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THE IMPORTANCE OF PROVIDING FULL SOLUTIONS IN MATHEMATICS ASSESSMENT

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This article explores the potential educational benefits of providing not only correct answers but also detailed solutions to each question in a mathematics assessment. The analysis is grounded in theoretical principles from mathematics didactics, formative assessment theory, and established practices in preparing for standardised tests. In the context of the Bulgarian educational system, it is argued that the lack of detailed feedback in mathematics tests limits the opportunities for error correction, the development of mathematical thinking, and the cultivation of metacognitive skills in students. A review of freely available materials and regulatory documents was conducted, and these were then compared with those of other educational systems. The necessity to publish solutions in accordance with current educational standards and for test-format tasks was formulated as a result. In conclusion, the findings of this study recommend that teachers, authors of educational content, and educational institutions take note of the implications of these observations.

Keywords: mathematics assessment, feedback in mathematics education, metacognitive skills, test preparation practices, mathematics didactics

ЗНАЧЕНИЕТО НА ПРЕДОСТАВЯНЕТО НА ПЪЛНИ РЕШЕНИЯ ПРИ ОЦЕНЯВАНЕТО ПО МАТЕМАТИКА

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Настоящата статия разглежда добавената образователна стойност на предоставянето не само на верните отговори, но и на подробни решения към всеки тестов въпрос по математика. Анализът се основава на теоретични постановки от дидактиката на

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математиката, теорията на формативното оценяване и утвърдените практиките при подготовката за стандартизирани изпити. В контекста на българската образователна система се аргументира, че липсата на подробна обратна връзка в математическите тестове ограничава възможностите за корекция на грешките, развитие на математическо мислене и формиране на метакогнитивни умения у учениците. Чрез преглед на свободно достъпни материали и нормативни документи и сравнение с други образователни системи се формулира нуждата от публикуване на решения според действащите образователни стандарти и за задачите в тестов формат. В заключение се формулират препоръки към учители, автори на учебно съдържание и образователни институции.

Ключови думи: оценяване по математика, обратна връзка в обучението по математика, метакогнитивни умения, подготовка за изпити, дидактика на математиката

Introduction. Assessment is of central importance in the domain of mathematics education, directly impacting both the learning process and students' motivation and attitudes towards learning. In the Bulgarian educational context, however, mathematical tests are predominantly utilized as a metric for evaluating outcomes rather than as a catalyst for learning and growth. In most cases, students are provided solely with the correct answers, without any elaboration on the rationale behind the solution or the reasons why the other options are incorrect [7]. Research in the field of mathematics education emphasizes that assessment which does not require or provide an explanation of the strategies and reasoning used fails to capture students' actual level of understanding and has a limited educational effect [10].

This approach is particularly appropriate for textbooks that contain solved problems and facilitate direct interaction with a teacher. In contrast, the majority of collections and educational materials provide merely a key with the correct answers. This practice is institutionally established, with detailed solutions published for exam materials, albeit exclusively for problems requiring an extended response. The phrase "Any other correct and complete solution will be awarded the maximum number of points" suggests that there may be alternative approaches to solving the problem, yet it is often unclear to students and parents what criteria categorize a solution as "correct and complete". It is evident that a discrete study of this topic is necessary.

Nevertheless, the present study focuses on the question of solutions to the tasks in the test section. The purpose of this article is not to engage in a discussion of the extent to which the test format is suitable for the assessment of mathematics. A salient issue is the substandard performance in mathematics examinations. A public debate has emerged on social media and among proactive parents regarding the emphasis placed on mechanising the process of solving tests by students. The results of the national external assessment following the 7th and 10th grade, in addition to the state matriculation exams, indicate systematic difficulties with tasks requiring reasoning and solutions involving sequential actions. In this context, the present article examines how the provision of detailed solutions could contribute to the improvement of the educational process within the current assessment standard.

Methodology and Research Design. This study adopts a qualitative research design combining document analysis and a comparative analytical approach to examine the presence, structure and educational role of detailed solutions in mathematics assessment materials. The analysis is based on freely available, officially published materials that reflect current mathematics assessment practices in secondary education. The sources are divided into three main groups: 1) Normative and official assessment documents from the Bulgarian education system; 2) Publicly available exam materials (2006–2025) with answer keys for students and teachers; 3) Assessment materials from international education systems (Austria, the International Baccalaureate and the SAT), selected for their structural comparability with Bulgarian assessments, and for their established practices for publishing detailed solutions and feedback. Materials were selected based on the following criteria: public accessibility; official or institutional status; relevance to mathematics education in lower and upper secondary education; availability of test-format assessment tasks; a direct link to practices for providing feedback; transparency of solutions; and compliance with curricula and assessment standards. These criteria ensure the inclusion of materials representative of established assessment practices that are suitable for qualitative comparative analysis. The analysis was conducted in three stages: 1) Qualitative content analysis of the assessment materials for determining the provision of solutions and the level of detail (e.g. whether only a final answer is provided or a step-by-step reasoned solution); 2) Comparative analysis of Bulgarian materials and selected international examples with an emphasis on the structure of solutions and the treatment of distractors; 3) Interpretative analysis based on the theories of formative assessment and constructivist mathematics pedagogy for the evaluation of the pedagogical implications of the provision or withholding of detailed solutions. While the study does not claim to be statistically generalisable, it aims to provide analytical insight and conceptual clarity. The qualitative approach enables detailed examination of assessment practices and their educational implications, thereby contributing to the broader scientific discourse on feedback, learning support and assessment design in mathematics education.

The utilization of exam materials as a pedagogical instrument. A considerable number of students and parents employ tests from prior years as a means of preparation. It is evident that the official website of the Ministry of Education and Science (MES) disseminates answer keys; however, there is an absence of official detailed solutions to the test questions.

Research in the field of formative assessment demonstrates that the most effective feedback is specific and provides an explanation of the problem-solving process [4]. Merely indicating the correct answer does not fulfil these criteria. As Hattie and Timperley observe, feedback constitutes a significant factor in academic achievement, with its efficacy contingent upon the nature of the feedback itself and its manner of presentation [8]. The provision of the correct answer in a mathematics test cannot be considered an effective pedagogical practice. Contemporary research in the field of education further supports this, demonstrating that feedback has the highest educational value when it is focused on the problem-solving process, includes explanations, and supports reasoning, rather than when it is reduced solely to information about the final result [11] [16].

It is important to establish effective mechanisms through which students and parents can access the knowledge and skills necessary to find solutions that are aligned with the curriculum for their respective grade. At present, students and teachers are offering freely accessible solutions online. It has been observed that certain educational institutions choose to furnish comprehensive solutions on a voluntary basis, or for the purpose of promotional activities. Parents and students are turning to forums and social networks for assistance, leveraging artificial intelligence-based tools to generate solutions. The substantial involvement of participants in the process is indicative of public engagement. However, it also highlights a discrepancy that demands resolution at a more authoritative level. Indeed, the authors of the tests possess solutions that are in accordance with current educational standards and which explicitly address the relevant knowledge and skills that each individual task aims to test and assess.

The constructivist approach in mathematics education. Mathematics, as a school subject, demands strict reasoning, a logical sequence of steps, and conceptual understanding. It is therefore of particular educational value to provide detailed solutions and analysis of distractors in mathematics tests.

In accordance with the principles of the constructivist approach in mathematics education, the importance of detailed solutions for students can be emphasised in the following aspects:

- It is evident that a considerable proportion of mathematical problems require a sequence of operations that are linked together. A detailed solution is instrumental in clarifying the individual steps and facilitating the identification of any potential errors.
- It has been demonstrated that a significant percentage of students use past examination papers for the purpose of self-preparation. When following the complete solution, the focus is directed towards the connections between concepts and the logical sequence of actions, rather than merely the final answer. This is an essential component of the sustainable acquisition of mathematical knowledge.
- The development of students' competencies is supported by detailed solutions, which facilitate self-regulated learning and reflection.

The constructivist paradigm in mathematics education is founded on the premise that learning is an active process of knowledge construction. This is characterized by the student interacting with mathematical tasks to construct understanding rather than merely reproducing end results passively. In the context of cognitive constructivism, as articulated by Jean Piaget, the process of cognitive development unfolds through the mechanisms of assimilation and accommodation [12] [13], which are predicated on the presence of cognitive conflict and the subsequent act of reflection. The review of detailed mathematical solutions is of particular significance, as it enables students to compare their own strategies with those employed by experts. The restriction of information to a simple final answer, without the availability of detailed solutions and reasoning, serves to limit opportunities for reflection and contradicts fundamental constructivist principles. This position is corroborated by empirical studies demonstrating that the analysis of ready-made, reasoned solutions facilitate conceptual understanding and reflection on the problem-solving process [20].

The analysis is further complemented by Lev Vygotsky's social constructivist theory, which underlines the significance of mediation and scaffolding in the zone of proximal development [19]. Within the Bulgarian education system, detailed solutions can be interpreted as a form of institutionalized didactic media that compensates for the limited individual feedback in mass education. The primary function of these tools is to facilitate the transition of students to a state of conscious and self-regulated problem solving. In his research on mathematical problem solving, Schoenfeld emphasizes that successful problem solving depends not only on the availability of formal knowledge, but also on strategic control, monitoring of actions, and conscious decision-making during the process. In this sense, access to detailed, reasoned solutions has been demonstrated to support the development of metacognitive skills and the understanding of the strategies used [15].

Research on worked examples by Atkinson and other scholars [1] [14] [20] demonstrates that the analysis of detailed solutions significantly enhances the acquisition of new mathematical knowledge when compared to practices where learners are only provided with final answers. Empirical studies conducted within a school environment demonstrate that the utilization of worked examples in the teaching of algebra contributes to the development of both procedural and conceptual understanding. This approach has been shown to reduce the cognitive load on students and to facilitate the transition to independent problem solving [5] [18].

In the context of Bulgarian assessment, this suggests that the publication of official solutions and comments on the tasks would fulfil an informative as well as an educational function.

The distinction between instrumental and relational understanding, as introduced by Richard Skemp [17], is of relevance in this context. The emphasis on the final answer has been shown to primarily stimulate instrumental understanding. Conversely, the systematic presentation and discussion of detailed solutions have been shown to support the development of relational understanding, which is essential for long-term mathematical literacy and the successful construction of knowledge in higher grades. From a constructivist perspective, mathematical understanding is manifested through the student's ability to justify and argue their solutions, rather than merely achieving the correct result. In this regard, explanations and reasoning represent an integral component of the learning and assessment process [10].

Despite the efforts made towards the digitization of educational resources, many of these electronic resources and tests offer only definitive conclusions, thereby failing to accommodate the need for ongoing academic development and critical thinking skills. This limitation renders them suboptimal from an educational standpoint. There is an increasing trend among teachers of utilizing automated tests within their professional practice. While it is evident that this approach fosters competencies in test-taking and time management, the question remains: in the absence of a definitive solution, how does this contribute to the development of cognitive skills, and to what extent are the students' solutions aligned with educational standards? It should be noted that some teachers insist on detailed solutions to test answers, thus enabling them to assess their students' actual academic performance and at the same time to apply the accepted test format scheme. It is noteworthy that certain automated tests offer feedback in the form of a reasoned and justified correct answer. Contemporary research indicates that digital environments have the potential to enhance the educational value of detailed solutions by incorporating interactive elements that encourage students to engage with the individual steps of a calculation, as opposed to merely reproducing pre-established algorithms [2]. It is evident that a coherent strategy is required at the institutional level to establish a set of standards that can then be applied universally.

Detailed solutions that are publicly available and meet educational standards, the curriculum, and the educational goals set for the relevant stage, as well as the tasks from national external assessments and state matriculation exams, are beneficial for students. Furthermore, they establish a benchmark for young teachers to follow, serve as educational resources for specialists from diverse fields who have retrained as teachers, and function as a support instrument for parents who provide assistance.

Examples are drawn from the State Matriculation examination in mathematics – profiled training, held on 25 August 2025 [21].

6. The sine of the angle that the line with equation $3x + y + 5 = 0$ *forms with the ordinate axis is:*

Possible solution paths

Option I. The solution according to the material provided involves applying the formula for the cosine of the angle between two lines (φ) – the given line and the abscissa axis – with equation $y = 0$, because it is equal to the sine of the angle sought (x): $x = 90^\circ - \varphi \Rightarrow \sin(90^\circ - \varphi) = \cos\varphi$. The following formula for an angle between two lines $g_1 : a_1 \cdot x + b_1 \cdot y + c_1 = 0$ and $g_2 : a_2 \cdot x + b_2 \cdot y + c_2 = 0$ is to be applied:

$$\cos\varphi = \left| \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}} \right| \Rightarrow \cos\varphi = \left| \frac{3 \cdot 0 + 1 \cdot 1}{\sqrt{3^2 + 1^2}\sqrt{0^2 + 1^2}} \right| \Rightarrow \cos\varphi = \frac{1}{\sqrt{10}}$$

$$\Rightarrow \sin x = \sin(90^\circ - \varphi) = \cos\varphi = \frac{\sqrt{10}}{10}$$

Option II. A different approach, which is also recommended by ChatGPT, is to calculate the gradient, i.e. the tangent of the angle (α) between the line and the positive direction of the x-axis. In this particular case:

$$\tan\alpha = -3$$

The angle sought is acute and has a magnitude of $x = \alpha - 90^\circ$. Through the application of consecutive trigonometric transformations, the following gradient value is calculated:

$$\tan x = \frac{1}{3} \Rightarrow \frac{\sin x}{\cos x} = \frac{1}{3}$$

The fundamental trigonometric identity is used to determine the value of $\sin x$.

Option III. The problem can also be resolved with knowledge from the ninth grade. A sketch of the line through its intersection points with the coordinate axes $(0; -5)$, $\left(-\frac{5}{3}; 0\right)$ illustrates the angle sought and the right-angled triangle enclosed between the line and the coordinate axes with sides of lengths 5 and $5/3$. The hypotenuse is determined by the Pythagorean theorem, and the sine sought is equal to the ratio of the opposite side to the hypotenuse.

9. Which of the common fractions is used to write the infinite repeating decimal $1.\overline{36}$*?*

$A)\frac{15}{11}$ $B)\frac{7}{5}$ $C)\frac{11}{7}$ $D)\frac{11}{4}$

Possible solutions

Option I. Many students have memorized a formula for such cases

$$1+\frac{36}{99}=1+\frac{4}{11}=\frac{15}{11}$$

Option II. The authors' anticipated solution is most likely the one suggested in current textbooks, namely, to consider the number as the whole part plus the sum of an infinite geometric progression:

$$a_1=\frac{36}{100}, q=\frac{1}{100}$$

$$1,(36)=1+\frac{36}{100}+\frac{36}{10000}+\frac{36}{1000000}+\dots$$

$$1,(36)=1+S_\infty$$

$$S_\infty=\frac{36}{100}.\frac{1}{1-\frac{1}{100}}\Rightarrow S_\infty=\frac{36}{100}.\frac{1}{\frac{99}{100}}\Rightarrow S_\infty=\frac{4}{11}$$

$$1.\overline{36}=1+\frac{4}{11}\Rightarrow 1.\overline{36}=\frac{15}{11}$$

An alternative approach involves the writing of decimal fractions, for example, where the first term is equivalent to 0.36 and the quotient is 0.01.

$$1.\overline{36}=1+0.36+0.0036+0.000036+\dots$$

$$1.\overline{36}=1+S_\infty$$

$$S_\infty=0.36.\frac{1}{1-0.01}\Rightarrow S_\infty=\frac{0.36}{0.99}\Rightarrow S_\infty=\frac{4}{11}$$

Option III. The response provided by ChatGPT to this question is based on an alternative approach. The following notation is introduced:

$$x=1.3636363636\dots$$

$$100x=136.36363636\dots$$

$$100x-x=135\Rightarrow 99x=135$$

$$x=\frac{15}{11}\Rightarrow 1.\overline{36}=\frac{15}{11}$$

Option IV. Verification of the answers is also considered appropriate. Direct verification by division, i.e. 5th grade knowledge, demonstrates that the first distractor is the correct answer. Verification of the other distractors is also suggested: answer B is expressed as a decimal fraction, with a value of 1.4, answer D is given as a decimal notation of 2.75, and answer C is greater than 1.5.

It is evident that state matriculation examinations demand a comprehensive mathematical aptitude and a range of strategies. A thorough analysis of potential approaches can provide authors with valuable insights, particularly regarding the extent to which the task, as articulated, encompasses the pertinent knowledge and skills. This phenomenon is particularly evident in the tasks from the external assessments at the end of grades 7 and 10, which can be solved by examining all possible cases, i.e. by directly checking the distractors.

In parallel with other educational systems. In Austria [3], with the support of the Ministry of Education, two platforms have been created. The first platform comprises pairs of files, with problems and solutions. The second platform offers more flexible access to individual problems from previous years, with the option of thematic filtering and sorting by complexity, video files with solutions explanations are provided as well. The format employed is not a test, but rather a short-answer format.

The International Baccalaureate [9] provides access to the Mark scheme, which contains detailed solutions with clearly indicated individual steps and the corresponding scoring.

The Scholastic Assessment Test (SAT) [6] is the closest in format and mathematics curriculum to the national external assessment at the end of 10th grade in Bulgaria. Each sample test contains a detailed explanation of the solution to each problem, as well as an analysis of distractors and common mistakes in reasoning and calculations.

The presentation of detailed solutions to individual problems from mathematics tests is a key factor in improving the quality of mathematics education in Bulgaria. This approach aligns with the principles of effective international standardized exams, transforming assessment into a catalyst for learning rather than merely a tool for selection, thereby fostering the development of mathematical thinking. Consequently, it is crucial that, in addition to the answer key, comprehensive solutions to the national tests are also provided. It is recommended that authors and publishers who offer educational materials with assessments adopt this standard procedure.

Conclusion. The absence of detailed feedback following standardized assessments within the Bulgarian education system constitutes a missed learning opportunity that has been overlooked. From a constructivist perspective, mathematics tests should not function solely as a tool for selection and ranking, but as a means of forming knowledge, diagnosing typical errors, and developing metacognitive skills. The integration of comprehensive solutions and a thorough examination of distractors would align Bulgarian practices with established international models, thereby enhancing the educational value of mathematics assessment.

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