МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2026
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2026
*Proceedings of the Fifty-Fifth Spring Conference
of the Union of Bulgarian Mathematicians
Tryavna, Bulgaria, April 5–9, 2026*

GRIESMER CODES, GREY–RANKIN BOUND AND THE PROJECTIVE DUAL TRANSFORM [1]

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Abstract

Professor Stefan Dodunekov made influential contributions across several domains of coding theory, number theory, and algebra. In this survey we focus on three areas in which his scientific interests were particularly substantial. The first two concern families of linear codes meeting the Griesmer bound and the Grey–Rankin bound, which have long served as central benchmarks in the study of optimal codes. The third theme is the projective dual transform, which has proven essential in the study of codes related to these bounds, and which has offered a unifying geometric and algebraic framework. Based on his contributions to this field, we emphasize both their prominence and their continuing influence on contemporary research related to various classes and families of linear codes over a finite field.

2020 Mathematics Subject Classification: 11T71, 94B05, 51E30.

Key words: Griesmer bound, Grey–Rankin bound, projective dual transform.

Professor Dodunekov was one of the most influential figures in Bulgarian coding theory and made significant contributions to several interrelated mathematical disciplines. His research spanned coding theory, algebra, and number theory, but his most profound impact was arguably in the structural analysis of optimal linear codes and in the development of geometric and algebraic tools for their investigation.

This survey focuses on three research directions in which Dodunekov played an initiating and guiding role. Two of them concern codes related to classical bounds, namely the Griesmer bound and the Grey–Rankin bound, which define fundamental limitations on the parameters of linear codes. The third direction focuses on the projective dual transform, which emerges as an indispensable tool for studying codes related to these bounds. Our goal is to provide an overview of Dodunekov's contributions in each of these areas and to trace how the initial questions he posed influenced subsequent developments.

Let $\mathbb{F}_q^n$ denote the n-dimensional vector space over the finite field $\mathbb{F}_q$, where q is a prime power. A q-ary linear $[n,k,d]_q$ code C is a k-dimensional subspace of $\mathbb{F}_q^n$ with minimum Hamming distance d. One of Dodunekov's foundational contributions to the theory of linear codes concerns the problem of determining the minimal length $n_q(k,d)$,

[1]Partially supported by the Bulgarian National Science Fund under Contracts No KP-06-H62/2-2022 and No KP-06-H92/3 - 08.12.2025.

https://doi.org/10.55630/mem.2026.55.370-381

that is, the smallest integer n for which a linear $[n, k, d]_q$ code exists. One of the main bounds for the length of a linear q-ary code with dimension k and minimum distance d is the Griesmer bound

$$n \geq g_q(k, d) = \sum_{i=0}^{k-1} \left\lceil \frac{d}{q^i} \right\rceil .$$

Codes meeting this bound are traditionally called *Griesmer codes.* In a series of papers [8, 10], Dodunekov established the remarkable fact that for fixed q and k, and for all sufficiently large values of d, one has $n_q(k, d) = g_q(k, d)$, that is, Griesmer codes exist for all but finitely many distances. Consequently, the determination of $n_q(k, d)$ remains an open problem for only a finite set of exceptional values of d.

Another important result due to Dodunekov, crucial for the structure of binary optimal codes, asserts that for a binary Griesmer code, if the minimum distance is even, then *all* codewords have even weights. This structural property becomes fundamental in several classification results and influences the geometry of the corresponding projective configurations [11].

Two linear codes are considered equivalent if one can be obtained from the other by coordinate permutations, nonzero scalar multiplications of coordinates, and field automorphisms. To classify linear codes with given parameters means to find exactly one representative from each equivalence class.

Dodunekov's work initiated a broader line of research that led to the description of infinite families of Griesmer codes. In particular, every Griesmer code can be viewed as a member of an infinite family with the following properties:

- The lengths of successive codes in the family differ by a constant equal to the length of the simplex code of the same dimension.
- Although the number of inequivalent codes of a fixed length initially grows with the length, there exists a constant L such that for all lengths exceeding L, the number of inequivalent representatives stabilizes.
- All codes in the family can be transformed into a family Ψ of codes of fixed length, whose weights are multiples of a power of q. For several parameter sets, the structure of the codes in Ψ is explicit enough to enable complete classification.

A key tool in the development of this theory is the *projective dual transform*, which provides a geometric mechanism for transferring structural properties between the families and for understanding the invariant characteristics of optimal codes. Its role in the study of Griesmer codes and their generalizations has proven to be fundamental and continues to influence modern research in coding theory.

In addition to his contributions to the theory of Griesmer codes, Dodunekov made significant advances in the study of codes meeting the Grey–Rankin bound. Together with several collaborators, he obtained classifications of such optimal codes in dimensions 6 and 8 [9]. Let C be a binary self-complementary code with length n and minimum d. Then

$$|C| \leq \frac{8d(n-d)}{n-(n-2d)^2},$$

provided that $n > (n-2d)^2$.

A third major direction of Dodunekov's research, developed jointly with Simonis, concerns the systematic investigation of the *projective dual transform* [7]. Their studies examined the transform from several complementary perspectives, including conditions for self-duality and connections to various geometric and algebraic constructions.

This transform is based on the natural duality between points and hyperplanes in the projective geometry $\mathrm{PG}(k-1,q)$: any point $a = (a_1, a_2, \dots, a_k)$ defines a hyperplane H_a which consists of all points $x = (x_1, x_2, \dots, x_k)$ such that $(a, x) = \sum a_i x_i = 0$. The projective dual transform is a powerful tool in the classification of infinite families of Griesmer codes. It is based on the fact that all codes from a family can be mapped to projective dual codes with fixed length and weights. An important result in this direction was the demonstration of a direct correspondence between such families of codes and bent Boolean functions, including self-dual bent functions. This relation allowed the introduction of the notion of *projective self-polar codes*, which connect optimal linear codes, projective geometries, and bent functions.

1 The Projective Dual Transform. One of the central concepts unifying several of Dodunekov's contributions to coding theory is the *projective dual transform*. In this section we outline the projective dual transform and present the basic notions required in later parts of the survey. Our presentation emphasizes the structural ideas rather than technical details.

There are several definitions of the dual transform, depending on the purpose for which it is used. Most commonly, the transform is defined constructively using tools from projective geometry [7], or algebraically via matrix representations [1, 5]. In [2], we instead employed characteristic vectors. A convenient way to represent a linear code is via its *characteristic vector*, which is defined relative to the q-ary simplex code. The simplex code $\mathcal{S}_{q,k}$ is the unique (up to equivalence) $[\theta(q,k), k, q^{k-1}]_q$ one-weight code, where

$$\theta(q,k) = \frac{q^k - 1}{q - 1}.$$

Its generator matrix $S_{q,k}$ has as columns a complete set of representatives of the points of the projective space $\mathrm{PG}(k-1,q)$.

Let C be an $[n,k]_q$ linear code with generator matrix G. The *characteristic vector* of C with respect to G is the vector $\chi(C,G) = (\chi_1, \dots, \chi_{\theta(q,k)}) \in \mathbb{Z}^{\theta(q,k)}$, where χ_i is the number of columns of G that are equal or proportional to the i-th column of $S_{q,k}$. The characteristic vector records the multiplicities of projective points in the generator matrix and satisfies $\sum_{i=1}^{\theta(q,k)} \chi_i = n$.

Consider the $\theta(q,k) \times \theta(q,k)$ matrix $M_k = S_{q,k}^T S_{q,k}$ over $\mathbb{F}_q$ and let $\mathcal{N}(M_k)$ denote the binary matrix obtained by replacing each nonzero entry of M_k by 1. This matrix is symmetric and invertible over $\mathbb{Z}$. If χ is the characteristic vector of an $[n,k]_q$ code C, then the product $\chi \mathcal{N}(M_k)$ yields the weights of a maximal set of nonproportional codewords in C. Consequently, the full weight distribution of C can be computed without enumerating all codewords. This observation lies at the heart of several algorithmic and theoretical applications of the projective dual transform.

Definition 1.1. *Let C be an $[n,k]_q$ linear code with characteristic vector χ. Let $\alpha, \beta \in \mathbb{Q}$ be such that $\alpha w + \beta$ is a nonnegative integer for every nonzero weight w of C. The* projective dual code *of C with respect to the parameters (α, β) is the linear code $D_{\alpha,\beta}(C)$ whose characteristic vector is*

$$\chi_{\alpha,\beta} = \alpha(\chi\mathcal{N}(M_k)) + \beta.\mathbf{1},$$

where $\mathbf{1}$ *denotes the all-ones vector of length $\theta(q,k)$.*

Equivalently, the generator matrix of $D_{\alpha,\beta}(C)$ consists of $\alpha w_i + \beta$ copies of the i-th column of $S_{q,k}$, where w_i runs over the weights of a maximal set of nonproportional codewords in C. This formulation shows the close connection between the projective dual transform and geometric constructions in projective spaces.

The parameters α and β control the transformation of the weights and allow for considerable flexibility. In particular, divisibility properties of the original code naturally lead to fractional values of α that reduce the length of the dual code.

A key result gives explicit formulas for the parameters of the dual code.

Theorem 1.2. *Let C be an $[n,k]_q$ linear code and let $\alpha, \beta \in \mathbb{Q}$ satisfy the admissibility conditions above. Then the length of $D_{\alpha,\beta}(C)$ is*

$$n_D = \alpha n q^{k-1} + \beta \frac{q^k - 1}{q-1},$$

and its nonzero weights are

$$q^{k-2}\big(\alpha((q-1)n + \chi_i) + \beta q\big),$$

where χ_i ranges over the coordinates of the characteristic vector χ of C.

An immediate and important consequence is the following.

Corollary 1.3. *If C is a projective linear code, then its projective dual $D_{\alpha,\beta}(C)$ is a two-weight code.*

This result explains the central role of the projective dual transform in the construction and classification of two-weight and few-weight codes, which are closely related to optimal codes, strongly regular graphs, and finite geometries.

A linear code C is called *projective self-dual* if it is equivalent to its projective dual $D_{\alpha,\beta}(C)$ for suitable parameters α and β. A projective self-dual code is a σ-self-dual code with respect to a transformation function σ of degree one as defined by Dodunekov and Simonis in [7, Definition 17]. A weaker notion is that of *formal projective self-duality*, where only the parameters and the weight distribution coincide. In [2], we studied projective self-dual codes in greater detail and described their connections with self-dual and self-polar incidence structures. We introduced the notion of self-polar codes and established several of their properties. We also investigated the relationship between self-polar codes and self-dual bent functions. For the parameters α and β for which $D_{\alpha,\beta}(C)$ is equivalent to the projective self-dual code C we use the following proposition given as Proposition 6 in [7].

Proposition 1.4. *Suppose C is not a replicated simplex code. Then C is a formally projective self-dual code for some α and β if and only if*

$$\alpha = \pm q^{1-\frac{k}{2}}, \quad \beta = -\frac{q-1}{1+q^{k-1}\alpha} n. \tag{1}$$

Projective self-duality, formal projective self-duality, and self-polarity are discussed in more detail in Section 3. Let us mention that every self-polar code is projective self-dual, but the converse is not necessarily true.

2 Griesmer codes and the projective dual transform. One of the fundamental properties for Griesmer codes is that, for fixed field size q and dimension k, there exists a threshold d_0 such that for every $d \geq d_0$ there exists a Griesmer code with minimum distance d. Equivalently, beyond this point the Griesmer bound is not merely a theoretical limit but is achieved by actual codes.

In this section we prove that the classification problem for Griesmer codes exhibits a *stabilization* property: as the length increases within the natural Griesmer parameter families, the number of inequivalent Griesmer codes does not keep growing indefinitely. More precisely, after a certain length threshold the number of equivalence classes becomes constant. The projective dual transform provides a unifying framework for two closely related tasks:

(i) reducing the classification of long Griesmer codes to the classification of shorter divisible codes, and

(ii) determining the point at which the number of inequivalent codes in a Griesmer family stabilizes (the constant L).

We briefly summarize both aspects. For fixed dimension k, a full classification of Griesmer codes can be organized by the families $\{\Psi_t(k, d_0)\}_{t=0}^{\infty}$, $1 \leq d_0 \leq q^{k-1}$, where the codes in the family $\Psi_t(k, d_0)$ have parameters

$$\left[g_q(k, d_0) + t\frac{q^k - 1}{q - 1},\ k,\ d_0 + tq^{k-1}\right]_q. \tag{2}$$

Classifying codes of growing length in (2) is hard even for small q and k. The key reduction is that for each fixed d_0 the classification of Griesmer codes with minimum distance $d_0 + tq^{k-1}$ can be reduced to classifying codes of a fixed length n' whose weights are all divisible by a prescribed integer ξ (usually a power of the characteristic). When n' is small, this reduction makes classification feasible.

The reduction relies on the remarkable fact that divisibility of the minimum distance forces divisibility of all codeword weights.

Theorem 2.1 ([10]). *Let C be a Griesmer $[n, k, d]_p$ code over $\mathbb{F}_p$, where p is a prime. If $p^e \mid d$, then p^e divides all weights of C.*

The binary case was proved by Dodunekov and Manev (see [10]), and later generalized to all prime fields (e.g. [12]). Note that there is a similar conjecture for divisible codes over composite fields, but it still remains an open problem. A consequence that is particularly useful in this context is the following persistence principle: if p^i divides a projective Griesmer code with minimum distance d_0, then the same divisor divides every Griesmer code in the family with minimum distance $d_0 + tq^{k-1}$ for $t \geq 1$. Recall that a linear code is projective, if any two columns of its generator matrix are non-proportional, or equivalently, the coordinates of its characteristic vector are only 0's and 1's.

Let C be a projective Griesmer code and assume that C is δ-divisible with $\delta = p^s$, $s \geq 0$, which means that all weights in C are divisible by δ. Choose

$$\alpha = \frac{1}{\delta}, \qquad \beta = -\frac{d_0}{\delta}.$$

Applying the projective dual transform to C yields a two-weight code $C' = D_{\alpha,\beta}(C)$ of length

$$n' = \frac{n}{\delta}q^{k-1} - \frac{d_0}{\delta}\frac{q^k - 1}{q - 1}, \tag{3}$$

with nonzero weights

$$w_0 = \frac{q^{k-2}}{\delta}\big(n(q-1)+1\big) - \frac{q^{k-1}}{\delta}d_0, \quad w_1 = w_0 - \frac{q^{k-2}}{\delta}. \tag{4}$$

If we take a δ-divisible Griesmer code which is not projective, its projective dual code for the same values of α and β has the same length n' but the number of its nonzero weights is equal to the number of the different coordinates of the characteristic vector of C. In particular, the integer $\xi = \frac{q^{k-2}}{\delta}$ divides all nonzero weights of C'. This is the mechanism underlying the reduction: instead of classifying Griesmer codes at growing lengths $n + t\frac{q^k-1}{q-1}$, one classifies codes of a fixed length n' whose nonzero weights lie in an arithmetic progression with step ξ.

Denote by $\Omega_t(k, d_0)$ the family of linear codes of dimension k, effective length n' given by (3), and nonzero weights in the set $\{w_0, w_1, \ldots, w_t\}$, where

$$w_i = w_0 - i\frac{q^{k-2}}{\delta}, \; i = 1, \ldots, t, \qquad 1 \leq t \leq -\beta',$$

and by $\Omega_{1-\beta'}(k, d_0)$ the family, consisting of codes of effective length n' and nonzero weights from the set $\{w_0, \ldots, w_{-\beta'}\}$ but with dimension $\leq k$. Clearly,

$$\Omega_1 \subseteq \Omega_2 \subseteq \cdots \subseteq \Omega_{-\beta'} \subseteq \Omega_{1-\beta'}.$$

Theorem 2.2. *There is a one-to-one correspondence between the families $\Omega_1(k, d_0)$ and $\Psi_0(k, d_0)$, $\Omega_2(k, d_0)$ and $\Psi_1(k, d_0)$, …, $\Omega_{1-\beta'}(k, d_0)$ and $\Psi_{-\beta'}(k, d_0)$. Moreover, every code in $\Psi_{-\beta'+j}(k, d_0)$ for $j \geq 0$ can be obtained from a code in $\Psi_{-\beta'}(k, d_0)$ by concatenating j copies of the simplex code of dimension k.*

This theorem was proven in [1]. We illustrate the method with a classification of linear codes over $\mathbb{F}_5$.

Example 2.3. Consider the families $\Psi_t(3, 8)$, $t \geq 0$, of Griesmer codes with parameters $[11 + 31t, 3, 8 + 25t]_5$. The parameters are $k = 3$, $d_0 = 8$, $\delta = 1$, $\alpha = 1$, $\beta = -8$. If C is a projective $[11, 3, 8]_5$ code, then $D_{1,-8}(C)$ is a $[27, 3, 20]_5$ code whose weights are all divisible by $\xi = 5$.

To classify the codes in $\Psi_t(3, 8)$, it therefore suffices to classify all codes of length 27 whose weights are multiples of 5. There are two inequivalent $[27, 3]_5$ with nonzero weights $\{20, 25\}$. Hence $\Omega_1(3, 8)$ consists of two codes and therefore there are two inequivalent $[11, 3, 8]_5$ Griesmer codes.

Further, $\Omega_2 = \Omega_1 \cup [27, 3, \{15, 20, 25\}]$ (14 codes)}, $\Omega_3 = \Omega_2 \cup [27, 3, \{10, 15, 20, 25\}]$ (25 codes)}, $\Omega_4 = \Omega_3 \cup [27, 3, \{5, 10, 15, 20, 25\}]$ (9 codes)}, and finally $\Omega_5 = \Omega_4 \cup$ $\{[27, 2, \{20, 25\}]$ (one code)}, $[27, 2, \{15, 20, 25\}]$ (one code) }, $[27, 2, \{10, 15, 20, 25\}]$ (one code)}. By Theorem 2.2, these families correspond to the following classes of Griesmer codes (we list the number of inequivalent codes after the parameters):

$$\Psi_0 : \{[11, 3, 8] - 2\}, \quad \Psi_1 : \{[42, 3, 33] - 16\}, \quad \Psi_2 : \{[73, 3, 58] - 41\},$$

$$\Psi_3 : \{[104, 3, 83] - 50\}, \quad \Psi_4 : \{[135, 3, 108] - 53\}, \quad \Psi_5 : \{[166, 3, 133] - 53\}.$$

A recurring theme in the classification of Griesmer codes is that within the families $\Psi_t(k, d_0)$, the number of inequivalent codes does not grow indefinitely with t. Instead, after a certain threshold the number stabilizes. A convenient way to describe this phenomenon uses *concatenation* with the simplex code together with the projective dual transform. Let C_1 and C_2 be q-ary linear codes of dimension k, generated by G_1 and G_2, respectively. We say that a code C is obtained by *concatenation* (or gluing) of C_1 and C_2 if it has a generator matrix of the form $(G_1 \,|\, G_2)$. If C_2 is the q-ary simplex code of dimension k, then the particular choice of generator matrices is irrelevant: any two such concatenations yield equivalent codes.

The next lemma formalizes the fact that appending simplex blocks to a Griesmer code preserves Griesmer optimality and shifts the parameters in the expected way.

Lemma 2.4. *Let C be a Griesmer $[g_q(k, d), k, d]_q$ code. If we concatenate C with t copies of the q-ary simplex code of dimension k, then we obtain again a Griesmer code.*

A second ingredient is a restriction on column multiplicities in generator matrices of Griesmer codes, originally proved by Dodunekov and crucial in controlling when a simplex block must occur.

The following lemma presents a standard result in the literature on linear codes meeting the Griesmer bound that implies exactly this type of restriction on column multiplicities in a generator matrix of an optimal code.

Lemma 2.5. *If $d \le sq^{k-1}$, then any generator matrix of a code C with parameters $[t + g_q(k, d), k, d]_q$ contains not more than $t + s$ mutually proportional columns.*

These results motivate viewing Griesmer codes in intervals according to distance and length. The intervals for the minimum distance are $(tq^{k-1}, (t + 1)q^{k-1}]$, $t \ge 0$. For $d = (t + 1)q^{k-1}$, there is a unique $[\frac{q^k-1}{q-1}(t + 1), k, d]$ code and it is obtained by concatenating $t + 1$ copies of the simplex code. Correspondingly, the relevant intervals for the length are $(t\frac{q^k-1}{q-1}, (t + 1)\frac{q^k-1}{q-1}]$.

Fix k and $d_0 \le q^{k-1}$, and consider the Griesmer families $\Psi_t(k, d_0)$, consisting of all Griesmer codes with parameters $\left[g_q(k, d_0)+t\frac{q^k-1}{q-1},\ k,\ d_0+tq^{k-1}\right]_q$, $t \ge 0$. For each fixed k there is an integer t_0 such that for all $t \ge t_0$ there exists at least one Griesmer code with these parameters. More importantly for classification, the number of inequivalent codes in the family eventually stabilizes.

Theorem 2.6. *There exists a constant L such that for all $t \geq L$ (with fixed k and $d_0 < q^{k-1}$) the number of inequivalent Griesmer codes with parameters*

$$\left[g_q(k,d_0) + t\tfrac{q^k-1}{q-1},\ k,\ d_0 + tq^{k-1}\right]_q$$

is constant. Furthermore, $L \leq P(k,d_0) = g_q(k,d_0)(q-1) - d_0 q$.

Idea of proof. Write $d_0 = q^{k-1} - d'$ and $g_q(k,d_0) = \frac{q^k-1}{q-1} - n'$, so that

$$n' = \frac{q^k-1}{q-1} - g_q(k,d_0).$$

Then the parameters of the codes in $\Psi_t(k,d_0)$ can be rewritten as

$$\left[(t+1)\tfrac{q^k-1}{q-1} - n',\ k,\ d_0 + tq^{k-1}\right]_q.$$

Choosing $t = n'$ the corresponding length equals

$$n_t = (n'+1)\frac{q^k-1}{q-1} - n',$$

which can be realized by taking $n'+1$ simplex blocks and deleting n' columns. According to Lemma 2.5, any generator matrix of a code with these parameters can have at most $t+1$ mutually proportional columns. Hence at least one full simplex block remains inside the generator matrix, and therefore codes at level $t = n'$ are obtained from codes at level $t = n' - 1$ by concatenation with one simplex block. This shows that the number of inequivalent codes does not increase beyond $L = n' - 1$.

Using the correspondence between Griesmer families and the associated dual-transform families $\Omega_q(d_0,k)$, one obtains $L \leq P(k,d_0)$. □

As a consequence, for fixed k one can also bound the number of inequivalent Griesmer codes at a given length by a constant depending only on k. The constant L depends on both the dimension k and the base distance d_0. Intuitively, it is larger when d_0 is smaller, and its value is affected by the divisors of d_0.

3 Linear codes related to Grey–Rankin bound. The classes of codes considered in this part are binary linear projective two-weight codes which are subcodes of codes attaining the Grey–Rankin bound. Such codes play a distinguished role in coding theory, as they lie at the interface between linear codes, combinatorial designs, and spectral properties of Boolean functions. Together with Stefan Dodunekov, we constructed all binary self-complementary $[120,9,56]$ codes with nonzero weights 56, 64, and 120 which have an automorphism of order 3 [3].

Any characteristic vector of a projective linear code can naturally be interpreted as a truth table of a Boolean function. In this identification, the two-weight property corresponds to the Walsh spectrum taking only two values, a property characteristic of bent Boolean functions. Thus, codes provide a natural starting point for how coding theory can be used to study bent functions, and in particular, self-dual and anti-self-dual bent functions.

All code families and constructions discussed below, including the self-complementary lifting, are defined within the considered class of optimal two-weight codes. Consequently, the lifting procedures and duality properties preserve both the extremal nature of the codes and their interpretation in terms of Boolean functions.

Let C be a binary $[n, k]$ linear code with a characteristic vector χ with respect to the generator matrix G. Recall that the projective dual transform associates to (C, G) a new code $D_{\alpha,\beta}(C)$ whose characteristic vector is $\chi_{\alpha,\beta} = \alpha\,\chi M_k + \beta.\mathbf{1}$, where M_k is the incidence matrix of the simplex code. Since the code is binary, $\theta(2, k) = 2^k - 1$ and $\mathcal{N}(M_k) = M_k$.

Definition 3.1 (Projective self-polar code)**.** *A binary linear code C is called* projective self-polar *if there exists a characteristic vector χ of C and admissible parameters* (α, β) *such that the projective dual transform preserves the characteristic vector, i.e.*

$$\chi = \chi_{\alpha,\beta}.$$

This formulation emphasizes the existence of a *polarizing representation* of the code, rather than invariance only up to code equivalence. It is this fixed-point property that allows a direct connection to self-duality in combinatorics and Boolean function theory.

Further, we associate a $2^k \times 2^k$ binary matrix M_C with the code C which depends on the parameters (α, β), where

$$M_C = \begin{pmatrix} 0 & \chi \\ \chi_{\alpha,\beta}^T & M_k \end{pmatrix}. \qquad (5)$$

If $\chi_{\alpha,\beta} = \chi$, then M_C is symmetric. More generally, if M_C is equivalent to its transpose via row and column permutations, then M_C can be interpreted as an incidence matrix of a self-dual incidence structure. If, in addition, the same permutation realizes both row and column coincidence, the structure is self-polar. Recall that an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ with incidence matrix N is *self-polar* if there exists a permutation matrix P such that

$$PN = N^T P^T.$$

A projective self-polar code in the sense of Definition 3.1 gives rise to a self-polar incidence structure via the matrix M_C in (5). This provides a combinatorial interpretation of the self-polar property of the code.

Example 3.2. Consider the binary linear code C with characteristic vector

$$\chi = (011100110100000) \in \mathbb{Z}^{15}.$$

This code has parameters $[6, 4, 2]$ and weight enumerator $W(y) = 1 + 6y^2 + 9y^4$. For $\alpha = -\frac{1}{2}$ and $\beta = 2$ we obtain

$$\chi_{\alpha,\beta} = -\tfrac{1}{2}\,\chi M_4 + 2.\mathbf{1} = (011100110100000) = \chi,$$

hence C is projective self-polar. The associated matrix M_C is symmetric and defines a self-polar incidence structure.

4 From self-polar codes to bent Boolean functions. The matrix M_k is closely related to the Sylvester-type Hadamard matrix H_k, namely

$$H_k = (-1)^{\overline{M}_k}, \qquad \overline{M}_k = \begin{pmatrix} 0 & \mathbf{0} \\ \mathbf{0}^T & M_k \end{pmatrix}.$$

Consequently, multiplication by M_k in the characteristic-vector formalism corresponds to Walsh–Hadamard transforms of Boolean functions. This observation underlies the transition from codes to Boolean functions.

Let $f : \mathbb{F}_2^k \to \mathbb{F}_2$ be a Boolean function with truth table $[f]$. Associate to f the code C_f whose characteristic vector χ_f is the truth table without the coordinate at 0 (so $[f] = (f(0), \chi_f)$). For bent functions (which exist only for even k), the Walsh spectrum takes only the values $\pm 2^{k/2}$, and this forces C_f to be a two-weight code with weights $2^{k-1} \pm 2^{\frac{k}{2}-1}$. This property gives us that the length of the code C_f is

$$n_f = \begin{cases} 2^{k-1} \pm 2^{\frac{k}{2}-1}, & \text{if } f(0) = 0 \\ 2^{k-1} \pm 2^{\frac{k}{2}-1} - 1, & \text{if } f(0) = 1. \end{cases}$$

Denote by Φ_{k^ϵ} the family of binary two-weight codes with length $n = 2^{k-1} + \epsilon 2^{\frac{k}{2}-1}$ and nonzero weights $w_1 = 2^{k-2} + \epsilon 2^{\frac{k}{2}-1}$ and $w_2 = 2^{k-2}$, where $\epsilon = \pm 1$, and by Φ'_{k^ϵ} the family of codes of length $n = 2^{k-1} + \epsilon 2^{\frac{k}{2}-1} - 1$. These families of two-weight codes were studied in [4]. Take

$$\begin{aligned} \alpha &= \epsilon \frac{1}{2^{\frac{k}{2}-1}}, & \beta &= -\epsilon 2^{\frac{k}{2}-1}, & &\text{if } C \in \Phi_{k^\epsilon}, \\ \alpha &= -\epsilon \frac{1}{2^{\frac{k}{2}-1}}, & \beta &= 1 + \epsilon 2^{\frac{k}{2}-1}, & &\text{if } C \in \Phi'_{k^\epsilon}. \end{aligned}$$

A code $C \in \Phi_{k^\epsilon}$ is *self-polar* in the sense of Definition 3.1, when it has a characteristic vector χ such that

$$\chi = \epsilon \frac{1}{2^{\frac{k}{2}-1}} \chi M_k - \epsilon 2^{\frac{k}{2}-1}.\mathbf{1}.$$

Definition 4.1 ([6]). *The bent Boolean function $f : \mathbb{F}_2^k \to \mathbb{F}_2$ is called self-dual if $(-1)^f = \frac{1}{2^{k/2}} f^W$ and anti-self-dual if $(-1)^f = -\frac{1}{2^{k/2}} f^W$, where*

$$f^W(x) = \sum_{y \in \mathbb{F}_2^k} (-1)^{f(y) + x \cdot y}.$$

The following theorem gives us, that if f is a self-dual or anti-self-dual bent function, the corresponding two-weight code is self-polar.

Theorem 4.2 ([2], Theorem 5.2). *Let f be a self-dual or anti-self-dual bent function and C_f be its corresponding code with a characteristic vector χ_f as defined above. Then*

$$C_f \in \begin{cases} \Phi_{k^-}, & \text{if } f(0) = 0 \text{ and } f \text{ is self-dual}, \\ \Phi_{k^+}, & \text{if } f(0) = 0 \text{ and } f \text{ is anti-self-dual}, \\ \Phi'_{k^-}, & \text{if } f(0) = 1 \text{ and } f \text{ is anti-self-dual}, \\ \Phi'_{k^+}, & \text{if } f(0) = 1 \text{ and } f \text{ is self-dual}. \end{cases}$$

In all cases the characteristic vector $\chi_{f,\alpha,\beta}$ of its projective dual code coincides with χ_f for the corresponding suitable values of α and β.

We now describe the self-complementary lifting (SCL) construction purely on the level of codes. Let C be a binary code with characteristic vector χ. Define a new code B with characteristic vector

$$\chi_B = (\overline{\chi}, 0, \chi, 0, \chi, 0, \chi), \qquad \overline{\chi} = \mathbf{1} + \chi. \tag{6}$$

The code B has dimension $k+2$ and preserves the two-weight property. Moreover, if C is projective self-polar, then so is B (with respect to the appropriately lifted parameters).

Interpreting χ as a truth table, the construction (6) yields a Boolean function in $k+2$ variables. A direct computation using block Hadamard matrices shows that SCL maps self-dual bent functions to anti-self-dual bent functions and vice versa.

Theorem 4.3. *Let C be a projective self-polar code and let f be the Boolean function associated with its characteristic vector. Then f is a self-dual or anti-self-dual bent function. Conversely, every self-dual or anti-self-dual bent function arises from a projective self-polar code in one of the corresponding two-weight families.*

5 Conclusion. In this survey we have highlighted three interwoven research directions that illustrate the depth and lasting influence of Dodunekov's work in coding theory, and in which we had the privilege to collaborate with him and subsequently to continue developing his ideas. His results on Griesmer codes clarified the asymptotic attainability of the Griesmer bound and revealed strong divisibility and rigidity properties that underpin the structure and classification of optimal linear codes. These ideas were further enriched through joint and subsequent works on codes meeting the Grey–Rankin bound, including complete classifications in specific dimensions and the identification of extremal constructions. Central to these developments is the projective dual transform, introduced and systematically investigated in [7]. The later extensions of these methods, linking optimal linear codes with projective geometries and bent Boolean functions, demonstrate how the questions and tools initiated by Dodunekov have continued to generate new results and to shape ongoing research in coding theory.

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ГРИЙСМЪРОВИ КОДОВЕ, ГРАНИЦА НА ГРЕЙ-РАНКИН И ПРОЕКТИВНО-ДУАЛНА ТРАНСФОРМАЦИЯ

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Абстракт

Академик Стефан Додунеков има значителен принос в няколко области на математиката, които включват теория на кодирането, теория на числата и алгебра. В това изследване се фокусираме върху три области, в които научните му интереси са били особено значителни. Първите две се отнасят до семейства линейни кодове, достигащи границите на Грийсмър и на Грей-Ранкин, които отдавна служат като централни ориентири в изучаването на оптимални кодове. Третата тема се отнася до проективно-дуалната трансформация, която играе съществена роля при изучаването на кодове, свързани с тези граници, и която предлага обединяваща геометрична и алгебрична насока на изследванията. Въз основа на приноса му в тази област, ние подчертаваме както тяхната важност, така и продължаващото му влияние върху съвременните изследвания, свързани с различни класове и семейства линейни кодове над крайно поле.

Ключови думи: Граница на Грийсмър, граница на Грей-Ранкин, проективно-дуална трансформация.