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A COMPETENCE-BASED METHODOLOGICAL FRAMEWORK FOR MEANINGFUL DERIVATION OF FORMULAE FOR AREAS, SURFACE AREAS AND VOLUMES IN THE PREPARATION OF FUTURE MATHEMATICS TEACHERS

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The article focuses on the development of the methodological competence of future mathematics teachers through a systematic approach to teaching formulae for areas, surface areas and volumes of geometric figures in secondary school. A conceptual–methodological model is proposed, including a competence-based methodological framework for the meaningful introduction of the formulae. This framework reveals how the formulae should be introduced by the teacher and is therefore intended for university training in "Methodology of Mathematics Education"; it includes didactic technologies for the formation of methodological skills in students. The model employs object–activity modelling, geometric transformations, analogies and generalisations to achieve conceptual understanding and learning situations for professional reflection of future mathematics teachers.

Keywords: inductive-integrative didactic framework; methodological competence; geometric modelling; meaningful derivation of formulae; preparation of future mathematics teachers

КОМПЕТЕНТНОСТНА МЕТОДИЧЕСКА РАМКА ЗА СМИСЛОВО ИЗВЕЖДАНЕ НА ФОРМУЛИ ЗА ЛИЦА, ПОВЪРХНИНИ И ОБЕМИ В ПОДГОТОВКАТА НА БЪДЕЩИ УЧИТЕЛИ ПО МАТЕМАТИКА

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Статията се фокусира върху развитието на методическата компетентност на бъдещите учители по математика чрез системен подход за преподаване на формули за лица, повърхнини и обеми на геометрични фигури в средното училище. Предложен е концептуално-методически модел, включващ компетентностна методическа рамка за смислово въвеждане на формулите. Тази рамка разкрива *как да се въвеждат формулите от учителя* и затова е предназначена за университетското обучение по „Методика на обучението по математика“ и включва дидактически технологии за формиране на методически умения у студентите. Моделът използва предметно-действено моделиране, геометрични преобразувания, аналогии и обобщения за постигане на концептуално разбиране и обучаващи ситуации за професионална рефлексия на бъдещите учители по математика.

Ключови думи: индуктивно-интегративна дидактическа рамка; методическа компетентност; геометрично моделиране; смислово извеждане на формули; подготовка на бъдещи учители по математика

Introduction

The future mathematics teacher must be not merely a bearer of knowledge but a mediator of mathematical thinking. Their *methodological competence* is a complex, integrative characteristic. It also includes *specialised methodological skills* for using effective didactic techniques for introducing and studying mathematical concepts in an accessible, engaging and effective way, consistent with the age characteristics of the students and contemporary educational realities.

In the mathematics curricula for secondary school, state educational requirements stand out concerning the knowledge and application of formulae for areas of figures and volumes of solids. *The thesis* is that the unstable memorisation and reproduction of formulae by students is a symptom of a deeper problem – namely, the methodological competence of the teacher, based on dominant reproductive teaching methods. If the teachers themselves perceive formulae as "symbols to be memorised" or ready-made list elements, this will inevitably transfer into school practice. Therefore, university preparation plays an important role in overcoming this problem through the formation of the methodological competence of future teachers.

From a methodological perspective, a *formula* represents neither a concept nor a statement, but a symbolic record of a quantitative relationship between quantities, expressing a generalised result of mathematical reasoning or experiment. A formula has four levels of representation: action-based (through modelling, measuring, cutting); visual–schematic (through drawing, figure, net); symbolic (through literal–numerical notation); verbal (a verbal equivalent of the symbolic notation).

For brevity, the term "*formulae*" is used in the article to include formulae for: areas of plane figures, surface areas of solids and volumes of solids. These are studied across several educational stages at school, which determines their significant place in mathematics education.

This gives rise to the following research questions: According to what methodological model should the introduction of the formulae in school be organised? Which approaches support the revealing of their logic? What didactic technologies should student future mathematics teachers master?

The object of the study is the methodological competence of students – future mathematics teachers – in the discipline "Methodology of Mathematics Education". **The subject** of the study is specialised methodological skills, as an essential component of the professional–methodological competence of the future teacher, for mastering specific didactic techniques in introducing the formulae. **The aim** and tasks of the study relate to: theoretical analysis of specific didactic techniques for introducing the formulae; development of a methodological framework based on a combined approach for introducing the formulae; and illustration of corresponding models of didactic technologies.

Problematic and brief theoretical analysis

An analysis of the author's teaching practice in the discipline "Methodology of Mathematics Education" outlines a persistent tendency towards mechanical application of the formulae. A significant proportion of the students also encounter difficulties in arguing their derivation. Methodologically, this is associated with the widespread declarative approach in school education "presentation – memorisation – application". Formulae are often provided ready-made by the teacher, without tracing their origin and without connection to modelling activities and analogies with already studied formulae. As a result, students build a weak connection between the geometric figure and the corresponding formula, and algorithmic exercises do not require in-depth understanding. **It follows** that the sustainable mastery of formulae by students requires effective ***methodological competence*** on the part of the teacher – to possess knowledge of current educational approaches and developed methodological skills for applying methods, techniques and learning resources for activity-based and meaningful introduction of formulae.

Pedagogical and methodological studies by a number of authors investigate contemporary approaches to inquiry-based and activity-based learning grounded in constructivism. The scientific literature emphasises that through the inquiry approach learners are encouraged to seek solutions to a given problem rather than merely memorise rules and procedures [2]. Learning through discovery increases learners' interest in studying mathematics [8]. They understand mathematical concepts more deeply and are more capable of transferring their knowledge to new situations [7]. Activities of Grade 6 students with manipulative materials (tangram type) lead to better achievement in mathematics [6]. Higher results have been achieved in teaching students using a multisensory approach based on the "concrete – representational – abstract" (CRA) framework [5]. The importance of experimentation has been emphasised [4]. It is appropriate to combine computer dynamic models with manipulatives [1].

These and other studies show that contemporary education oriented towards the acquisition of skills and functional literacy requires teachers to demonstrate approaches and technologies grounded in constructivism. The methodology of mathematics education, as a scientific field and academic discipline, is also charged with this task: to improve students' pedagogical competence as a basis of their professional–methodological competence. Overcoming the aforementioned problems directs us towards the development of a methodological model for teaching the topic "Areas and Volumes of Figures". In the context of university instruction, the proposed framework is considered not as a direct technology for work with pupils, but as a means for developing the methodological competences of students – future mathematics teachers – in planning and implementing the introduction of geometric formulae.

Competence-oriented methodological framework for developing the methodological competences of future teachers when introducing geometric formulae

In university instruction, this framework is regarded as a meta-methodological model aimed at developing the methodological competences of student future mathematics teachers: competence for methodological planning of the introduction of new knowledge; competence for organising productive cognitive activity of students; competence for reasoned geometric explanation and justification of relationships.

The proposed framework is based on an inductive-empirical integrated approach that combines:

- an inquiry approach (experiment, rediscovery);
- a constructive and activity-based approach (modelling, constructing);
- general scientific methods of cognition – comparison, analogy and generalisation.

The methodological novelty of the framework lies in the systematic unification of these approaches into a single sequence of cognitive and meta-methodological actions through which students simultaneously experience the process of conceptual comprehension and reflect on its pedagogical organisation.

The framework is based on the following principles: learner activity; visualisation and modelling; analogy and generalisation (transition from particular cases to a general relationship); reflection (comprehension both of the mathematical results and of the methodological decisions).

The didactic implementation includes consecutive interrelated stages:

1. **A problem situation** motivating the need for the formula and directing students towards planning an appropriate introduction.
2. **Manipulative and experimental activity** with material, schematic and dynamic models for activity-based construction of the formula, accompanied by discussion of possibilities for organising student activity.
3. **Visual–logical comprehension** and comparison with already known relationships, emphasising methodological decisions for supporting generalisation.
4. **Formulation of the relationship** between quantities as a generalisation of the performed actions and observations, combined with analysis of linguistic and symbolic precision.
5. **Bidirectional transition** between symbolic and verbal formulation of the formula, considered as a means for developing the future teacher's skills in mathematical communication.

It follows that mastering the proposed didactic technologies functions as a means for developing students' professional-methodological competence.

Within the model, the ***formula*** is regarded as the result of an inquiry process, in which students analyse how in the school environment it can be:

- derived through geometric actions (cutting, rearranging, completing, comparing figures with equal areas or volumes);
- connected with real and modelled situations;

- interpreted geometrically (each quantity has a visual meaning, and the factors in the formulae are interpreted as geometric characteristics);
- presented as the result of purposeful investigation rather than as a ready-made rule.

An essential point is that students not only perform the activities but also conduct subsequent methodological reflection on the conditions for their effective use in school practice.

The role of the instructor in the university environment is to model professional actions: to organise an environment for inquiry, to direct attention to essential relationships through questions, to support generalisation and mathematical formalisation, and to stimulate students' methodological reflection. Hence, the framework is aimed at:

- understanding formulae as the result of activity rather than ready-made rules;
- building skills for geometric reasoning and argumentation;
- purposeful development of future teachers' methodological competences for planning and implementing the introduction of formulae through rediscovery technologies.

In accordance with this framework, didactic technologies for justified derivation of the formulae have been developed, which students analyse, modify and plan for application in their future pedagogical practice.

Areas of plane figures – didactic technologies

The preparation of student future teachers begins with methodological updating and rediscovery of the area of a rectangle as basic knowledge for deriving the other formulae. Covering the rectangle with unit squares ensures the transition from concrete measurement to a generalised relationship. Students analyse the cutting of the rectangle along the diagonal and justify the methodological advantages of the activity-based approach. The subsequent folding forms two triangles whose area is half the area of the rectangle (Fig. 1). The practical activity aims at awareness of the didactic logic of the derivation.

In a parallelogram, an internal height is dropped and a right triangle is separated and moved in order to transform the parallelogram into a rectangle (Fig. 2). The area of the rhombus is considered as a special case of the parallelogram. Students master a universal model for transforming figures applicable in pedagogical practice. They independently investigate transformations of a trapezium (Figs. 3, 4 and 5), formulating methodological decisions and using analogy for memorisation. The generalisations include: "The area of an arbitrary figure is found as the sum or difference of the areas of its component figures" and "The areas of quadrilaterals – rectangle, parallelogram, rhombus and trapezium – are found as the product of a side (or the arithmetic mean of the bases) and the corresponding height; for a triangle the result is divided by two".

Students design their own didactic implementations, deriving the formula for the area of a regular polygon as the sum of the areas of its constituent triangles. Under the guidance of the instructor, they examine images in textbooks and dynamic visualisation of the transformation of the circle into a rectangle, selecting appropriate digital tools and presenting the results through brief methodological presentations. This develops skills for didactic design and reasoned selection of technologies.

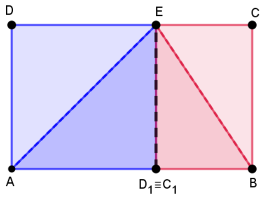

Fig. 1. Triangle

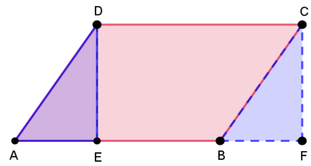

Fig. 2. Parallelogram

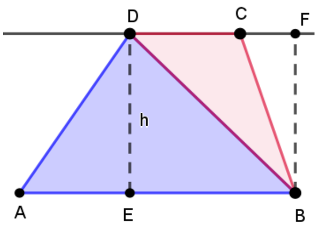

Fig. 3. Trapezium – Method 1

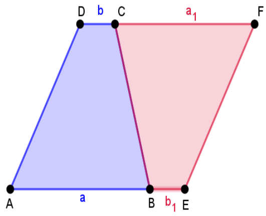

Fig. 4. Trapezium – Method 2

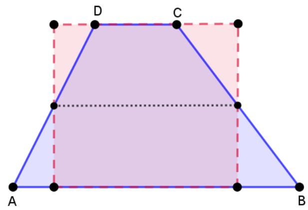

Fig. 5. Trapezium – Method 3

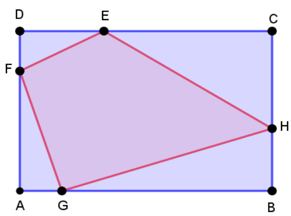

Fig. 6. Quadrilateral

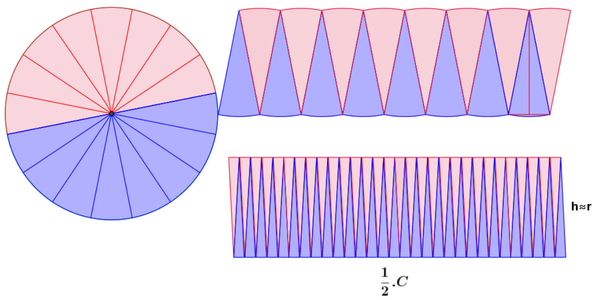

Fig. 7. Circle

Didactic technology – surface areas of solids

The net of a right prism is constructed, with students analysing the number and type of the obtained faces and establishing the relationship between the perimeter of the base and the lateral surface area. In this way the formula $S = P.h$ is comprehended as a structural relationship between quantities rather than as a ready-made rule.

Through modelling of a right circular cylinder, future teachers discover the analogy with the prism and on this basis justify the formulae for lateral and total surface area.

An analogous approach is applied when constructing nets of a pyramid and a cone, where students trace the common relationships between the formulae. Methodologically it is appropriate to compare the surface areas of the prism and the cylinder in order to highlight both the similarities and the essential differences.

When deriving the formula for the surface area of a sphere, students use modelling and experimentation. A polystyrene ball is cut into two hemispheres. At the centre of the cut a string is fixed and wound so as to cover the entire circle of the section. A second string is fixed at the "pole" and wound along the surface of the hemisphere. By comparing the lengths, students establish that the second string is twice as long, which leads to the conclusion that the surface area of the sphere is four times the area of the great circle. Students understand the role of the teaching experiment.

Didactic technology – volumes of solids

Through modelling of a rectangular parallelepiped and filling it with unit cubes (or through instructional videos), students trace the number of cubes in one layer and the number of layers. In this motivated way the formula for volume as the product of the area of the base and the height is reached. On this basis, an analogy is established with the volumes of a right prism and a right circular cylinder.

From cardboard, a regular quadrilateral pyramid and a prism with equal bases and equal heights are constructed. Through experimental filling with granular material (or through animation) it is established that three pyramids fill one prism. From this, students trace the logic for discovering the relationship for the volume of a pyramid and verify its validity for a cone and a cylinder.

Additionally, a cylinder is modelled with a diameter twice that of a tennis ball. After successive pouring of granular material and placing the ball, the change in level is recorded, which leads to the derivation of the formula $V_{ball} = \frac{4}{3}\pi r^3$.

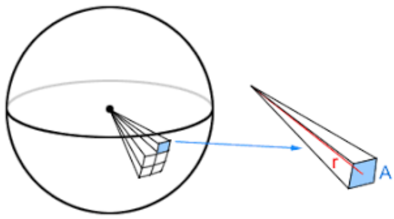

Fig. 8. Volume of a ball

Alternatively, the volume of the sphere is justified through its decomposition into many "pyramids" with convex bases (Fig. 8). The volume of each of them is approximated by $V \approx \frac{1}{3}Ar$, where A is the area of the base. Through logical generalisation students reach the conclusion that the sum of the bases corresponds to the surface area of the sphere, from which the formula follows: $V_{ball} = \frac{1}{3}\ r.S = \frac{1}{3}\ r.\ 4\pi r^2 = \frac{4}{3}\pi r^3$.

At the contemporary stage, the methodological competence of the teacher is expanding with skills for blended learning (online learning, application of digital technologies and educational software, flipped classroom, etc.). Therefore, in the methodological implementation it is appropriate to integrate modern digital tools, for example GeoGebra, which extend work with manipulatives through dynamic visualisations and support conceptual understanding.

The proposed didactic solutions show how, through transformation, cutting, completing and comparison of figures, meaningful derivation of formulae can be achieved. The pedagogical value of the approach is expressed in ensuring a transition from concrete action to theoretical generalisation and in developing reasoned geometric thinking in future teachers.

Conclusion

The developed competence-based methodological framework offers a systematically organised methodological model for introducing formulae for areas, surface areas and volumes in the university preparation of future mathematics teachers. Its leading characteristic is the transition from declarative presentation to meaningful productive discovery through modelling, geometric transformations, analogies and generalisations.

The author's contributions can be summarised in the following directions: an inductive–integrative didactic framework for introducing geometric formulae has been synthesised; a sequence of cognitive actions ensuring their meaningful derivation has been structured; didactic solutions aimed at enriching the methodological competence of future mathematics teachers have been proposed.

The proposed framework has a conceptual–methodological character and outlines prospects for subsequent empirical verification of its effectiveness.

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