, България
e-mails: [1]c.dinkova@shu.bg, [2]r.encheva@shu.bg

В тази работа са изследвани фокалните повърхнини на развиваеми линейни повърхнини, получени от крива на Френе, параметризирана спрямо естествен параметър, и нейната фокална крива в Евклидовото тримерно пространство. За всяка крива конструираме съответната тангенциална повърхнина и извеждаме параметризация

https://doi.org/10.55630/mem.2026.55.049-059

2020 Mathematics Subject Classification: 53A04; 53A05; 51M15.

* This research is partially supported by Scientific Research Grant RD-08-58/27.01.2026 of Konstantin Preslavsky University of Shumen.

на нейната фокална повърхнина. Работейки относно естествения параметър на базовата крива, получаваме зависимости между коефициентите на двете основни форми на оригиналните развиваеми повърхнини и тези на техните фокални повърхнини. Получените и в двата случая фокални повърхнини са развиваеми. Установени са формули, свързващи средните им кривини.

Ключови думи: Крива на Френе, Фокална крива, Праволинейна повърхнина, Развиваема повърхнина, Фокална повърхнина, Гаусова кривина, Средна кривина

1 Introduction

Ruled surfaces are important in both classical and contemporary differential geometry because of their intricate structures and their extensive use in geometric modelling, computer-aided design, and kinematic surface creation. Their geometry is determined by the invariants of the basic base curve, and these invariants—namely, curvature and torsion—offer a fundamental framework for evaluating the behaviour of such surfaces under Euclidean motions. An interesting approach in this context is the examination of focal surfaces, which occur as envelopes of normal lines and include essential curvature information for the original surface. For cylinders, cones, and offset surfaces, focal surfaces show important geometric relations, and hence they have been extensively studied. Focal and generalized focal surfaces are crucial for curvature-based constructs like manufacturing, optical modelling, and computer-aided geometric design, according to recent studies. Georgiev and Pavlov (see [4], [5], [6], [7], [8]), in particular, show how focal surfaces reflect classical surfaces' intrinsic geometry and how their invariants can characterise curvature-dependent offset surfaces. In [11], Güler provides an intriguing perspective on the focal surfaces of offset surfaces. Despite this progress, the focal surfaces of developable ruled surfaces generated by space curves remain less explored. Tangential surfaces—ruled surfaces formed by attaching tangent lines along a Frenet curve—are a fundamental class of developable surfaces. Their focal surfaces obviously preserve developability, a classical result in the differential geometry of surfaces (see [2],[3],[15]), but it is not immediately clear how their fundamental forms and curvature invariants relate to those of the original ruled surfaces.

In this paper, we construct new surfaces in the Euclidean space $\mathbb{E}^3$ as tangential surfaces generated by a pair of spatial curves, a Frenet curve $\boldsymbol{\gamma}$ and its focal curve $\boldsymbol{C}_\gamma$. By examining tangential surfaces obtained by a Frenet curve and by its focal curve, we aim to clarify the geometric structure of their focal surfaces and to establish explicit relationships between their invariants. The final section illustrates the theoretical findings with an explicit example involving a cylindrical curve and its focal curve, together with visualizations of the corresponding developable and focal surfaces.

The following section presents key terminology and important information regarding focal curves and focal surfaces.

2 Preliminaries

2.1 Frenet curves in the Euclidean three-space

The Euclidean three-dimensional space $\mathbb{E}^3$ is regarded as an affine space, and it has a vector space $\mathbb{R}^3$ that is related to it. The position column vector from $\mathbb{R}^3$ can be used to represent any point in $\mathbb{E}^3$. The scalar (or dot) product $\langle \boldsymbol{a}, \boldsymbol{b} \rangle \in \mathbb{R}$ and the vector cross product $\boldsymbol{a} \times \boldsymbol{b} \in \mathbb{R}^3$ are well-known operations for any two column vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$. The norm of the vector $\boldsymbol{a}$ is $\|\boldsymbol{a}\| = \sqrt{\langle \boldsymbol{a}, \boldsymbol{a} \rangle} = \sqrt{\boldsymbol{a}^2}$.

Assume $\gamma : I \longrightarrow \mathbb{E}^3$ be a curve described by a vector parametric equation

$$\boldsymbol{\gamma}(q) = (x(q),\, y(q),\, z(q))^T, \quad q \in I \tag{1}$$

on an interval $I \subseteq \mathbb{R}$. The coordinate functions $x(q)$, $y(q)$, $z(q)$ are supposed to have continuous derivatives up to order 3. We will refer to that curve as a regular C^3 space curve. If the derivatives $\boldsymbol{\gamma}'(q)$, $\boldsymbol{\gamma}''(q)$ and $\boldsymbol{\gamma}'''(q)$ are linearly independent vectors in $\mathbb{E}^3$ for every $q \in I$, then the curves are known as Frenet curves. For a Frenet curve γ, the Euclidean curvatures of γ in $\mathbb{E}^3$ (a curvature $\varkappa$ and a torsion τ) are determined by

$$\varkappa(q) = \frac{\|\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)\|}{\|\boldsymbol{\gamma}'(q)\|^3} > 0,\ \tau(q) = \frac{\langle \boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q), \boldsymbol{\gamma}'''(q) \rangle}{\|\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)\|^2} \neq 0 \tag{2}$$

Moreover, there are three unit vectors

$$\boldsymbol{t}(q) = \frac{\boldsymbol{\gamma}'(q)}{\|\boldsymbol{\gamma}'(q)\|}, \boldsymbol{n}(q) = \frac{\left(\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)\right) \times \boldsymbol{\gamma}'(q)}{\|\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)\|.\|\boldsymbol{\gamma}'(q)\|}, \boldsymbol{b}(q) = \frac{\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)}{\|\boldsymbol{\gamma}'(q) \times \boldsymbol{\gamma}''(q)\|} \tag{3}$$

defined at any point $\boldsymbol{\gamma}(q)$ of the curve. A positively orientated orthonormal basis known as a Frenet frame is formed by these three vectors.

In the case that the Frenet curve $\gamma : I \longrightarrow \mathbb{E}^3$ is a unit speed curve, namely, $\|\boldsymbol{\gamma}'(q)\| = 1$ for any $q \in I$, the parameter q is usually substituted with the parameter s, and the vector equation (1) is referred to as an arc-length parametrisation of γ. In this instance, a more straightforward form of the formulas (2) and (3) can be used. Especially, the torsion and the curvature are defined by the equations

$$\varkappa(s) = \|\boldsymbol{\gamma}''(s)\| > 0 \quad \text{and} \quad \tau(s) = \frac{\langle \boldsymbol{\gamma}'(s) \times \boldsymbol{\gamma}''(s), \boldsymbol{\gamma}'''(s) \rangle}{\|\boldsymbol{\gamma}''(t)\|^2} \neq 0,$$

respectively. Further, the Frenet-Serret equations

$$\boldsymbol{t}'(s) = \varkappa(s)\boldsymbol{n}(s), \quad \boldsymbol{n}'(s) = -\varkappa(s)\boldsymbol{t}(s) + \tau(s)\boldsymbol{b}(s), \quad \boldsymbol{b}'(s) = -\tau(s)\boldsymbol{n}(s).$$

for the unit speed curve γ are fulfilled.

Henceforth, we shall use "′" for the differentiation concerning a natural parameter and "·" as the differentiation concerning an arbitrary parameter.

Definition 1. *[10, p.241] The **focal curve** of a regular C^3 space curve $\gamma : I \to \mathbb{E}^3$ is the curve given by*

$$\boldsymbol{C}_{\boldsymbol{\gamma}}(q) = \boldsymbol{\gamma}(q) + c_1(q)\boldsymbol{n}(q) + c_2(q)\boldsymbol{b}(q),$$

where $\boldsymbol{n}$ is a principal unit normal vector field of γ, $\boldsymbol{b}$ is a binormal unit vector field of $\boldsymbol{\gamma}$. The coefficients $c_1(q)$ and $c_2(q)$ are smooth functions called focal curvatures of γ, given by

$$c_1(q) = \frac{1}{\varkappa(q)}, \quad c_2(q) = -\frac{\frac{d}{dq}\varkappa(q)}{\| \frac{d\boldsymbol{\gamma}(q)}{dq} \| \, \varkappa(q)^2 \tau(q)} = \frac{\frac{dc_1(q)}{dq}}{\| \frac{d\boldsymbol{\gamma}(q)}{dq} \| \, \tau(q)},$$

where $\varkappa(q) > 0$ and $\tau(q) \neq 0$ are the Euclidean curvatures of γ.

In other words, the focal curve of an immersed smooth curve γ, in the Euclidean space $\mathbb{E}^3$, consists of the centres of its osculating spheres. The functions $c_1(q)$ and $c_2(q)$ are well defined because $\varkappa(q)$ and $\tau(q)$ are non-zero functions. In addition, the parametrization of the focal curve of the unit speed curve $\boldsymbol{\gamma}$ becomes

$$C_\gamma(s) = \boldsymbol{\gamma}(s) + \frac{1}{\varkappa(s)}\boldsymbol{n}(s) + \left(\frac{1}{\varkappa(s)}\right)' \frac{1}{\tau(s)}\boldsymbol{b}(s). \tag{4}$$

According to [16], for the Frenet frame $\boldsymbol{T}$, $\boldsymbol{N}$, and $\boldsymbol{B}$, as well as the Euclidean curvatures $\varkappa_{C_\gamma}$ and τ_{C_γ} of the curve C_γ, we have

$$\boldsymbol{T} = \delta\boldsymbol{b},\ \boldsymbol{N} = -\varepsilon\boldsymbol{n},\ \boldsymbol{B} = \delta\varepsilon\boldsymbol{t};\ \varkappa_{C_\gamma} = \frac{|\tau|}{|c_1\tau + c_2'|} = \varepsilon\frac{\tau}{c_1\tau + c_2'},\ \tau_{C_\gamma} = \frac{\varkappa}{c_1\tau + c_2'}, \tag{5}$$

where $\delta = sign(c_1\tau + c_2')$ and $\varepsilon = sign(\tau)\delta$. We consider only non-spherical curves γ, where $c_1\tau + c_2' \neq 0$.

The differential geometry of space curves is described in more detail in [10].

2.2 Ruled surfaces associated with a space curve

This section presents precise formulas for the invariants of ruled surfaces related to a space curve. A ruled surface in $\mathbb{E}^3$ is (locally) represented by the mapping $S(u,v) : I \times I \longrightarrow \mathbb{E}^3$ defined as $S(u,v) = \boldsymbol{\gamma}(u) + v\boldsymbol{l}(u)$, where $\gamma : I \longrightarrow \mathbb{E}^3$ and $l : I \longrightarrow \mathbb{E}^3\backslash\{0\}$ are smooth functions, and I is an open interval. We designate γ as the base curve and l as the director curve. The straight lines $v \to \boldsymbol{\gamma}(u) + v\boldsymbol{l}(u)$ are referred to as the rulings of $S(u,v)$ (see [10]). Let $S(u,v)$ represents a ruled surface. We define $S(u,v)$ as *developable* if its Gaussian curvature is zero. From this point further on, we shall presume that $\|\boldsymbol{l}(u)\| = 1$. It is easy to show that the Gaussian curvature $K(u,v)$ and the Mean curvature $H(u,v)$ of a ruled surface $S(u,v)$ are

$$K(u,v) = -\left(\frac{\det(\boldsymbol{\gamma}'(u), \boldsymbol{l}(u), \boldsymbol{l}'(u))}{EG - F^2}\right)^2,$$

$$H(u,v) = \frac{\det(\boldsymbol{\gamma}'(u) + v.\boldsymbol{l}'(u), \boldsymbol{l}(u), \boldsymbol{\gamma}''(u) + v.\boldsymbol{l}''(u)) - 2\langle\boldsymbol{\gamma}'(u), \boldsymbol{l}(u)\rangle.\det(\boldsymbol{\gamma}'(u), \boldsymbol{l}(u), \boldsymbol{l}'(u))}{2\sqrt{EG - F^2}^3}$$

where $E = E(u,v) = S_u^2 = \|\boldsymbol{\gamma}'(u) + v\boldsymbol{l}'(u)\|^2$, $F = F(u,v) = \langle S_u, S_v\rangle = \langle\boldsymbol{\gamma}'(u), \boldsymbol{l}(u)\rangle$ and $G = G(u,v) = S_v^2 = \|\boldsymbol{l}(u)\|^2 = 1$ are the coefficients of the First Fundamental Form of $S(u,v)$. By $\det(\cdot,\cdot,\cdot)$ we denote the determinant of the corresponding column vectors. The ruled surface $S(u,v)$ is a smooth surface if and only if $\boldsymbol{\gamma}'(u) \times \boldsymbol{l}(u) + v.\boldsymbol{l}'(u) \times \boldsymbol{l}(u) \neq 0$ for all $(u,v) \in I \times I$. Some special types of developable ruled surfaces can be found in [1] and [12].

2.3 Focal surfaces of regular surfaces

Definition 2. *Let $S(u,v)$ be a parametric surface of class $\mathcal{C}^2$, and let $\mathfrak{N}(u,v)$ be its unit normal vector field. The parametric offset surface is defined as*

$$\overline{S}(u,v) = S(u,v) + d \cdot \mathfrak{N}(u,v),$$

where d is a nonzero real constant.

Definition 3. *Let $S(u,v)$ be parametric surface of class $\mathcal{C}^2$ and $\mathfrak{N}(u,v)$ be its unit normal vector field. The parametric representations of the focal surfaces of $S(u,v)$ is given by*

$$\overline{S}_1(u,v) = S(u,v) + \frac{1}{\kappa_1}.\mathfrak{N}(u,v),\ \overline{S}_2(u,v) = S(u,v) + \frac{1}{\kappa_2}.\mathfrak{N}(u,v), \tag{6}$$

where κ_1 and κ_2 are principal curvatures of the surface $S(u,v)$.

3 Main results

3.1 Focal surfaces of developable ruled surfaces

The focal surfaces of developable ruled surfaces are likewise developable surfaces. This assertion is a known fact in the classical differential geometry of surfaces. It can be proven using approaches based on the differential geometry of line congruences (see [13],[14]). Let $S_1(s,v) = \boldsymbol{\gamma}(s) + v.\boldsymbol{t}(s)$ be a tangential ruled surface with a base curve γ and a director curve represented by the unit tangent vector $\boldsymbol{t}$ of γ. The obtained surface is developable because the determinat $\det(\boldsymbol{\gamma}'(s), \boldsymbol{l}(s), \boldsymbol{l}'(s)) = \det(\boldsymbol{t}(s), \boldsymbol{t}(s), \boldsymbol{t}'(s))$ is equal to zero. The following theorem will give us an explicit reprezentation of the focal surface of the tangential ruled surface $S_1(s,v)$. We will utilise this representation to provide an alternative proof that the generated surface is also developable and to determine the relation between the mean curvatures of the associated surfaces.

Theorem 1. *Let* (1) *be a parametrization of a unit speed Frenet curve $\gamma : I \longrightarrow \mathbb{E}^3$ of class C^3 with Frenet frame $\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$, and let $\varkappa(s) \neq 0$ and $\tau(s) \neq 0$ be the Euclidean curvature and the Euclidean torsion of γ, respectively. Suppose that $S_1(s,v) = \boldsymbol{\gamma}(s) + v.\boldsymbol{t}(s)$ is the developable tangential surface associated with the unit speed Frenet curve γ. Then the focal surface $\overline{S}_1(s,v)$ of $S_1(s,v)$ is also developable and has representation*

$$\overline{S}_1(s,v) = \boldsymbol{\gamma}(s) + v.\left(\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right), \ v \in \mathbb{R} \setminus \{0\}.$$

The Mean curvature $\overline{H}_1(s,v)$ of $\overline{S}_1(s,v)$ can be expressed by the Mean curvature $H_1(s,v)$ of $S_1(s,v)$ with the following equation

$$\overline{H}_1(s,v) = -\frac{\tau(1 + 4v^2 H_1^2)}{4|vH_1 + v^2 H_{1s}|}, \tag{7}$$

where H_{1s} is the partial derivative of $H_1(s,v)$ with respect s.

Proof. The coefficients of the First Fundamental Form of $S_1(s,v)$ are given by the following equalities

$$\begin{aligned} E_1 &= E_1(s,v) = S_{1s}^2 = \|\boldsymbol{\gamma}'(s) + v\boldsymbol{t}'(s)\|^2 = \|\boldsymbol{t}(s) + v.\varkappa(s)\boldsymbol{n}(s)\|^2 = 1 + v^2\varkappa^2(s), \\ F_1 &= F_1(s,v) = \langle S_{1s}, S_{1v}\rangle = \langle \gamma'(s), \boldsymbol{t}(s)\rangle = \langle \boldsymbol{t}(s), \boldsymbol{t}(s)\rangle = 1, \\ G_1 &= G_1(s,v) = S_{1v}^2 = \|\boldsymbol{t}(s)\|^2 = 1, \end{aligned} \tag{8}$$

where $S_{1s} = \boldsymbol{\gamma}'(s) + v\boldsymbol{t}'(s)$, $S_{1v} = \boldsymbol{t}(s)$ are the partial derivatives of $S_1(s,v)$ with respect to s and v, respectively. Hence we get that

$$S_{1s} \times S_{1v} = -v\varkappa(s)\boldsymbol{b}(s) \text{ and } \|S_{1s} \times S_{1v}\| = |v|\varkappa(s).$$

Then the unit normal vector function $\mathfrak{N}_1(s,v) = \dfrac{S_{1s} \times S_{1v}}{\|S_{1s} \times S_{1v}\|}$ of $S_1(s,v)$ is

$$\mathfrak{N}_1(s,v) = -\varepsilon_1\boldsymbol{b}(s), \ \varepsilon_1 = sign(v). \tag{9}$$

From $S_{1ss} = \dfrac{\partial S_{1s}}{\partial s} = -\varkappa^2(s).v.\boldsymbol{t}(s) + (\varkappa(s) + v.\varkappa'(s))\boldsymbol{n}(s) + v.\tau(s)\varkappa(s).\boldsymbol{b}(s)$, $S_{1sv} = \dfrac{\partial S_{1s}}{\partial v} = \boldsymbol{t}'(s) = \varkappa(s)\boldsymbol{n}(s)$ and $S_{1vv} = \dfrac{\partial S_{1v}}{\partial v} = \dfrac{\partial \boldsymbol{t}(s)}{\partial v} = \vec{o}$ the coefficients of the

Second Fundamental Form of $S_1(s,v)$ are

$$(10)\qquad \begin{aligned} L_1 &= L_1(s,v) = \langle \mathfrak{N}_1(s,v), S_{1uu}\rangle = -|v|\varkappa(s).\tau(s),\\ M_1 &= M_1(s,v) = \langle \mathfrak{N}_1(s,v), S_{1uv}\rangle = -\varepsilon_1\varkappa(s).\langle \boldsymbol{b}(s), \boldsymbol{n}(s)\rangle = 0,\\ N_1 &= N_1(s,v) = \langle \mathfrak{N}_1(s,v), S_{1vv}\rangle = 0. \end{aligned}$$

It is clear that for any parameter values (s,v), the Gaussian curvature $K_1(s,v)$ of the developable surface $S_1(s,v)$ is vanished and the Mean curvature $H_1(s,v)$ can be expressed by

$$(11)\qquad H_1(s,v) = \frac{-|v|\varkappa(s).\tau(s)}{2v^2\varkappa^2(s)} = \frac{-\tau(s)}{2|v|\varkappa(s)}.$$

If $\kappa_1(s,v)$ and $\kappa_2(s,v)$ are the principal curvatures of the surface $S_1(s,v)$ then from $K_1(s,v) = \kappa_1(s,v).\kappa_2(s,v) = 0$ and $H_1(s,v) = \dfrac{\kappa_1(s,v)+\kappa_2(s,v)}{2}$ it follows that

$$(12)\qquad \kappa_1(s,v) = 2H_1(s,v) = \frac{-\tau(s)}{|v|\varkappa(s)} \text{ and } \kappa_2(s,v) = 0.$$

Then from the equation (6) in **Definition 3** and the equations (9) and (12) it follows

$$\overline{S}_1(s,v) = S_1(s,v) + \frac{1}{\kappa_1}.\mathfrak{N}_1(s,v) = \boldsymbol{\gamma}(s) + v.\left(\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right).$$

From the equalities

$$\det(\boldsymbol{\gamma}'(s), \boldsymbol{l}(s), \boldsymbol{l}'(s)) = \det\left(\boldsymbol{t}(s), \boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s), \left(\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right)'\right) =$$

$$\det\left(\boldsymbol{t}(s), \boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s), \left(\frac{\varkappa(s)}{\tau(s)}\right)'\boldsymbol{b}(s)\right) = 0.$$

it follows that the surface $\overline{S}_1(s,v)$ is developable. Let us now compute the coefficients of the First Fundamental Form of $\overline{S}_1(s,v)$. We obtain that

$$(13)\qquad \begin{aligned} \overline{E}_1 &= \overline{E}_1(s,v) = \overline{S}_{1s}^2 = \left\| \boldsymbol{t}(s) + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\boldsymbol{b}(s)\right\|^2 = 1 + v^2\left(\frac{\varkappa(s)}{\tau(s)}\right)'^2,\\ \overline{F}_1(s,v) &= \langle \overline{S}_{1s}, \overline{S}_{1v}\rangle = \langle \boldsymbol{t}(s) + v.\left(\frac{\varkappa}{\tau}\right)'\boldsymbol{b}(s), \boldsymbol{t}(s) + \frac{\varkappa}{\tau}\boldsymbol{b}(s)\rangle = 1 + v.\frac{\varkappa}{\tau}\left(\frac{\varkappa}{\tau}\right)',\\ \overline{G}_1 &= \overline{G}_1(s,v) = \overline{S}_{1v}^2 = \left\|\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right\|^2 = 1 + \left(\frac{\varkappa(s)}{\tau(s)}\right)^2, \end{aligned}$$

where $\overline{S}_{1s} = \boldsymbol{t}(s) + v.\left(\dfrac{\varkappa(s)}{\tau(s)}\right)'\boldsymbol{b}(s)$ and $\overline{S}_{1v} = \boldsymbol{t}(s) + \dfrac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)$ are the partial derivatives of $\overline{S}_1(s,v)$ with respect to s and v, respectively. Consequently, we find that

$$\overline{S}_{1s}\times\overline{S}_{1v} = \left(-\frac{\varkappa(s)}{\tau(s)} + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\right)\boldsymbol{n}(s) \text{ and } \|\overline{S}_{1s}\times\overline{S}_{1v}\| = \left|-\frac{\varkappa(s)}{\tau(s)} + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\right|.$$

Then the unit normal vector function $\overline{\mathfrak{N}}_1(s,v) = \dfrac{\overline{S}_{1s}\times\overline{S}_{1v}}{\|\overline{S}_{1s}\times\overline{S}_{1v}\|}$ of $\overline{S}_1(s,v)$ is determined by

$$\overline{\mathfrak{N}}_1(s,v) = \bar{\varepsilon}_1 \boldsymbol{n}(s), \text{ where } \bar{\varepsilon}_1 = sign\left(-\frac{\varkappa(s)}{\tau(s)} + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\right). \tag{14}$$

From

$$\overline{S}_{1ss} = \frac{\partial \overline{S}_{1s}}{\partial s} = -\tau(s)\left(-\frac{\varkappa(s)}{\tau(s)} + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\right)\boldsymbol{n}(s) + v\left(\frac{\varkappa(s)}{\tau(s)}\right)''\boldsymbol{b}(s),$$

$$\overline{S}_{1sv} = \frac{\partial \overline{S}_{1s}}{\partial v} = \left(\frac{\varkappa(s)}{\tau(s)}\right)'\boldsymbol{b}(s) \text{ and } \overline{S}_{1vv} = \frac{\partial \overline{S}_{1v}}{\partial v} = \frac{\partial}{\partial v}\left(\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right) = \vec{o}$$

and the equation (14), the coefficients of the Second Fundamental Form of $\overline{S}_1(s,v)$ are given by

$$\begin{aligned} \overline{L}_1 = \overline{L}_1(s,v) &= \langle \overline{\mathfrak{N}}_1(s,v), \overline{S}_{1uu}\rangle = -\tau\left|-\frac{\varkappa}{\tau} + v.\left(\frac{\varkappa}{\tau}\right)'\right|, \\ \overline{M}_1 = \overline{M}_1(s,v) &= \langle \overline{\mathfrak{N}}_1(s,v), \overline{S}_{1uv}\rangle = 0,\ \overline{N}_1 = \overline{N}_1(s,v) = \langle \overline{\mathfrak{N}}_1(s,v), \overline{S}_{1vv}\rangle = 0. \end{aligned} \tag{15}$$

It is easy to see that the Mean curvature $\overline{H}_1(s,v)$ of the developable surface $\overline{S}_1(s,v)$ is

$$\overline{H}_1(s,v) = \frac{-\tau(s)\left(1 + \left(\frac{\varkappa}{\tau}\right)^2\right)}{2\left|-\frac{\varkappa}{\tau} + v.\left(\frac{\varkappa}{\tau}\right)'\right|}. \tag{16}$$

Finally from equations (11) and (16) we reached to the equation (7). □

The next statement gives us relations between the coefficients of the First and the Second Fundamental Form of $\overline{S}_1(s,v)$ and $S_1(s,v)$.

Theorem 2. *Let (1) be a parametrization of a unit speed Frenet curve $\gamma : I \longrightarrow \mathbb{E}^3$ of class C^3 with a Frenet frame $\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)$, and let $\varkappa(s) \neq 0$ and $\tau(s) \neq 0$ be the Euclidean curvature and the Euclidean torsion of γ, respectively. Suppose that $S_1(s,v) = \boldsymbol{\gamma}(s) + v.\boldsymbol{t}(s),\ v \in \mathbb{R} \setminus \{0\}$ is the developable tangential surface associated with γ and $\overline{S}_1(s,v) = \boldsymbol{\gamma}(s) + v.\left(\boldsymbol{t}(s) + \frac{\varkappa(s)}{\tau(s)}\boldsymbol{b}(s)\right)$ is its focal surface. Then the coefficients of the First and the Second Fundamental Form of $\overline{S}_1(s,v)$ can be expressed by the coefficients of the First and the Second Fundamental Form of $S_1(s,v)$ as follows*

$$\overline{E}_1(s,v) = E_1(s,v) + v^2\left(\left(\frac{\varkappa(s)}{\tau(s)}\right)'^2 - \varkappa^2(s)\right),$$

$$\overline{F}_1(s,v) = F_1(s,v) + v\left(\frac{\varkappa(s)}{\tau(s)}\right)\left(\frac{\varkappa(s)}{\tau(s)}\right)',\ \overline{G}_1(s,v) = G_1(s,v) + \left(\frac{\varkappa(s)}{\tau(s)}\right)^2,$$

$$\overline{L}_1(s,v) = \frac{L_1(s,v)}{|v|\varkappa(s)}\left|-\frac{\varkappa(s)}{\tau(s)} + v.\left(\frac{\varkappa(s)}{\tau(s)}\right)'\right|,\ \overline{M}_1(s,v) = 0,\ \overline{N}_1(s,v) = 0.$$

Proof. The proof follows immediately from equations (8), (10), (13) and (15). □

Corollary 1. *Let (1) be an arbitrary parametrization of a Frenet curve $\gamma : J \longrightarrow \mathbb{E}^3$ of class C^3 with a Frenet frame $\boldsymbol{t}(u), \boldsymbol{n}(u), \boldsymbol{b}(u)$, and let $\varkappa(u) \neq 0$ and $\tau(u) \neq 0$ be the Euclidean curvature and the Euclidean torsion of γ, respectively. If $\overline{S}_1(u,v)$ is the* *focal surface of the tangential surface* $S_1(u,v)$ *of* γ *then the Mean curvature* $\overline{H}_1(u,v)$ *of* $\overline{S}_1(u,v)$ *can be expressed by the Mean curvature* $H_1(u,v)$ *of* $S_1(u,v)$ *as follows*

$$\overline{H}_1(u,v) = -\frac{\tau\dot{s}(1+4v^2H_1^2)}{4|v\dot{s}H_1 + v^2H_{1u}|}, \tag{17}$$

$$\begin{gathered}\overline{E}_1(u,v) = E_1(u,v) + v^2\left(\left(\frac{\varkappa(u)}{\tau(u)}\right)_u^2 - \dot{s}^2\varkappa^2(u)\right),\\ \overline{F}_1(u,v) = F_1(u,v) + v\frac{\varkappa(u)}{\tau(u)}\left(\frac{\varkappa(u)}{\tau(u)}\right)_u,\ \overline{G}_1(u,v) = G_1(u,v) + \left(\frac{\varkappa(u)}{\tau(u)}\right)^2,\\ \overline{L}_1(u,v) = \frac{L_1(u,v)}{|v|\varkappa}\left|-\frac{\varkappa(u)}{\tau(u)} + \frac{v}{\dot{s}}\left(\frac{\varkappa(u)}{\tau(u)}\right)_u\right|,\ \overline{M}_1(u,v) = \overline{N}_1(u,v) = 0.\end{gathered} \tag{18}$$

where H_{1u} *and* $\left(\frac{\varkappa(u)}{\tau(u)}\right)_u$ *are the derivatives of* $H_1(u,v)$ *and* $\frac{\varkappa(u)}{\tau(u)}$ *with respect* u *and* $s = s(u)$ *is the arc-length function of* γ.

Proof. The proof follows immediately from Theorem 1 replacing the parameter s with the arc-length function $s = s(u)$ of γ. □

Following the above construction, we shall examine an additional tangential surface closely associated with the curve γ. Utilising a unit speed parametrization of the curve γ with an arc-length parameter u and applying the equality (4) for its focal curve C_γ, we differentiate with regard to u and find that $\frac{dC_\gamma(u)}{du} = (c_1\tau + c_2')\boldsymbol{b}(u)$. Thus, the surface $S_2(u,v) = C_\gamma(u) + v\delta\boldsymbol{b}(u)$, where $\delta = sign(c_1\tau + c_2')$, constitutes a tangential ruled surface, characterised by a base curve C_γ and a director curve represented by the unit binormal vector $\boldsymbol{b}$ of γ. The acquired surface is evidently developable, and its focal surface also maintains this property of developability according to Theorem 1.

Theorem 3. *Let* (1) *be a parametrization of a unit speed Frenet curve* $\gamma : I \longrightarrow \mathbb{E}^3$ *of class* C^3 *with a Frenet frame* $\boldsymbol{t}(u), \boldsymbol{n}(u), \boldsymbol{b}(u)$, *and let* $\varkappa(u) \neq 0$ *and* $\tau(u) \neq 0$ *be the Euclidean curvature and the Euclidean torsion of* γ, *respectively. Suppose that* $S_2(u,v) = C_\gamma(u) + v\delta\boldsymbol{b}(u)$ *is the developable surface associated with the Focal curve* $\boldsymbol{C}_\gamma$ *of the unit speed Frenet curve* γ. *Then the focal surface* $\overline{S}_2(u,v)$ *of* $S_2(u,v)$ *is also developable and is represented as*

$$\overline{S}_2(u,v) = \boldsymbol{C}_\gamma(u) + v\delta\widetilde{\boldsymbol{D}}(u),\ \ where\ \widetilde{\boldsymbol{D}}(u) = \frac{\tau(u)}{\varkappa(u)}\boldsymbol{t}(u) + \boldsymbol{b}(u),\ v \in \mathbb{R}\setminus\{0\}. \tag{19}$$

The Mean curvature $\overline{H}_2(u,v)$ *of* $\overline{S}_2(u,v)$ *can be expressed in terms of the Mean curvature* $H_2(u,v)$ *of* $S_2(u,v)$ *by the next equation*

$$\overline{H}_2(u,v) = -\frac{\varkappa\delta(1+4v^2H_2^2)}{4|v(c_1\tau + c_2')H_2 + \delta v^2H_{2u}|}, \tag{20}$$

where H_{2u} *is the partial derivative of* $H_2(u,v)$ *with respect* u.

Proof. Applying Theorem 1 for the curve C_γ we get that

$$\overline{S}_2(u,v) = C_\gamma(u) + v\left(\boldsymbol{T}(u) + \frac{\varkappa_{C_\gamma}(u)}{\tau_{C_\gamma}(u)}\boldsymbol{B}(u)\right).$$

Replacing with the expressions (5) we obtain (19). The surface $\overline{S}_2(u,v)$ is obviously developable. Next, the equation (20) follows immediately from the equation (17) in the Corollary 1 having in mind that the arc-length prime $\dot{s}_{C_\gamma}$ of C_γ is equal to $|c_1\tau + c_2'| = \delta(c_1\tau + c_2')$. □

The vector $\widetilde{\boldsymbol{D}}(u)$ is referred to as the modified Darboux vector field along the curve γ. The next statement gives us the relations between the coefficients of the First and the Second Fundamental Form of $\overline{S}_2(u,v)$ and $S_2(u,v)$.

Theorem 4. *Let* (1) *be a parametrization of a unit speed Frenet curve* $\gamma : I \longrightarrow \mathbb{E}^3$ *of class* C^3 *with Frenet frame* $\boldsymbol{t}(u), \boldsymbol{n}(u), \boldsymbol{b}(u)$, *and let* $\varkappa(u) \neq 0$ *and* $\tau(u) \neq 0$ *be the Euclidean curvature and the Euclidean torsion of* γ, *respectively. Suppose that* $S_2(u,v) = C_\gamma(u) + v\delta\boldsymbol{b}(u)$ *is the developable surface associated with* γ *and* $\overline{S}_2(u,v)$ *is its focal surface. Then the coefficients of the First and the Second Fundamental Form of* $\overline{S}_2(u,v)$ *can be expressed by the coefficients of the First and the Second Fundamental Form of* $S_2(u,v)$ *as follows*

$$\overline{E}_2(u,v) = E_2(u,v) + v^2\left(\left(\frac{\tau(u)}{\varkappa(u)}\right)'^2 - \tau^2(u)\right),$$

$$\overline{F}_2(u,v) = F_2(u,v) + v\left(\frac{\tau(u)}{\varkappa(u)}\right)\left(\frac{\tau(u)}{\varkappa(u)}\right)', \; \overline{G}_2(u,v) = G_2(u,v) + \left(\frac{\tau(u)}{\varkappa(u)}\right)^2,$$

$$\overline{L}_2(u,v) = L_2(u,v)\left|\frac{\delta(c_1\tau + c_2')\frac{\tau(u)}{\varkappa(u)} - v.\left(\frac{\tau(u)}{\varkappa(u)}\right)'}{v\tau}\right|, \; \overline{M}_2(u,v) = 0, \; \overline{N}_2(u,v) = 0.$$

Proof. Let us denote by $\widetilde{s}$ the arc-length parameter of the curve C_γ. Then $\dot{\widetilde{s}} = \delta(c_1\tau + c_2')$. Applying Theorem 1, the equalities (18) in Corollary 1 and (5) the proof is completed. □

4 Application

Let $\boldsymbol{\gamma}(u) = (a(u\sin(u) + \cos(u)), a(\sin(u) - u\cos(u)), bu)$, where a and b are nonzero constant, is a cylindrical curve with a constant speed parametrization. According to [9], the corresponding focal curve C_γ is defined as follows:

$$\boldsymbol{C}_\gamma(u) = \left(\frac{(a^2+b^2)(2\cos(u) - u\sin(u))}{a(u^2+2)}, \frac{(a^2+b^2)(2\sin(u) + u\cos(u))}{a(u^2+2)}, \frac{u^3(a^2+b^2)}{b(u^2+2)}\right).$$

The pictures below depict the curves γ and C_γ, the surfaces $S_1(u,v)$ and $S_2(u,v)$, together with their corresponding focal surfaces $\overline{S}_1(u,v)$ and $\overline{S}_2(u,v)$.

Conclusion

In this paper, we investigated the focal surfaces of developable ruled surfaces generated by a unit speed Frenet curve and by its associated focal curve. Constructing the tangential surfaces corresponding to these curves, we derived explicit parametrizations of their focal surfaces and examined the relations between their geometric invariants. Working with respect to the natural parameter of the base curve allowed us to obtain precise expressions for the coefficients of the first and second fundamental forms of both the original developable surfaces and their focal surfaces.

Figure 1: From left to right: The developable surface $S_1(u,v)$, its focal surface $\overline{S}_1(u,v)$ together with the curve γ for $a=1, b=2$ and both surfaces

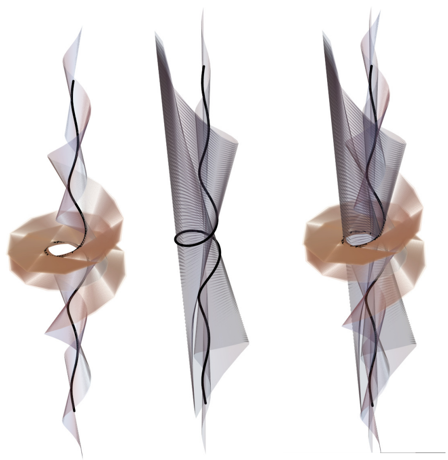

Figure 2: From left to right: The developable surface $S_2(u,v)$, its focal surface $\overline{S}_2(u,v)$ together with the curve C_γ for $a=1, b=2$ and both surfaces

The study establishes a geometric correspondence between a Frenet curve, its tangential surface, and the focal surface of that tangential surface, revealing a deeper structural connection within the family of developable surfaces. Furthermore, we obtained explicit formulas relating the mean curvatures of the original surfaces to those of their focal surfaces, showing how curvature information is transferred through the focal construction.

The theoretical results were illustrated through an example involving a cylindrical curve and its focal curve, together with visualizations of the associated developable and focal surfaces. This example highlights the practical applicability of the derived formulas and confirms the geometric behaviour predicted by the theory.

The results provide a basis for further research on the focal surfaces in applications such as geometric modelling, computer-aided design and construction, and analysis of new surfaces.

Acknowledgments

The authors express their gratitude to the reviewer for the perceptive remarks and useful suggestions.

References

[1] P. Alegre, K. Arslan, A. Carriazo, C. Murathan, G. Ozturk. Some special types of developable ruled surface, *Hacettepe Journal of Mathematics and Statistics*, **39** (3), (2010), 319 – 325.

[2] M. P. do Carmo. Differential geometry of curves and surfaces, (Revised and updated second ed.), Dover Publications, 2016.

[3] L. P. Eisenhart, A Treatise in Differential Geometry of Curves and Surfaces, New York, Ginn Camp., 1969.

[4] G. H. Georgiev, M. D. Pavlov. Focal surfaces of hyperbolic cylinders, *AIP Conf. Proc.*, **1910** (1), (2017), 050005. https://doi.org/10.1063/1.5013987.

[5] G. H. Georgiev, M. D. Pavlov. Curvature Dependent Offsets to Elliptic Cones, *Far East Journal of Mathematical Sciences*, **102** (11), (2017), 2757–2783. https://doi.org/10.17654/MS102112757

[6] G. H. Georgiev. Generalized offset surfaces to a torus, *AIP Conference Proceedings*, **2048** (1), (2018), 0500071–0500078.

[7] G. H. Georgiev, M. D. Pavlov. Focal and generalized focal surfaces of parabolic cylinders, *ARPN Journal of Engineering and Applied Sciences*, **13** (15), (2018), 4458-4465.

[8] G. H. Georgiev, M. D. Pavlov. Generalized offset surfaces to a pseudosphere, *AIP Conf. Proc.*, **2172** (1), (2019), 060002. https://doi.org/10.1063/1.5133530

[9] G. H. Georgiev, R. P. Encheva, C. L. Dinkova. Geometry of cylindrical curves over plane curves. *Applied Mathematical Sciences*, **113** (9) (2015), 5637–5649. http://dx.doi.org/10.12988/ams.2015.56456

[10] A. Gray, E. Abbena, S. Salamon. Modern Differential Geometry of Curves and Surfaces. New York, Chapman and Hall/CRC, 2006.

[11] F. Güler. The focal surfaces of offset surface. *Optik-International Journal for Light and Electron Optics*, **271**, (2022), 170053. https://doi.org/10.1016/j.ijleo.2022.170053.

[12] S. Izumiya, N. Takeuchi. Special curves and ruled surfaces. *Cotributions to Algebra and Geometry*, **44** (2003), 203–212.

[13] H. Pottmann, J. Wallner. Computational Line Geometry. Mathematics and Visualization. Springer Berlin, Heidelberg. 2010 doi:10.1007/978-3-642-04018-4.

[14] M.D. Shepherd. Line congruences as surfaces in the space of lines, *Differential Geometry and its Applications*, (1999), 1–26.

[15] Dirk J. Struik. Lectures on Classical Differential Geometry. 2nd ed., Dover Publications, 1961.

[16] R. Uribe-Vargas. On vertices, focal curvatures and differential geometry of space curves. *Bull. of the Brazilian Math. Soc.* **36** (2005), 285–307.