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A DIOPHANTINE INEQUALITY WITH MIXED POWERS OF FOUR PRIMES OF A SPECIFIC TYPE

Tatiana L. Todorova
Faculty of Mathematics and Informatics, Sofia University "St. Kliment Ohridski", Sofia, Bulgaria
e-mail: tlt@fmi.uni-sofia.bg

In this paper we study a Diophantine inequality with one prime and three prime squares. We prove that it has infinitely many solutions in primes p_i such that each $p_i + 2$ $(i = 1, \dots, 4)$ is an almost-prime of prescribed order.

Keywords: Diophantine inequality, Davenport-Heilbronn method, Rosser's weights, vector sieve, almost primes.

ДИОФАНТОВО НЕРАВЕНСТВО СЪС СМЕСЕНИ СТЕПЕНИ НА ЧЕТИРИ ПРОСТИ ЧИСЛА ОТ СПЕЦИАЛЕН ТИП

Татяна Л. Тодорова
Факултет по математика и информатика,
Софийски университет „Св. Климент Охридски", София, България
e-mail: tlt@fmi.uni-sofia.bg

В статията разглеждаме диофантово неравенство с едно просто и квадратите на три прости числа. Доказваме, че то има безбройно много решения в прости числа p_i, такива че всяко от числата $p_i + 2$, $i = 1, ..., 4$, е почти просто от някакъв ред.

Ключови думи: Диофантово неравенство, метод на Девънпорт-Хейлброн, тегла на Розер, векторно решето, почти прости числа.

1 Introduction

In 1967 Baker [1] demonstrate that for real numbers $\eta, \lambda_i \in \mathbb{R},\ \lambda_i \neq 0,\ i = 1, 2, 3$ which are not all of the same sign, $\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}$ there are infinitely many ordered triples of primes p_1, p_2, p_3 such that

(1) $$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3 + \eta| < \mathcal{E}$$

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with $\mathcal{E} = (\log max\, p_j)^{-A}$, $A > 0$. Baker's result was refined several times, and the best-known result belongs to Matomäki [12] with $\mathcal{E} = (\max p_j)^{-2/9+\theta}$ as here $\theta > 0$. Baker and Harman [2] proved that under the generalized Riemann hypothesis, it is possible to reach $\mathcal{E} = (\max p_j)^{-1/4+\theta}$.

Let

$$\lambda_i \in \mathbb{R},\ \lambda_i \neq 0,\ i = 1, 2, 3, 4, \tag{2}$$

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \quad \text{not all of the same sign}, \tag{3}$$

$$\lambda_1/\lambda_2 \in \mathbb{R} \setminus \mathbb{Q}, \tag{4}$$

$$\eta \in \mathbb{R}\,. \tag{5}$$

Under conditions (2)-(5) Languasco and Zaccagnini [11] considered the diophantine inequality

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta| < \mathcal{E}$$

and proved that it has infinitely many solutions in primes with $\mathcal{E} = (\max p_j)^{-1/18+\theta}$. Wang and Yao [19] improved their result with $\mathcal{E} = (\max p_j)^{-1/14+\theta}$. Under the conditions $\lambda_1/\lambda_2 \in \mathbb{R}\setminus\mathbb{Q}$, and if both λ_1/λ_2 and λ_1/λ_3 are algebraic, they proved that the exponent can be replaced by $-\dfrac{3}{40} + \theta$. The known best result, due to Ge, Zhao, and Wang [9] is $\mathcal{E} = (\max p_j)^{-5/64+\theta}$. Later, Gambini [8] generalised this problem and proved the existence of infinitely many primes $p_1, ..., p_4$ such that

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^k + \eta| < \mathcal{E}$$

with $\mathcal{E} = (\max p_1, p_2^2, p_3^2, p_4^2)^{-\psi(k)+\theta}$, $\psi(k) = \min\{\frac{1}{14}, \frac{14-5k}{28k}\}$ and $1 < k < 14/5$.

Diophantine inequalities in which the prime variables p_i satisfy additional restrictions – for example, that $p_i + 2$ is an almost-prime of given order – form another important category. The well-known hypothesis states that there are infinitely many prime numbers p such that $p + 2$ is also prime. In 1973, Chen [5] demonstrated that there are infinitely many primes p for which $p + 2 = P_2$. Here, P_r represents an integer that has no more than r prime factors, counted with their multiplicities.

In 2015, Dimitrov and Todorova [7] merged the problems studied by Baker and Chen. For $\mathcal{E} = (\log \max p_j)^{-B}$, they showed that inequality (1) admits solutions under the conditions $p_1+2 = P_8$, $p_2+2 = P_8'$, and $p_3+2 = P_8''$. Subsequently, Dimitrov [6] improved the exponent to $\mathcal{E} = (\max p_j)^{-1/12+\theta}$, while requiring $p_1 + 2 = P_{28}$, $p_2 + 2 = P_{28}'$, and $p_3 + 2 = P_{28}''$. Later, the author [15] established the existence of infinitely many triples of primes satisfying (1) with $\mathcal{E} = (\max p_j)^{-1/18+\theta}$ and conditions $p_1 + 2 = P_7$, $p_2 + 2 = P_7'$, $p_3 + 2 = P_7''$.

In 2025, Todorova and Georgieva [17] proved that there are infinitely many prime triples for which

$$|\lambda_1 p_1 + \lambda_2 p_2 + \lambda_3 p_3^2 + \eta| < (\max p_j)^{-1/12+\theta}$$

and

$$p_1 + 2 = P_{20}'\,, \quad p_2 + 2 = P_{20}''\,, \quad p_3 + 2 = P_{42}'''\,.$$

This paper is concerned with a Diophantine inequality in mixed powers of primes, requiring that each $p_i + 2$, $i = 1, 2, 3, 4$, is almost a prime. Specifically, we prove the theorem below.

Theorem 1. If conditions (2), (3), (4), (5) are satisfied and $\theta > 0$, then there exist infinitely many ordered quadruples of primes p_1, p_2, p_3, p_4 with

$$|\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta| < (\max p_j)^{-\frac{1}{200}+\theta} \tag{6}$$

and

$$p_1 + 2 = P'_{20}\,, \quad p_2 + 2 = P''_{41}\,, \quad p_3 + 2 = P'''_{41}\,, \quad p_4 + 2 = P^{\mathrm{iv}}_{41}\,. \tag{7}$$

2 Notations

Throughout, the symbols p, p_1, p_2, p_3, p_4 denote prime numbers. We write $\varphi(n)$ for Euler's totient function and $\mu(n)$ for the Möbius function. The greatest common divisor and the least common multiple of m_1 and m_2 are denoted by (m_1, m_2) and $[m_1, m_2]$, respectively. Congruence modulo k is abbreviated as $m \equiv n\,(k)$ instead of $m \equiv n \pmod{k}$. The integer part of a real number y is written $[y]$, and we set $e(y) = e^{2\pi i y}$.

The notation $k \sim K$ means $K/2 < k \le K$. The letter ε stands for an arbitrarily small positive constant, not necessarily the same at each occurrence. This convention allows us, for example, to write $x^\varepsilon \log x \ll x^\varepsilon$.

3 Auxiliary Results

Our argument makes use of a vector sieve and relies on the theorem stated below.

Lemma 1. Suppose that $\mathcal{D} \in \mathbb{R}, \mathcal{D} > 4$. There exists an arithmetical functions $\lambda^{\pm}(d)$ (called Rosser's functions of level $\mathcal{D}$) with the following properties:

1. For any positive integer d, we have
$$|\lambda^{\pm}(d)| \le 1, \qquad \lambda^{\pm}(d) = 0 \quad \text{if} \quad d > \mathcal{D} \quad \text{or} \quad \mu(d) = 0.$$

2. If $n \in \mathbb{N}$ then
$$\sum_{d|n} \lambda^{-}(d) \le \sum_{d|n} \mu(d) \le \sum_{d|n} \lambda^{+}(d).$$

3. If $\mathbf{z} \in \mathbb{R}$ is such that $\mathbf{z}^2 \le \mathcal{D}$ and if
$$\mathcal{B}^{\pm}(\mathcal{D},\, \mathbf{z}) = \sum_{d|P(\mathbf{z})} \frac{\lambda^{\pm}(d)}{\varphi(d)}, \quad s = \frac{\log \mathcal{D}}{\log \mathbf{z}}, \tag{8}$$
then we have
$$\mathcal{B}(\mathbf{z}) \le \mathcal{B}^{+}(\mathcal{D},\, \mathbf{z}) \le \mathcal{B}(\mathbf{z}) \left(F(s) + O\left((\log \mathcal{D})^{-\frac{1}{3}}\right)\right), \tag{9}$$
$$\mathcal{B}(\mathbf{z}) \ge \mathcal{B}^{-}(\mathcal{D},\, \mathbf{z}) \ge \mathcal{B}(\mathbf{z}) \left(f(s) + O\left((\log \mathcal{D})^{-\frac{1}{3}}\right)\right), \tag{10}$$

where $F(s)$ and $f(s)$ satisfy

$$
\begin{aligned}
&F(s) = \frac{2e^{\gamma}}{s} && \text{if} \quad 0 < s \le 3;\\
&F(s) = \frac{2e^{\gamma}}{s}\left(1 + \int\limits_{2}^{s-1} \frac{\log(t-1)}{t}\,dt\right) && \text{if} \quad 3 \le s \le 5;\\
&f(s) = \frac{2e^{\gamma}\log(s-1)}{s}, && \text{if} \quad 2 \le s \le 4;\\
&(sF(s))' = f(s-1) && \text{if} \quad s > 3;\\
&(sf(s))' = F(s-1) && \text{if} \quad s > 2.
\end{aligned} \tag{11}
$$

Here, γ is Euler's constant $\gamma \approx 0.577$.

Proof: See Greaves [10, Chapter 4] and [4]. □

Mertens' theorem yields

$$\mathcal{B}(\mathbf{z}) = \frac{2C_2 e^{-\gamma}}{\log \mathbf{z}}\Big(1 + O\Big(\frac{1}{\log \mathbf{z}}\Big)\Big), \tag{12}$$

with

$$C_2 = \prod_{p \ge 3}\Big(1 - \frac{1}{(p-1)^2}\Big) \approx 0.66016\ldots$$

being the twin prime constant. From the basic estimates of the linear sieve, namely (9) and (10), it follows that

$$0 \le \mathcal{B}^{-}(\mathcal{D}, \mathbf{z}) \le \mathcal{B}(\mathbf{z}) \le \mathcal{B}^{+}(\mathcal{D}, \mathbf{z}). \tag{13}$$

The proof of our theorem requires bounds for two kinds of exponential sums taken over primes; these bounds are supplied by the next two lemmas.

Lemma 2. Suppose $\alpha \in \mathbb{R}$ and α satisfy conditions

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}\,, \quad a \in \mathbb{Z}\,, \quad q \in \mathbb{N}\,, \quad (a, q) = 1\,, \quad q \ge 1\,. \tag{14}$$

Let $\xi(d)$ be complex numbers defined for $d \le D$ and $\xi(d) \ll 1$. If

$$W(X) = \sum_{d \sim D} \xi(d) \sum_{\substack{p \sim X \\ p+2 \equiv 0\,(d)}} e(\alpha p)\log p \tag{15}$$

then for any arbitrary small $\varepsilon > 0$ we have

$$W(X) \ll X^{\varepsilon}\left(\frac{X}{q^{1/2}} + X^{1/2}q^{1/2} + X^{2/3}D + X^{5/6}\right).$$

Proof: The proof is contained in the proof of Lemma 1 from [18]. □

Lemma 3. Suppose $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ satisfies condition (14), $D \in \mathbb{R}$, $D \le x^{1/6}$, $\lambda(d)$ are complex numbers defined for $d \le D$ and $\lambda(d) \ll \tau(d)$. Then for any arbitrary small $\varepsilon > 0$ and $b \in \mathbb{Z}$ for the sum

$$W = \sum_{d \le D} \lambda(d) \sum_{\substack{n \sim x \\ n \equiv b\,(d)}} e\big(\alpha n^2\big)\Lambda(n) \tag{16}$$

we have

$$W \ll x^{\varepsilon}\left(\frac{xD^{1/2}}{q^{1/4}} + x^{1/2}D^{1/2}q^{1/4} + x^{7/8}D^{3/4} + x^{1/3}D^{1/2}q^{1/2} + \frac{x^{7/6}D^{1/2}}{q^{1/2}}\right). \tag{17}$$

Proof: The proof follows the argument of Theorem 1 in [16]. Using Vaughan's identity, we decompose the sum W into $O(\log x)$ type I sums

$$W_1 = \sum_{\substack{d\sim D\\(b,\,d)=1}} \lambda(d) \sum_{m\sim M} a(m) \sum_{\substack{\ell\sim L\\ m\ell\equiv b\,(d)}} e(\alpha(m\ell)^2)$$

or

$$W_1' = \sum_{\substack{d\sim D\\(b,\,d)=1}} \lambda(d) \sum_{m\sim M} a(m) \sum_{\substack{\ell\sim L\\ m\ell\equiv b\,(d)}} \log(\ell)e(\alpha(m\ell)^2)$$

with $M \leq x^{1/3}$ and $O(\log x)$ type II sums

$$W_2 = \sum_{\substack{d\sim D\\(b,\,d)=1}} \lambda(d) \sum_{\substack{\ell\sim L\\ m\ell\equiv b\,(d)}} b(\ell) \sum_{m\sim M} a(m)e(\alpha(m\ell)^2)$$

with $M \in [x^{1/3},\, x^{2/3}]$ and

$$ML \sim x, \quad a(m) \ll \tau(m)\log m, \quad b(\ell) \ll \tau(\ell)\log \ell\,. \tag{18}$$

To evaluate the sum W_2 we will consider only the case $x^{1/2} \leq M \leq x^{2/3}$. The evaluation in the case $x^{1/2} \leq L \leq x^{2/3}$ is the same. Following the same steps as in the evaluation of W_2 for $D < x^{1/6}$ from the proof of Theorem 1 in [16], we successively obtain

$$W_2^2 \leq x^{\varepsilon}\Big(xMD + V_2\Big),$$

where

$$V_2^2 \ll x^{\varepsilon}\left(x^3LD + x^2D^2 \sum_{z\leq xLD} \min\left\{\frac{x^2}{z},\, \frac{1}{||\alpha z||}\right\}\right).$$

So, from Lemma 2.2 from [20], ch. 2,§2.1 we obtain

$$W_2 \ll x^{\varepsilon}\left(\frac{xD^{1/2}}{q^{1/4}} + x^{1/2}D^{1/2}q^{1/4} + x^{7/8}D^{3/4}\right). \tag{19}$$

Again, proceeding as in the evaluation of W_1 for $D < x^{1/6}$ from Theorem 1 [16], we successively obtain

$$W_1^2 \ll x^{\varepsilon}\left(xMD + MD \sum_{z\ll xMD} \min\left\{\frac{x^2}{z},\, \frac{1}{||\alpha z||}\right\}\right)$$

and from Lemma 2.2 from [20], ch. 2,§2.1 we receive

$$W_1 \ll x^{\varepsilon}\left(x^{1/3}D^{1/2}q^{1/2} + x^{5/6}D + \frac{x^{7/6}D^{1/2}}{q^{1/2}}\right). \tag{20}$$

Therefore, equations (19) and (20) yield

$$W \ll x^{\varepsilon}\left(\frac{xD^{1/2}}{q^{1/4}} + x^{1/2}D^{1/2}q^{1/4} + x^{7/8}D^{3/4} + x^{1/3}D^{1/2}q^{1/2} + \frac{x^{7/6}D^{1/2}}{q^{1/2}}\right).$$

□

Remmark 1. By applying the properties of the von Mangoldt Λ function and Lemma

3 in the range $D \le x^{1/12}$, we find that

$$\sum_{d\le \mathcal{D}} \lambda(d) \sum_{\substack{p^2\sim x\\ p\equiv b\,(d)}} e\big(\alpha p^2\big) \ll x^{\varepsilon}\bigg(\frac{x^{1/2}D^{1/2}}{q^{1/4}} + x^{1/4}D^{1/2}q^{1/4} + x^{7/16}D^{3/4} + x^{1/6}D^{1/2}q^{1/2} + \frac{x^{7/12}D^{1/2}}{q^{1/2}}\bigg).$$

The next lemma gives the necessary estimate.

Lemma 4. Let D, $\widetilde{D}$, Δ, $L(\beta)$ and $\widetilde{L}(\beta)$, $I(\beta)$ and $\widetilde{I}(\beta)$ be defined by (26), (25), (43), (44), (29), (30). If $|\beta| < \Delta$, restriction (24) are fulfilled, then for large enough X and an arbitrarily large fixed positive real number A, the following equalities

$$L(\beta) - I(\beta)\sum_{d\le D}\frac{\lambda(d)}{\varphi(d)} = O\bigg(\frac{X}{(\log X)^{A/2-3}}\bigg) + O\Big(X^{4/5+\xi/2}(\log X)^{14}\Big),$$

$$\widetilde{L}(\beta) - \widetilde{I}(\beta)\sum_{d\le \widetilde{D}}\frac{\lambda(d)}{\varphi(d)} = O\bigg(\frac{X^{1/2}}{(\log X)^{A/2-3}}\bigg) + O\Big(X^{2/5+\xi/2}(\log X)^{14}\Big)$$

are fulfilled.

Proof: This is Lemma 13 from [17]. □

The following lemmas provide estimates for the integrals $I(\beta)$ and $\widetilde{I}(\beta)$ and for the integrals derived from them.

Lemma 5. For integrals $I(\beta)$ and $\widetilde{I}(\beta)$, defined by (29) and (30), we have

$$I(\beta) \ll \frac{1}{|\beta|},$$

$$\widetilde{I}(\beta) \ll \frac{1}{|\beta|\sqrt{X}}\,. \qquad (21)$$

Proof: The statement is followed by partial integration. □

Remmark 2. We notice that the trivial estimate

$$\widetilde{I}(\beta) \ll X^{1/2} \qquad (22)$$

is valid.

Lemma 6. Let Δ be defined by (25). Then for integrals $I(\beta)$ and $\widetilde{I}(\beta)$, defined by (29) and (30), we have

$$\int_{-\Delta}^{\Delta} |I(\beta)|^2 d\beta \ll X \quad \text{and} \quad \int_{-\Delta}^{\Delta} |\widetilde{I}(\beta)|^2 d\beta \ll 1.$$

Proof: This is Lemma 15 from [17]. □

We shall also require the estimates provided by the following Lemma.

Lemma 7. Let Δ, D, $\widetilde{D}$ be defined by (25), (26) and condition (24) are fulfilled. Then for sums $L(\beta)$ and $\widetilde{L}(\beta)$ (see (43) and (44)), we have

$$\int\limits_{-\Delta}^{\Delta} |L(\beta)|^2 d\beta \ll X \log^5 X \quad \text{and} \quad \int\limits_{-\Delta}^{\Delta} |\widetilde{L}(\beta)|^2 d\beta \ll \sqrt{X} \log^5 X\,.$$

Proof: This is Lemma 16 from [17] □

Remmark 3. We notice that the trivial estimate

(23) $$|\widetilde{L}(\beta)| \ll X^{1/2} \log X$$

is valid.

4 Beginning of the proof

Let $0 < \lambda_0 < 1$, $\xi, \delta, \alpha, \widetilde{\delta}, \widetilde{\alpha}$, be positive real numbers that we will specify later, but for now only assume the conditions

(24) $$\xi + 3\delta < \frac{12}{25}, \quad \xi + \delta < \frac{3}{8}, \quad \xi + 3\widetilde{\delta} < \frac{6}{25}, \quad \xi + \widetilde{\delta} < \frac{3}{16}, \quad 0 < \alpha < \delta/2, \quad 0 < \widetilde{\alpha} < \widetilde{\delta}/2.$$

We define

(25) $$\Delta = X^{\xi - 1},$$

(26) $$D = X^{\delta}, \quad \widetilde{D} = X^{\widetilde{\delta}},$$

(27) $$\mathcal{E} = X^{-\frac{1}{200} + \theta}, \ \ \theta > 0 \text{ is arbitrary small}, \quad H = \frac{1000 \log^2 X}{\mathcal{E}},$$

(28) $$z = X^{\alpha}, \quad \widetilde{z} = X^{\widetilde{\alpha}},$$

(29) $$I(\beta) = \int\limits_{\lambda_0 X}^{X} e(\beta y) dy,$$

(30) $$\widetilde{I}(\beta) = \int\limits_{\lambda_0^{\frac{1}{2}} X^{\frac{1}{2}}}^{X^{\frac{1}{2}}} e(\beta y^2) dy,$$

(31) $$P(z) = \prod_{2 < p < \mathbf{z}} p, \quad \mathcal{B}(\mathbf{z}) = \prod_{2 < p < \mathbf{z}} \left(1 - \frac{1}{p-1}\right).$$

Consider the sum

$$\Gamma(X) = \sum_{\substack{\lambda_0 X < p_1, p_2, p_3, p_4 \leq X \\ (p_1 + 2, P(z)) = 1, \\ (p_i + 2, P(\widetilde{z})) = 1, i = 2,3,4 \\ |\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta| < \mathcal{E}}} \log p_1 \log p_2 \log p_3 \log p_4\,,$$

with $\mathcal{E}$ given by (27). If we prove the inequality $\Gamma(X) > 0$ then the inequality (6) would have a solution in primes p_1, p_2, p_3, p_4 satisfying $(p_1 + 2, P(z)) = 1$, $(p_i + 2, P(\widetilde{z})) = 1, i = 2, 3, 4$. If the number $p_i + 2$ has ℓ_i prime factors counted with multiplicity, then from

(28) we find that $\ell_1 < \dfrac{1}{\alpha}$ and $\ell_i < \dfrac{1}{\widetilde{\alpha}}$ for $i = 2, 3, 4$. This means that $p_1 + 2$ would be an almost prime of order $[1/\alpha]$ and $p_i + 2$, $i = 2, 3, 4$ would be an almost prime of order $[1/\widetilde{\alpha}]$.

To transform the sum $\Gamma(X)$ we take a function υ such that

$$\begin{array}{lll} \upsilon(x) = 1 & \text{for} & |x| \le 3\mathcal{E}/4 \\ 0 < \upsilon(x) < 1 & \text{for} & 3\mathcal{E}/4 < |x| < \mathcal{E} \\ \upsilon(x) = 0 & \text{for} & |x| \ge \mathcal{E}\,. \end{array} \tag{32}$$

The function υ has derivatives of sufficiently large order, and its Fourier transform

$$\Upsilon(x) = \int\limits_{-\infty}^{\infty} \upsilon(t) e(-xt) dt \tag{33}$$

satisfy

$$|\Upsilon(x)| \le \min\left(\frac{7\mathcal{E}}{4}, \frac{1}{\pi|x|}, \frac{1}{\pi|x|}\left(\frac{k}{2\pi|x|\mathcal{E}/8}\right)^k\right) \tag{34}$$

for all $k \in \mathbb{N}$. For the existence of such a function, see [13].

Using the function $\upsilon(x)$ we get

$$\Gamma(X) \ge \Gamma_0(X) = \sum_{\substack{\lambda_0 X < p_1, p_2, p_3, p_4 \le X \\ (p_1+2, P(z))=1, \\ (p_i+2, P(\widetilde{z}))=1, i=2,3,4}} \upsilon(\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta) \times \log p_1 \log p_2 \log p_3 \log p_4\,. \tag{35}$$

Our goal is to demonstrate that for specific values of α and $\widetilde{\alpha}$ (as large as possible), there exists a sequence $X_1, X_2, \ldots \to \infty$ such that $\Gamma_0(X_j) > 0$. Then the number of prime solutions p_i of (6) in the interval $(\lambda_0 X_j, X_j]$ with $p_1 + 2 = P_{[1/\alpha]}$ and $p_i + 2 = P_{[1/\widetilde{\alpha}]}$ for $i = 2, 3, 4$ is positive. This approach allows us to generate an infinite sequence of quadruples of primes p_1, p_2, p_3, p_4 that satisfy the desired properties.

Let $\Lambda_1 = \sum\limits_{d|(p_1+2, P(z))} \mu(d)$ and $\Lambda_i = \sum\limits_{d|(p_i+2, P(\widetilde{z}))} \mu(d)$, $i = 2, 3, 4$ be the characteristic functions of primes p_i, such that $(p_1 + 2,\ P(z)) = 1$ and $(p_i + 2,\ P(\widetilde{z})) = 1$ for $i = 2, 3, 4$, respectively. Then from (35) it follows that

$$\Gamma_0(X) = \sum_{\substack{\lambda_0 X < p_i \le X \\ i=1,2,3,4}} \upsilon(\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta) \times \Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4 \log p_1 \log p_2 \log p_3 \log p_4\,. \tag{36}$$

Let $\lambda^{\pm}(d)$ and $\widetilde{\lambda}^{\pm}(d)$ represent the lower and upper bounds of Rosser's weights at levels D and $\widetilde{D}$, respectively (see Lemma 1). If

$$\Lambda_1^{\pm} = \sum_{d|(p_1+2, P(z))} \lambda^{\pm}(d) \quad \text{and} \quad \Lambda_i^{\pm} = \sum_{d|(p_i+2, P(\widetilde{z}))} \widetilde{\lambda}^{\pm}(d)\,, i = 2, 3, 4 \tag{37}$$

then, from Lemma 1 we have $\Lambda_i^{-} \le \Lambda_i \le \Lambda_i^{+}$, $i = 1,\ 2,\ 3,\ 4$.

We will utilize the following simple inequality

$$\Lambda_1\Lambda_2\Lambda_3\Lambda_4 \geq \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^- - 3\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+ \tag{38}$$

see [[3], Lemma 13]. Using (36) and (38) we get

$$\Gamma_0(X) \geq \Gamma(X) = \sum_{\lambda_0 X < p_1,p_2,p_3,p_4 \leq X} \upsilon(\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta) \times \big(\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^- - 3\Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+\big) \log p_1 \log p_2 \log p_3 \log p_4 \,. \tag{39}$$

We substitute the function $\upsilon(\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta)$ in (39) with its inverse Fourier transform (33) and get

$$\Gamma(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\substack{\lambda_0 X < p_i \leq X \\ i=1,2,3,4}} e\big((\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta)t\big) \times \log p_1 \log p_2 \log p_3 \log p_4 \big(\Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^-\Lambda_4^+ + \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^- - 3\Lambda_1^+\Lambda_2^+\Lambda_3^+\big)\, dt \,. \tag{40}$$

Thus,

$$\Gamma(X) = \Gamma_1(X) + \Gamma_2(X) + \Gamma_3(X) + \Gamma_4(X) - 3\Gamma_5(X) \tag{41}$$

where $\Gamma_1(X)$, $\Gamma_2(X)$, $\Gamma_3(X)$, $\Gamma_4(X)$, and $\Gamma_5(X)$ are the contributions of the consecutive terms on the right side of (40). It is clear that

$$\Gamma_1(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1,p_2,p_3,p_4 \leq X} e\big((\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta)t\big) \times \Lambda_1^-\Lambda_2^+\Lambda_3^+\Lambda_4^+ \log p_1 \log p_2 \log p_3 \log p_4 \, dt \,,$$

$$\Gamma_2(X) = \Gamma_3(X) = \Gamma_4(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1,p_2,p_3,p_4 \leq X} e\big((\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta)t\big) \times \Lambda_1^+\Lambda_2^-\Lambda_3^+\Lambda_4^+ \log p_1 \log p_2 \log p_3 \log p_4 \, dt \,,$$

$$\Gamma_5(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) \sum_{\lambda_0 X < p_1,p_2,p_3,p_4 \leq X} e\big((\lambda_1 p_1 + \lambda_2 p_2^2 + \lambda_3 p_3^2 + \lambda_4 p_4^2 + \eta)t\big) \times \Lambda_1^+\Lambda_2^+\Lambda_3^+\Lambda_4^+ \log p_1 \log p_2 \log p_3 \log p_4 \, dt \,.$$

Therefore,

$$\Gamma(X) \geq \Gamma_1(X) + 3\Gamma_2(X) - 3\Gamma_5(X) \,. \tag{42}$$

We proceed to estimate $\Gamma_1(X)$. The integrals $\Gamma_i(X)$, $i = 2, ..., 5$ can be handled analogously. Changing the order of summation and taking into account (37), we obtain

$$\Gamma_1(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t) e(\eta t) L^-(\lambda_1 t) \widetilde{L}^+(\lambda_2 t) \widetilde{L}^+(\lambda_3 t) \widetilde{L}^+(\lambda_4 t) dt \,,$$

where

(43)
$$L^{\pm}(t)=\sum_{d|P(z)}\lambda^{\pm}(d)\sum_{\substack{\lambda_0X<p\le X\\ p+2\equiv 0\,(d)}}e(pt)\log p$$

and

(44)
$$\widetilde{L}^{\pm}(t)=\sum_{d|P(\widetilde{z})}\widetilde{\lambda}^{\pm}(d)\sum_{\substack{\lambda_0X<p^2\le X\\ p+2\equiv 0\,(d)}}e(p^2t)\log p\,.$$

We denote that $\lambda^{\pm}(d),\widetilde{\lambda}^{\pm}(d)$ are real numbers such that $|\lambda^{\pm}(d)|\le 1$ and $|\widetilde{\lambda}^{\pm}(d)|\le 1$. Furthermore, $\lambda^{\pm}(d)=0$ and $\widetilde{\lambda}^{\pm}(d)=0$ if $2|d$ or $\mu(d)=0$. Following the Davenport–Heilbronn version of the circle method (see [[20], Ch. 11]), we split the integral $\Gamma_1(X)$ into three distinct parts:

(45)
$$\Gamma_1(X)=\Gamma_1^{(1)}(X)+\Gamma_1^{(2)}(X)+\Gamma_1^{(3)}(X)\,,$$

where

(46)
$$\Gamma_1^{(1)}(X)=\int\limits_{|t|\le\Delta}\Upsilon(t)e(\eta t)L^{-}(\lambda_1t)\widetilde{L}^{+}(\lambda_2t)\widetilde{L}^{+}(\lambda_3t)\widetilde{L}^{+}(\lambda_4t)dt,$$

(47)
$$\Gamma_1^{(2)}(X)=\int\limits_{\Delta<|t|<H}\Upsilon(t)e(\eta t)L^{-}(\lambda_1t)\widetilde{L}^{+}(\lambda_2t)\widetilde{L}^{+}(\lambda_3t)\widetilde{L}^{+}(\lambda_4t)dt,$$

(48)
$$\Gamma_1^{(3)}(X)=\int\limits_{|t|\ge H}\Upsilon(t)e(\eta t)L^{-}(\lambda_1t)\widetilde{L}^{+}(\lambda_2t)\widetilde{L}^{+}(\lambda_3t)\widetilde{L}^{+}(\lambda_4t)dt\,.$$

The quantities $\Delta=\Delta(X)$ and $H=H(X)$ are defined in equations (25) and (27), respectively. Lemma 2 of [7] provides the following result:

(49)
$$\Gamma_1^{(3)}(X)\ll 1.$$

The estimate of $\Gamma_i^{(3)}(X), i=2,3,4$ is the same. It is easy to see from equation (42) that

(50)
$$\Gamma\ge\big|\Gamma_1^{(1)}(X)+3\Gamma_3^{(1)}(X)-3\Gamma_5^{(1)}(X)\big|$$
$$+O\big(|\Gamma_1^{(2)}(X)|\big)+O\big(|\Gamma_2^{(2)}(X)|\big)+O\big(|\Gamma_5^{(2)}(X)|\big)+O\big(1\big)\,.$$

We will estimate $\Gamma_1^{(1)}(X)$ and $\Gamma_1^{(2)}(X)$ in Sections 5 and 6, respectively. The estimation of the sums $\Gamma_i^{(1)}(X)$, $\Gamma_i^{(2)}(X)$, $i=2,3,4$ is carried out in the same way. In Section 7, we will finalize the proof of the theorem.

5 Asymptotic formula for $\mathbf{\Gamma_1^{(1)}(X)}$

From now on, we will put

(51)
$$\mathcal{B}^{\pm}=\mathcal{B}^{\pm}(D,\,z),\quad \widetilde{\mathcal{B}}^{\pm}=\mathcal{B}^{\pm}(\widetilde{D},\,\widetilde{z}).$$

To find asymptotic formulae for $\Gamma_1^{(1)}(X)$ we need the following

Lemma 8. Let Δ, D, $\widetilde{D}$, z, $\tilde{z}$, $I(\beta)$, $\widetilde{I}(\beta)$, $L(\beta)$ and $\widetilde{L}(\beta)$ be defined by (25), (26), (28), (29), (30), (43), (44) and

$$R = \int_{-\Delta}^{\Delta} \Upsilon(t)e(\eta t)\Big(\mathcal{B}^{\pm}\big(\widetilde{\mathcal{B}}^{\pm}\big)^3 I(\lambda_1 t)\widetilde{I}(\lambda_2 t)\widetilde{I}(\lambda_3 t)\widetilde{I}(\lambda_4 t) - L^{\pm}(\lambda_1 t)\widetilde{L}^{\pm}(\lambda_2 t)\widetilde{L}^{\pm}(\lambda_3 t)\widetilde{L}^{\pm}(\lambda_4 t)\Big)\, dt\,.$$

Then

$$R \ll \mathcal{E}\bigg(\frac{X^{3/2}}{(\log X)^{A/2-8}} + X^{7/5+\xi/2}(\log X)^{19}\bigg)\,.$$

Proof: Using estimate (34) we found that

$$R \ll \mathcal{E}\big(J_1 + J_2 + J_3 + J_4\big)\,, \tag{52}$$

where

$$J_1 = \int_0^{\Delta} \Big|\mathcal{B}^{\pm} I(\lambda_1 t) - L^{\pm}(\lambda_1 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_2 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_3 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_4 t)\Big| dt,$$

$$J_2 = \int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_2 t) - \widetilde{L}^{\pm}(\lambda_2 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_3 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_4 t)\Big| dt,$$

$$J_3 = \int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big|\big|\widetilde{L}^{\pm}(\lambda_2 t)\big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_3 t) - \widetilde{L}^{\pm}(\lambda_3 t)\Big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_4 t)\Big| dt,$$

$$J_4 = \int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big|\big|\widetilde{L}^{\pm}(\lambda_2 t)\big|\big|\widetilde{L}^{\pm}(\lambda_3 t)\big|\Big|\widetilde{\mathcal{B}}^{\pm}\widetilde{I}(\lambda_4 t) - \widetilde{L}^{\pm}(\lambda_4 t)\Big| dt.$$

From Lemma 4, (24), (13) and (12) follows

$$J_1 \ll \bigg(\frac{X}{(\log X)^{A/2}} + X^{4/5+\xi/2}(\log X)^{11}\bigg) J_{11}, \tag{53}$$

where

$$J_{11} = \int_0^{\Delta} \big|\widetilde{I}(\lambda_2 t)\big|\big|\widetilde{I}(\lambda_3 t)\big|\big|\widetilde{I}(\lambda_4 t)\big| dt\,.$$

Using the Cauchy-Schwarz inequality and Lemma 6, we get

$$J_{11} \ll \bigg(\int_0^{\Delta} \big|\widetilde{I}(\lambda_2 t)\big|^2 dt\bigg)^{1/2}\bigg(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^2\big|\widetilde{I}(\lambda_4 t)\big|^2 dt\bigg)^{1/2}$$

$$\ll \bigg(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^2\big|\widetilde{I}(\lambda_4 t)\big|^2 dt\bigg)^{1/2}.$$

Repeated application of the Cauchy–Schwarz inequality together with the trivial bound

(22) yields

$$J_{11} \ll \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^4 dt \Big)^{1/4} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_4 t)\big|^4 dt \Big)^{1/4} \tag{54}$$

$$\ll X^{1/2} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^2 dt \Big)^{1/4} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_4 t)\big|^2 dt \Big)^{1/4}.$$

From Lemma 6 we have $J_{11} \ll X^{1/2}$, and (53) implies

$$J_1 \ll \frac{X^{3/2}}{(\log X)^{A/2}} + X^{13/10+\xi/2} (\log X)^{11}. \tag{55}$$

To estimate J_2 we will use Lemma 4 and (12):

$$J_2 \ll \Big(\frac{X^{1/2}}{(\log X)^{A/2-1}} + X^{2/5+\xi/2} (\log X)^{12} \Big) J_{21}, \tag{56}$$

where

$$J_{21} = \int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big| \big|\widetilde{I}(\lambda_3 t)\big| \big|\widetilde{I}(\lambda_4 t)\big| dt.$$

Applying twice the Cauchy-Schwarz inequality and Lemma 7 we obtain

$$J_{21} = \Big(\int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big|^2 dt \Big)^{1/2} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^2 \big|\widetilde{I}(\lambda_4 t)\big|^2 dt \Big)^{1/2}$$

$$\ll X^{1/2} \big(\log X \big)^{5/2} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_3 t)\big|^4 dt \Big)^{1/4} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_4 t)\big|^4 dt \Big)^{1/4}.$$

We estimate the last two integrals above using an argument analogous to the one for (54). So, from (56) we get

$$J_2 \ll \frac{X^{3/2}}{(\log X)^{A/2-7/2}} + X^{7/5+\xi/2} (\log X)^{29/2}. \tag{57}$$

By the same reasoning, we obtain

$$J_3 \ll \Big(\frac{X^{1/2}}{(\log X)^{A/2-2}} + X^{2/5+\xi/2} (\log X)^{13} \Big) J_{31}, \tag{58}$$

where

$$J_{31} = \int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big| \big|\widetilde{L}^{\pm}(\lambda_2 t)\big| \big|\widetilde{I}(\lambda_4 t)\big| dt.$$

Using trivial estimate (23) and the Cauchy-Schwarz inequality we obtain

$$J_{31} = X^{1/2} \log X \Big(\int_0^{\Delta} \big|L^{\pm}(\lambda_1 t)\big|^2 dt \Big)^{1/2} \Big(\int_0^{\Delta} \big|\widetilde{I}(\lambda_4 t)\big|^2 dt \Big)^{1/2}.$$

Applying Lemma 7 and Lemma 6 yields

$$J_{31} \ll X (\log X)^{7/2}.$$

Combined with (58) we receive

$$J_3 \ll \frac{X^{3/2}}{(\log X)^{A/2-11/2}} + X^{7/5+\xi/2}(\log X)^{33/2}. \tag{59}$$

An analogous treatment of J_4 gives

$$J_4 \ll \left(\frac{X^{1/2}}{(\log X)^{A/2-3}} + X^{2/5+\xi/2}(\log X)^{14}\right) J_{41}, \tag{60}$$

where

$$J_{41} = \int\limits_0^\Delta \left|L^\pm(\lambda_1 t)\right|\left|\widetilde{L}^\pm(\lambda_2 t)\right|\left|\widetilde{L}^\pm(\lambda_3 t)\right| dt.$$

Applying the Cauchy–Schwarz inequality twice and using Lemma 7, we obtain

$$J_{41} \ll \left(\int\limits_0^\Delta \left|L^\pm(\lambda_1 t)\right|^2 dt\right)^{1/2} \left(\int\limits_0^\Delta \left|\widetilde{L}^\pm(\lambda_2 t)\right|^4 dt\right)^{1/4} \left(\int\limits_0^\Delta \left|\widetilde{L}^\pm(\lambda_3 t)\right|^4 dt\right)^{1/4} \tag{61}$$

$$\ll X^{1/2}(\log X)^{5/2} \left(\int\limits_0^\Delta \left|\widetilde{L}^\pm(\lambda_2 t)\right|^4 dt\right)^{1/4} \left(\int\limits_0^\Delta \left|\widetilde{L}^\pm(\lambda_3 t)\right|^4 dt\right)^{1/4}.$$

Let $i = 2$ or $i = 3$. From $\Delta \ll 1$ (see (25)) and λ_i-fixed follows that

$$\int\limits_0^\Delta \left|\widetilde{L}^+(\lambda_i t)\right|^4 dt = \frac{1}{\lambda_i} \int\limits_0^{\lambda_i \Delta} \left|\widetilde{L}^+(t)\right|^4 dt \ll \int\limits_0^1 \left|\widetilde{L}^+(t)\right|^4 dt$$

$$\ll \sum_{\substack{d_i | P(z) \\ i=1,2,3,4}} \sum_{\substack{(\lambda_0 X)^{\frac{1}{2}} < p_j \le X^{\frac{1}{2}} \\ p_j + 2 \equiv 0\, (d_j) \\ j=1,2,3,4}} \log p_1 \log p_2 \log p_3 \log p_4 \int\limits_0^1 e((p_1^2 + p_2^2 - p_3^2 - p_4^2)t) dt$$

Arguing as in §6 [17] we obtain that

$$\int\limits_0^\Delta \left|\widetilde{L}^+(\lambda_i t)\right|^4 dt \ll X \log^5 X.$$

Therefore, from (61) and above inequality we get

$$J_{41} \ll X \log^5 X,$$

so from (60) we receive

$$J_4 \ll \frac{X^{3/2}}{(\log X)^{A/2-8}} + X^{7/5+\xi/2}(\log X)^{19}. \tag{62}$$

From (55), (57), (59) and (62) follows the statement of the lemma. □

From Lemma 8 and from (46) we get

$$\Gamma_1^{(1)}(X) = \mathcal{B}^-\left(\widetilde{\mathcal{B}}^+\right)^3 \int\limits_{|t|\leq\Delta} \Upsilon(t)e(\eta t)I(\lambda_1 t)\widetilde{I}(\lambda_2 t)\widetilde{I}(\lambda_3 t)\widetilde{I}(\lambda_4 t)dt + O\left(\frac{\mathcal{E}X^{3/2}}{(\log X)^{A/2-8}}\right) + O\left(\mathcal{E}X^{7/5+\xi/2}(\log X)^{19}\right). \tag{63}$$

Let

$$\mathcal{J}(X) = \int\limits_{-\infty}^{\infty} \Upsilon(t)e(\eta t)I(\lambda_1 t)\widetilde{I}(\lambda_2 t)\widetilde{I}(\lambda_3 t)\widetilde{I}(\lambda_4 t)dt\,,$$

$$\mathcal{J}_1(X) = \int\limits_{\Delta}^{\infty} \Upsilon(t)e(\eta t)I(\lambda_1 t)\widetilde{I}(\lambda_2 t)\widetilde{I}(\lambda_3 t)\widetilde{I}(\lambda_4 t)dt\,,$$

$$\mathcal{J}_2(X) = \int\limits_{-\infty}^{-\Delta} \Upsilon(t)e(\eta t)I(\lambda_1 t)\widetilde{I}(\lambda_2 t)\widetilde{I}(\lambda_3 t)\widetilde{I}(\lambda_4 t)dt\,.$$

We evaluate $\mathcal{J}_1(X)$; the integral $\mathcal{J}_2(X)$ is estimated in the same way. Using Lemma 5, (34) and (25) we get

$$\mathcal{J}_1(X) \ll \frac{\mathcal{E}}{X^{3/2}} \int\limits_{\Delta}^{\infty} \frac{dt}{|\lambda_1\lambda_2\lambda_3\lambda_4|t^4} \ll \frac{\mathcal{E}}{\Delta^3 X^{3/2}} = \mathcal{E}X^{3/2-3\xi}\,. \tag{64}$$

Using restriction (24) we choose

$$\xi = \frac{1}{100}\,, \quad \delta = \frac{47}{300} - \varepsilon \quad \text{and} \quad \widetilde{\delta} = \frac{23}{300} - \varepsilon\,, \tag{65}$$

where ε is an arbitrarily small positive number. From (63), (64), (13), (12) we get

$$\Gamma_1^{(1)}(X) = \mathcal{J}(X)\mathcal{B}^-\left(\widetilde{\mathcal{B}}^+\right)^3 + O\left(\frac{\mathcal{E}X^{3/2}}{(\log X)^{A/2-8}}\right). \tag{66}$$

Applying the method of Lemma 4 in [7] with the parameter restriction

$$\lambda_0 < \min\left(\frac{\lambda_1}{4|\lambda_4|},\ \frac{\lambda_2}{4|\lambda_4|},\ \frac{\lambda_3}{4|\lambda_4|},\ \frac{1}{16}\right),$$

we derive the lower bound

$$\mathcal{J} \gg \mathcal{E}X^{3/2}.$$

The constant implicit in the Vinogradov symbol $\gg$ depends solely on λ_1, λ_2, λ_3 and λ_4.

6 Asymptotic formula for $\mathbf{\Gamma_1^{(2)}(X)}$

Relation (50) shows that obtaining a nontrivial lower bound for Γ requires proving that the integrals $\Gamma_i^{(2)}$, $i = 1, ..., 5$ are sufficiently small. To achieve this, we exploit the irrationality of the ratio λ_1/λ_2. This property will enable us to give a nontrivial estimate for at least one of the sums $L^{\pm}(\lambda_1 t)$ or $\widetilde{L}^{\pm}(\lambda_2 t)$.

Let

$$V(t,\,X) = \min\left\{\left|L^{\pm}(\lambda_1 t)\right|^{1/2},\left|\widetilde{L}^{\pm}(\lambda_2 t)\right|\right\}\,. \tag{67}$$

Given our choice of parameters in (65) and by Lemma 2, if $\lambda_1 t$ is irrational, we obtain

$$|L^{\pm}(\lambda_1 t)|^{1/2} \ll X^{\varepsilon}\bigg(\frac{X^{1/2}}{q^{1/4}} + X^{1/4}q^{1/4} + X^{5/12}\bigg).$$

One readily verifies that for

$$q \in \Big[X^{19/75},\, X^{87/150}\Big], \tag{68}$$

the bound

$$|L^{\pm}(\lambda_1 t)|^{1/2} \ll X^{99/200+\varepsilon}$$

holds. From the choice made in (65) and by Remark 1, if $\lambda_2 t$ is irrational, we obtain the estimate

$$|\widetilde{L}^{\pm}(\lambda_2 t)| \ll X^{\varepsilon}\bigg(\frac{X^{323/600}}{q^{1/4}} + X^{173/600}q^{1/4} + X^{99/200} + X^{123/600}q^{1/2} + \frac{X^{373/600}}{q^{1/2}}\bigg).$$

Hence, for any q, satisfying condition (68),

$$|\widetilde{L}^{\pm}(\lambda_2 t)| \ll X^{99/200+\varepsilon}.$$

The following statement holds.

Lemma 9. Let $t,\, X,\, \lambda_1,\, \lambda_2 \in \mathbb{R}$,

$$|t| \in (\Delta,\, H)\,, \tag{69}$$

where Δ and H are denoted by (25) and (27), $\lambda_1,\, \lambda_2$ satisfies (4) and $V(t,\, X)$ is defined by (67). Then there exists a sequence of real numbers $X_1,\, X_2, \ldots \to \infty$ such that

$$V(t\,, X_j) \ll X_j^{\frac{99}{200}+\varepsilon}\,, \quad j = 1, 2, \ldots\,. \tag{70}$$

Proof: The proof is identical to that of Lemma 8 in [15]. Since $\lambda_1/\lambda_2 \in \mathbb{R}\setminus\mathbb{Q}$, Corollary 1B of [14] guarantees that there are infinitely many coprime pairs (a, q) with q arbitrarily large, satisfying

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| < \frac{1}{q^2}. \tag{71}$$

For sufficiently large q we choose X such that

$$q = X^{1/3}. \tag{72}$$

The argument of Lemma 8 in [15] produces an infinite sequence $q^{(1)}, q^{(2)}, \ldots$ satisfying (71). In view of (72), this generates an infinite sequence $X_1, X_2, \ldots$ such that, for each X_j, either $\lambda_1 t$ or $\lambda_2 t$ admits a rational approximation with a denominator in the interval (68). Therefore, (70) is satisfied and the proof is complete. □

To estimate the integral $\Gamma_1^{(2)}$ we will use (67) to notice that

$$|L^{-}(\lambda_1 t)||\widetilde{L}^{+}(\lambda_2 t)| \leq V(t\,, X_j)\bigg(|L^{-}(\lambda_1 t)|^{\frac{1}{2}}|\widetilde{L}^{+}(\lambda_2 t)| + |L^{-}(\lambda_1 t)|\bigg).$$

Next, from (34), above inequalities and estimate (70) for integral $\Gamma_1^{(2)}(X_j)$, denoted by (47) we find

$$\begin{aligned} \Gamma_1^{(2)}(X_j) &\ll \mathcal{E}V(t\,, X_j)\Big(\mathcal{I}_1(X_j) + \mathcal{I}_2(X_j)\Big) \\ &\ll \mathcal{E}X_j^{99/200+\varepsilon}\Big(\mathcal{I}_1(X_j) + \mathcal{I}_2(X_j)\Big)\,, \end{aligned} \tag{73}$$

where

$$\mathcal{I}_1(X_j) = \int\limits_{\Delta}^{H} |L^-(\lambda_1 t)|^{\frac{1}{2}} |\widetilde{L}^+(\lambda_2 t)||\widetilde{L}^+(\lambda_3 t)||\widetilde{L}^+(\lambda_4 t)|dt,$$

$$\mathcal{I}_2(X_j) = \int\limits_{\Delta}^{H} |L^-(\lambda_1 t)||\widetilde{L}^+(\lambda_3 t)||\widetilde{L}^+(\lambda_4 t)|dt\,.$$

Using twice the Cauchy-Schwarz inequality, we get

$$\mathcal{I}_1(X_j) \ll \Bigg(\int\limits_{\Delta}^{H} \big|L^-(\lambda_1 t)\big|^2 dt\Bigg)^{\frac{1}{4}} \Bigg(\int\limits_{\Delta}^{H} \Big|\widetilde{L}^+(\lambda_2 t)\Big|^4 dt\Bigg)^{\frac{1}{4}} \times \Bigg(\int\limits_{\Delta}^{H} \Big|\widetilde{L}^+(\lambda_3 t)\Big|^4 dt\Bigg)^{\frac{1}{4}} \Bigg(\int\limits_{\Delta}^{H} \Big|\widetilde{L}^+(\lambda_4 t)\Big|^4 dt\Bigg)^{\frac{1}{4}}. \tag{74}$$

Arguing as in §6 [7] and §6 [17] we obtain that

$$\begin{aligned} &\int\limits_{\Delta}^{H} \big|L^{\pm}(\lambda_j t)\big|^2 dt \ll H X_j \log^5 X_j, \\ &\int\limits_{\Delta}^{H} \Big|\widetilde{L}^{\pm}(\lambda_i t)\Big|^4 dt \ll H X_j \log^5 X_j. \end{aligned} \tag{75}$$

So

$$\mathcal{I}_1(X_j) \ll H X_j \log^5 X_j. \tag{76}$$

To estimate $\mathcal{I}_2(X_j)$ we use twice the Cauchy-Schwarz inequality and get

$$\mathcal{I}_2(X_j) \ll \Bigg(\int\limits_{\Delta}^{H} \big|L^-(\lambda_1 t)\big|^2 dt\Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{\Delta}^{H} \Big|\widetilde{L}^+(\lambda_3 t)\Big|^4 dt\Bigg)^{\frac{1}{4}} \Bigg(\int\limits_{\Delta}^{H} \Big|\widetilde{L}^+(\lambda_4 t)\Big|^4 dt\Bigg)^{\frac{1}{4}}.$$

Using estimates (75) we get

$$\mathcal{I}_2(X_j) \ll H X_j \log^5 X_j. \tag{77}$$

From (73), (76), (77) and (27) with $\theta > 2\varepsilon$ follows

$$\begin{aligned} \Gamma_1^{(2)}(X_j) &\ll \mathcal{E} H X_j^{\frac{3}{2} - \frac{1}{200} + \varepsilon} \log^5 X_j \\ &\ll \mathcal{E} X_j^{\frac{1}{200} - \theta} X_j^{\frac{3}{2} - \frac{1}{200} + \varepsilon} \log^7 X_j \ll \mathcal{E} X_j^{3/2 - \varepsilon/2}. \end{aligned} \tag{78}$$

7 End of the proof

From (45), (49), (66) and (78) we obtain

$$\Gamma_1(X_j) = \mathcal{J}(X_j)\mathcal{B}^- \big(\widetilde{\mathcal{B}}^+\big)^3 + O\Bigg(\frac{\mathcal{E} X_j^{3/2}}{(\log X_j)^{A/2-8}}\Bigg). \tag{79}$$

Similarly, we can determine $\Gamma_2(X_j)$ and $\Gamma_5(X_j)$. From (42) and (79) we get

$$\Gamma(X_j) \geq \mathcal{J}(X_j)\bigg(\mathcal{B}^-\big(\widetilde{\mathcal{B}}^+\big)^3 + 3\mathcal{B}^+\big(\widetilde{\mathcal{B}}^+\big)^2\widetilde{\mathcal{B}}^- - 3\mathcal{B}^+\big(\widetilde{\mathcal{B}}^+\big)^3\bigg).$$

Using (9), (10), (12), (13) and (51)we obtain

$$\Gamma(X_j) \geq \mathcal{J}(X_j)\mathcal{B}(z)\mathcal{B}(\widetilde{z})^3\bigg(f(s) + 3f(\tilde{s}) - 3F(s)F(\tilde{s})^3\bigg),$$

where f and F are functions of the linear sieve as described in Lemma 1. Setting $s = \tilde{s} = 3,14999$ and employing relations (11) yields

$$\Gamma(X_j) \geq 0,000178\mathcal{J}(X_j)\mathcal{B}(z)\mathcal{B}(\widetilde{z})^3.$$

From (65) with $\varepsilon = 0,00001$ and from

$$s = \frac{\log D}{\log z}, \quad \tilde{s} = \frac{\log \widetilde{D}}{\log \widetilde{z}}$$

we receive $\frac{1}{\alpha} = 20,1063...$ and $\frac{1}{\widetilde{\alpha}} = 41,0868....$ Therefore,

$$p_1 + 2 = P'_{20}, \quad p_2 + 2 = P''_{41}, \quad p_3 + 2 = P'''_{41}, \quad p_4 + 2 = P^{\mathrm{iv}}_{41}$$

and the proof of Theorem 1 is complete.

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