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THE EDGE-ISOPERIMETRIC INEQUALITY  
FOR POWERS OF CYCLES

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This note provides a complete solution to a certain version of the edge-isoperimetric problem for powers of cycle graphs. Namely, it shows that the maximum number of edges induced by a vertex subset of  $C_n^s$  of size  $k$  is achieved by a set of  $k$  consecutive vertices.

**Keywords:** edge-isoperimetric inequality, cycle graph

ИЗОПЕРИМЕТРИЧНОТО НЕРАВЕНСТВО ЗА РЕБРАТА  
НА СТЕПЕНИТЕ НА ЦИКЛИЧЕН ГРАФ

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В тази статия се дава пълно решение на определена версия на реброво-изопериметричната задача за степени на циклични графи. По-точно се доказва, че максималният брой ребра в индуциран подграф на  $C_n^s$ , породен от множество от  $k$  върха, се постига, когато това множество се състои от  $k$  последователни върха.

**Ключови думи:** реброво-изопериметрично неравенство, цикличен граф

## 1 Introduction

Let  $G = (V, E)$  be a simple undirected graph. We consider the graph edge-isoperimetric problem in the following form: given a graph  $G$  and  $k \in \mathbb{N}$ , what is the largest number

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of edges in an induced subgraph of  $G$  with  $k$  vertices, that is

$$\max_{|U|=k, U \subset V} e(U).$$

For a  $d$ -regular graph, a minimizer of this quantity has the smallest graph perimeter (number of edges between a vertex subset and its complement) among  $k$ -sets. Recently, this question (in an equivalent supersaturation setting) for Johnson graphs was studied by A. M. Raigorodskii and his students [3] (see also references therein). Various results on this and related quantities can be found in [2].

The  $s$ th power  $G^s$  of graph  $G$  is a graph that has the same set of vertices, but in which two vertices are adjacent when their distance in  $G$  is at most  $s$ . This note solves the edge-isoperimetric problem for the powers  $C_n^s$  of the cycle graph  $C_n$ .

**Theorem 1.** *If  $n$ ,  $k$ , and  $s$  are positive integers such that  $n \geq k$  and  $n > s$ , then the maximum*

$$\max_{U \subset V(C_n^s), |U|=k} e(U)$$

*is attained by any set of  $k$  consecutive vertices of  $C_n^s$ .*

Let  $n$ ,  $k$ , and  $s$  be positive integers. We are interested in the maximal number of edges in an induced subgraph of  $C_n^s$  with  $k$  vertices. Note that if  $s \geq n/2$ , then  $C_n^s \cong K_n$ , so without loss of generality we may assume throughout this paper that  $s < n/2$ . We may also assume that  $k \geq s + 2$ , since otherwise any set of  $k$  consecutive vertices is optimal because all pairs of vertices are adjacent in  $C_n^s$ .

Note that Theorem 1 does not provide the classification of maximizers, which remains an open question. An example  $U = \{1, 3, 5\}$  for  $n = 6$ ,  $s = 2$  and  $k = 3$  shows that the complete classification may be complicated.

**Remark 1.** *After the paper was sent, the author was informed that this result was recently (and independently) obtained by Makai, Pasch, Petrova and Schiller [4]. The proof is different, although the main idea is the same.*

**Notation.** From now on we will abuse notation slightly and we will associate vertex set of  $C_n^s$  with the elements of  $\mathbb{Z}/n\mathbb{Z}$ , which we will also identify with the elements of  $[n]$  by their natural correspondence. We also define the following distances on  $[n]$ : the counterclockwise, clockwise, and cyclic distances  $d_+(i, j) := |i - j|$ ,  $d_-(i, j) = n - |i - j|$ , and  $d = \min(d_+, d_-)$ . Also, for a subset  $U$  of  $\mathbb{Z}/n\mathbb{Z}$  and an element  $v \in \mathbb{Z}/n\mathbb{Z}$ , let  $d_+(v, U) := \min\{k \in [n] : v + k \in U\}$ . For a vertex  $u \in \mathbb{Z}/n\mathbb{Z}$  and a subset  $U \subset \mathbb{Z}/n\mathbb{Z}$  such that  $u \in U$ , we denote the set of the clockwise neighbors of  $u$  in the set  $U$  by

$$N_+^U(u) := \{v \in U, v \neq u : d_+(u, v) \leq s\}.$$

Similarly,

$$N_-^U(u) := \{v \in U, v \neq u : d_-(u, v) \leq s\}$$

and

$$N^U(u) := N_-^U(u) \cup N_+^U(u).$$

Note that since  $n \geq 2s + 1$ , we have  $N_-^U(u) \cap N_+^U(u) = \emptyset$  for all  $u \in C_n^s$  and all  $U \subset C_n^s$ .

## 2 Application of general bounds

### 2.1 Bound from Turán's theorem

A classical method for finding the maximal number of edges in a subgraph is the celebrated Turán's theorem.

**Theorem 2** (Turán's). *Let  $G$  be a graph with clique number  $\omega(G)$  and let  $k > \omega(G)$ . Then*

$$\max_{U \subset V, |U|=k} e(U) \leq \binom{k}{2} - \frac{\omega(G)}{2} \left\lfloor \frac{k}{\omega(G)} \right\rfloor \left( \left\lfloor \frac{k}{\omega(G)} \right\rfloor - 1 \right).$$

The following lemma was used by Katona in his celebrated proof of the Erdős-Ko-Rado theorem.

**Lemma 1.** *For  $2s + 1 \leq n$  one has  $\omega(C_n^s) = s + 1$ .*

*Proof.* Let  $K$  be a clique in  $C_n^s$  and let  $u \in K$ . Then all vertices of  $K$  are among the  $2s$  neighbors of  $u$  in  $C_n^s$ . Let  $N_-(u) := \{u_1, u_2, \dots, u_s\}$  and  $N_+(u) := \{v_1, v_2, \dots, v_s\}$ , where  $d_-(u, u_i) = i$  and  $d_+(u, v_i) = i$ . Then note that for  $i = 1, 2, \dots, s$ , we have  $d(u_{s+1-i}, v_i) = s+1$ , so at most one of  $u_{s+1-i}$  and  $v_i$  is in  $K$ . This means that  $|K| \leq s+1$ .

On the other hand, any  $s + 1$  consecutive vertices of  $C_n^s$  form a clique, so indeed  $\omega(C_n^s) = s + 1$ .  $\square$

Now the application of Theorem 2 to  $C_n^s$  gives the following bound:

**Corollary 1.** *For  $k \geq s + 2$  we have*

$$\max_{U \subset V, |U|=k} e(U) \leq \binom{k}{2} - \frac{s+1}{2} \left\lfloor \frac{k}{s+1} \right\rfloor \left( \left\lfloor \frac{k}{s+1} \right\rfloor - 1 \right).$$

This gives

$$\max_{U \subset V, |U|=k} e(U) \leq \frac{sk^2}{2(s+1)} + \mathcal{O}(k).$$

### 2.2 Spectral bound

Another classical method for bounding the number of edges in a subgraph of a graph  $G$  is to use the spectral decomposition of the adjacency matrix  $A$  of  $G$ . Despite the fact that the following machinery is well known [1], it is complicated to find the bound in a pure form. We use the following

**Proposition 1.** *Let  $G = (V, E)$  be a graph and  $U \subset V$ . If  $A$  is the adjacency matrix of  $G$  and  $\chi_U$  is the characteristic vector of  $U$ , then*

$$2e(U) = \langle A\chi_U, \chi_U \rangle.$$

*Proof.* Note that  $(A\chi_U)_i$  is exactly the number of neighbors of  $i$  among the vertices of  $U$ . Therefore  $\langle A\chi_U, \chi_U \rangle$  is equal to the sum of the degrees of the vertices of  $U$  with respect to the subgraph of  $G$  induced by  $U$ , which is equal to  $2e(U)$ .  $\square$

Since  $A$  is real and symmetric, it admits an orthonormal eigenbasis  $v_1, v_2, \dots, v_n$ . Let  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  be the corresponding eigenvalues.

**Lemma 2.** *Let  $U \subset V$  and  $c_i := \langle v_i, \chi_U \rangle$  for  $i = 1, 2, \dots, n$ . Then*

$$\sum_{i=1}^n c_i^2 = |U|.$$

*Proof.* Note that  $\chi_U = \sum_{i=1}^n c_i v_i$ . Therefore,

$$|U| = \langle \chi_U, \chi_U \rangle = \sum_{i,j=1}^n c_i c_j \langle v_i, v_j \rangle = \sum_{i=1}^n c_i^2.$$

□

**Lemma 3.** *If  $U \subset V$  and  $c_i := \langle v_i, \chi_U \rangle$ , we have*

$$2e(U) = \sum_{i=1}^n \lambda_i c_i^2.$$

*Proof.* By Proposition 1 we have

$$2e(U) = \langle A\chi_U, \chi_U \rangle = \left\langle \sum_{i=1}^n \lambda_i c_i v_i, \sum_{j=1}^n c_j v_j \right\rangle = \sum_{i,j=1}^n \lambda_i c_i c_j \langle v_i, v_j \rangle = \sum_{i=1}^n \lambda_i c_i^2.$$

□

For regular graphs, the largest eigenvalue of the adjacency matrix, as well as its corresponding eigenvector, are known explicitly, [1].

**Lemma 4.** *For a  $d$ -regular graph  $G$ , we have  $\lambda_1 = d$  and  $v_1 = (1, 1, 1, \dots, 1)^T$ .*

We can now apply the lemmas to obtain the following:

**Corollary 2.** *Let  $U \subset C_n^s$  with  $|U| = k$ , then*

$$e(U) \leq sk^2/n + \lambda_2(k/2 - k^2/2n).$$

*Proof.* We have

$$2e(U) = \sum_{i=1}^n \lambda_i c_i^2 \leq 2sc_1^2 + \lambda_2 \left( \sum_{i=2}^n c_i^2 \right) \leq 2sc_1^2 + \lambda_2 (|U| - c_1^2) = \frac{2sk^2}{n} + \lambda_2 \left( k - \frac{k^2}{n} \right).$$

□

## 2.3 Numerical comparison

We present a numerical comparison of the general methods with the exact value for various values of  $k$  and  $s$  when  $n = 1000$ . As observed from the data, the spectral bound

$k$	$s$	Exact maximum	Spectral bound	Turán's bound
54	37	1295	1980	1431
118	53	4823	6149	6849
359	16	5608	5737	60691
210	115	17480	22511	21945
243	175	27125	36369	29403
313	295	48675	61627	48828
433	196	65562	73512	93331
404	372	80910	88116	81406
439	384	94656	99895	96141
473	462	111573	112499	111628

Table 1: Comparison of the exact answer and bounds for various  $k$  and  $s$

is particularly accurate when  $s \ll k$  holds.

Conversely, when  $k$  and  $s$  are close, i.e.,  $s \approx k$ , Turán's bound provides a good approximation. Recall that Turán's theorem gives

$$\text{Turán's bound} = \binom{k}{2} - \frac{s+1}{2} \left\lfloor \frac{k}{s+1} \right\rfloor \left( \left\lfloor \frac{k}{s+1} \right\rfloor - 1 \right) = \frac{k(k-1)}{2},$$

which is close to the exact maximum

$$sk - \frac{s(s+1)}{2} \approx k \left( k - \frac{1}{2} \right) - \frac{k^2}{2} \approx \frac{k(k-1)}{2},$$

since in this regime  $k - \frac{s+1}{2} \approx \frac{k-1}{2}$  and  $s \approx k$ .

However, in the intermediate regime where both  $k$  and  $s$  are large but  $k/s \gtrsim 2$ , neither bound provides an accurate approximation, as both the spectral and Turán's estimates deviate significantly from the exact maximum.

### 3 Proof of the main result

*Proof of Theorem 1.* Recall that we may assume  $n/2 + 1, k \geq s + 2$ . Let  $\mathcal{S}$  be the family of all sets  $S \subset \mathbb{Z}/n\mathbb{Z}$  with  $k$  elements that maximize the number of edges in the induced subgraph. A subset  $P \subseteq \mathbb{Z}/n\mathbb{Z}$  is called a *block* if it consists of consecutive elements of  $\mathbb{Z}/n\mathbb{Z}$ , i.e.  $P = \{p_1, p_2, \dots, p_r\}$ , where  $d(p_i, p_{i+1}) = 1$  for  $i = 1, 2, \dots, r-1$ . For a set  $U \in \mathcal{S}$  define  $f(U)$  as the maximal number of elements of  $U$  contained in a block of size  $s$ . For a subset  $T \in \mathbb{Z}/n\mathbb{Z}$ , which is contained in a block of size  $s$ , let  $l_T, r_T \in T$  be the endpoints of  $T$  such that  $T$  is located clockwise from  $l_T$  and counterclockwise starting from  $r_T$ , see Fig. 1. Note that  $r_T$  and  $l_T$  are well-defined, since  $s < n/2$ .

In the figures below, black points indicate elements of the set under consideration, white points indicate non-elements, and gray points represent points whose membership is irrelevant to the argument.

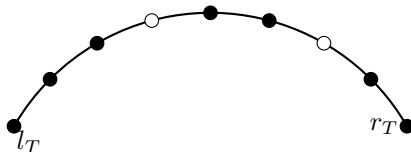


Figure 1: The endpoints of  $T$

For a vertex  $u \in \mathbb{Z}/n\mathbb{Z}$  and a subset  $U$  of  $\mathbb{Z}/n\mathbb{Z}$ , such that  $u \in U$ , define  $h(u, U)$  as the size of the largest block consisting of elements of  $U$  that contains  $u$ .

Let  $g(U) \subset 2^U$  be the set of all subsets of  $U$  with  $f(U)$  elements that are contained in a block of size  $s$ . Let

$$h(U) := \max_{T \in g(U)} h(l_T, U)$$

and let  $w(U)$  be a subset  $T \in g(U)$  such that  $h(U) = h(l_T, U)$ , see Fig. 2.

We are now ready to choose an extremal element. Let  $\mathcal{S}_1$  be the family of all sets  $S \in \mathcal{S}$  that maximize  $f(S)$ . Let  $\mathcal{S}_2$  be the family of all sets  $S \in \mathcal{S}_1$  for which the quantity  $h(S)$  is maximized. Pick any  $M \in \mathcal{S}_2$  and let for simplicity  $\mathcal{B}$  be a block of size  $s$  that

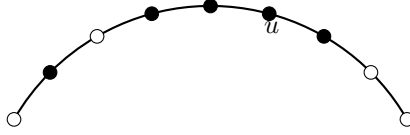


Figure 2: An example with  $h(u, U) = 4$

contains  $w(M)$ ,  $A := w(M) \cap \mathcal{B}$ ,  $u := l_A$ , and let  $\mathcal{A} \subset M$  be the block of size  $h(M)$  that contains  $u$ . Note that  $h(M)$  may be more than  $s$  as  $M$  may contain  $\mathcal{B}$ .

Without loss of generality we may assume that  $\mathcal{B} \subseteq \mathcal{A}$  or  $\mathcal{A} \subseteq \mathcal{B}$  as otherwise we can just shift  $w(M)$  counterclockwise until we reach the end of  $\mathcal{B}$  while preserving the maximality of  $w(M)$ , so we may also assume that  $u - 1 \notin M$ .

Assume for contradiction that the set  $C := \{w \in M \setminus A : d_+(w, A) > s\}$  is nonempty and let  $v \in C$  be such that  $d_+(v, A)$  is minimal, see Fig. 3. Consider the set  $M' := M \setminus \{v\} \cup \{u - 1\}$ . Note that  $u - 1$  is adjacent to all vertices in  $N_+^M(v)$  as all clockwise neighbors of  $v$  lie in the set  $\{w \in M \setminus A : d_+(w, A) \leq s\}$  by the minimality of  $d_+(v, M)$  on  $C$ . We further have that  $|N_-^M(v)| + 1 \leq |A|$  as  $v$  and all of its counterclockwise neighbors lie in a block of size  $s$  and so by maximality of  $A$ , it must have at least as many elements. Since  $u - 1$  is adjacent to at least  $|A| - 1$  elements from  $\mathcal{B}$  (as  $|\mathcal{B}| \leq s$ , at most one vertex is  $s + 1$  units apart from  $u - 1$ ), which shows that replacing  $v$  with  $u - 1$  gains  $|A| - 1 - |N_-^M(v)| \geq 0$  edges in  $M'$ . However, the latter contradicts the maximality of  $h(M)$  as adding  $u - 1$  increases  $\mathcal{A}$ .

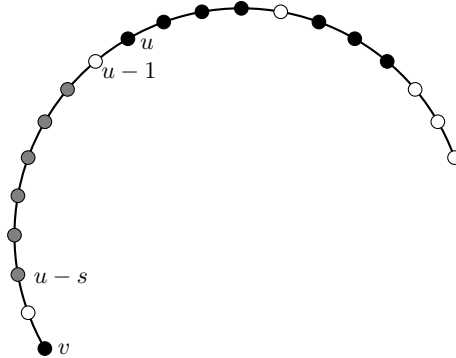


Figure 3: Swapping  $v$  and  $u - 1$

Thus, all  $v \in M$  satisfy at least one of the following:

1.  $v \in A$ ;
2.  $v \in \mathcal{A}$ ;
3.  $v \in N_-^M(u)$ .

We now show that  $A \subseteq \mathcal{A}$ . Assume for contradiction that  $r_A \notin \mathcal{A}$ . Then replace  $r_A$  by  $u - 1$ . We have that  $u - 1$  is adjacent to all the other vertices in  $A$  (except maybe

$r_{\mathcal{A}}$ , which we removed) and is adjacent to all other vertices in  $M$  as all of them are on  $N_-^M(u)$ . Hence, the number of edges has not decreased, while  $\mathcal{A}$  has increased, which again contradicts the maximality of  $h(M)$ . Thus,  $A \subseteq \mathcal{A}$ .

Finally, if  $v \in N_-^M(u) \setminus \mathcal{A}$ , then note that we can replace  $v$  by  $u - 1$  as all elements of  $N_-^M(u)$  are neighbors of  $u - 1$  and  $u - 1$  has  $\min(|\mathcal{A}|, s)$  neighbors from  $\mathcal{A}$ , which is at least the number of neighbors of  $v$  in  $\mathcal{A}$ . This once again contradicts the maximality of  $\mathcal{A}$ . Hence  $|\mathcal{A}| = k$  as required. □

**Corollary 3.** *If  $k + s < n$ , then we have*

$$\max_{U \subset C_k^s} e(U) = sk - \frac{s(s+1)}{2}.$$

*Proof.* Let  $U := \{1, 2, \dots, k\}$ . By the previous theorem it suffices to compute  $e(U)$ , which we will do by counting the degree of each vertex.

*The case  $k \geq 2s$ .* Each of the vertices  $s + 1, \dots, k - s$  has  $2s$  neighbors. The vertices  $i \in \{1, 2, \dots, s\}$  and  $k - i \in \{k - s + 1, \dots, k\}$  are of degree  $s + i - 1$ . This gives a total of

$$\frac{2s(k - 2s) + 2 \sum_{i=1}^s (s + i - 1)}{2} = sk - \frac{s(s+1)}{2}$$

edges.

*The case  $k < 2s$ .* Here each of the vertices  $i, k - i \in \{1, 2, \dots, k - s\}$  is of degree  $i - 1 + s$ . Also each vertex  $i$  for  $i = k - s + 1, k - s + 2, \dots, s$  has  $i - 1 + k - i = k - 1$  neighbors, which gives a total of

$$\frac{(2s - k)(k - 1) + 2 \sum_{i=1}^{k-s} (i - 1 + s)}{2} = sk - \frac{s(s+1)}{2}$$

edges. □

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