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ON T-AVOIDING SPHERICAL CODES AND DESIGNS IN 32-DIMENSIONAL EUCLIDEAN SPACE [1]

P. Boyvalenkov, D. Cherkashin, P. Dragnev,
D. Yorgov, and V. Yorgov

Abstract

In this article, we show that the minimal vectors of the extremal even unimodular lattices in $\mathbb{R}^{32}$ define T-avoiding universally optimal spherical codes for suitable sets T. Moreover, these codes are minimal T-avoiding spherical designs and maximal T-avoiding codes for appropriate choices of T.

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1 Introduction. Let $\mathbb{S}^{n-1} = \{\mathbf{x} = (x_1, \dots, x_n) : x_1^2 + \cdots + x_n^2 = 1\}$ be the unit sphere in $\mathbb{R}^n$. A spherical code is a non-empty finite set $C \subset \mathbb{S}^{n-1}$. Consider

$$I(C) := \{\mathbf{x} \cdot \mathbf{y} : \mathbf{x}, \mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}\},$$

the set of all inner products of distinct points of C, and denote by $s = \max I(C)$ the largest inner product (also known as a maximal cosine) of C. We refer to C as an s-code or as a spherical (n, N, s)-code, where $|C| = N$.

We are interested in spherical codes and designs (in particular, s-codes and τ-designs) which avoid some sets of inner products in $[-1, 1)$.

Definition 1.1. *Let $T \subset [-1, 1)$. A spherical code $C \subset \mathbb{S}^{n-1}$ is called T-avoiding if $I(C) \cap T = \varnothing$. If C is an s-code, then we assume $T \subset [-1, s)$ to avoid trivial cases.*

Spherical codes that are *well-distributed* have many applications in various fields. Of course, well-distributed can have many meanings (and so many applications). To quantify it, the notion of the discrete h-energy of a spherical code C for a given interaction potential h is considered (see, e.g., [1, 10] and references therein).

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Definition 1.2. *Given a function* $h : [-1,1] \to (-\infty,+\infty]$*, continuous on* $(-1,1)$*, we consider the* h*-energy of a spherical code* C *as*

$$E_h(C) := \sum_{\mathbf{x}\neq\mathbf{y}\in C} h(\mathbf{x}\cdot\mathbf{y}). \tag{1}$$

A spherical code C *is called* universally optimal *[6], if it has minimum* h*-energy among all codes on* $\mathbb{S}^{n-1}$ *of cardinality* $|C|$ *for all absolutely monotone potentials*[2] h.

The notion of T-avoiding universally optimal codes is then introduced in a natural way.

Definition 1.3. *Given* $T \subset [-1,1)$*, a* T*-avoiding code* C *is called* T-avoiding universally optimal *if for any absolutely monotone potential* h *it has minimum* h*-energy among all* T*-avoiding codes on* $\mathbb{S}^{n-1}$ *with cardinality* $|C|$.

An important meaning of good distribution is given by the concept of spherical designs [11].

Definition 1.4. *A spherical* τ*-design is a spherical code* $C \subset \mathbb{S}^{n-1}$ *such that*

$$\int_{\mathbb{S}^{n-1}} p(\mathbf{x})d\sigma_n(\mathbf{x}) = \frac{1}{|C|}\sum_{\mathbf{x}\in C} p(\mathbf{x}) \tag{2}$$

(σ_n *is the normalized surface measure) holds for all polynomials* $p(\mathbf{x}) = p(x_1, x_2, \ldots, x_n)$ *of total degree at most* τ.

In particular, codes with small $|I(C)|$ which are spherical τ-designs with large τ are good candidates for being well-distributed. In this paper we consider a class of codes on $\mathbb{S}^{31}$ which arise from even extremal unimodular lattices in $\mathbb{R}^{32}$. Such codes are spherical 7-designs and have $|I(C)| = 6$ as $0, -1 \in I(C)$ and the set $I(C) \setminus \{-1\}$ is symmetric about 0. With suitably chosen T, these codes are T-avoiding universally optimal and enjoy optimality as maximal codes and minimal designs in their class. We provide corresponding statements in Sections 4–6.

The interest in T-avoiding spherical codes came apparently after Gonçalves and Vedana [12] proved that four known extremal even unimodular lattices in $\mathbb{R}^{48}$ have the maximal possible density among the lattices with certain forbidden distances. The proof in [12] continues the research of Viazovska [22] and Cohn–Kumar–Miller–Radchenko–Viazovska [7, 8] who show the sphere packing optimality of E_8 and Leech lattices. In [3], T-avoiding $(48, 52416000, 1/2)$-codes with $T = (-1/3, -1/6) \cup (1/6, 1/3)$ were proved to be maximal as T-avoiding 1/2-codes (with certain mild assumptions) and minimal as T-avoiding spherical 11-designs. The universal optimality of these codes has been studied in [5]. The paper [4] considers T-avoiding codes in other dimensions.

2 Preliminaries.

We apply a linear programming technique which is briefly explained in this section.

For a given dimension n, we associate the sequence of Gegenbauer polynomials $\left(P_i^{(n)}\right)_{i=0}^{\infty}$ (see [19]), that are orthogonal on $[-1,1]$ with weight $w(t) := (1-t^2)^{(n-3)/2}$,

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[2] A potential function h is absolutely monotone if its derivatives are nonnegative, i.e. $h^{(j)} \geq 0$, $j \geq 1$.

which we normalize by $P_i^{(n)}(1) = 1$. Given a code $C \subset \mathbb{S}^{n-1}$, the i-th moment of C, $i \geq 1$, is defined by

$$M_i(C) := \sum_{\mathbf{x},\mathbf{y} \in C} P_i^{(n)}(\mathbf{x} \cdot \mathbf{y}).$$

A key property of Gegenbauer polynomials is their positive definiteness [18], which implies that $M_i(C) \geq 0$ for every code C and positive integer i.

A code C is a spherical τ-design if and only if $M_i(C) = 0$, $i = 1, \ldots, \tau$. Note also that for an antipodal C (such that $C = -C$) it follows that $M_i(C) = 0$ for all odd i.

The linear programming technique we use is based on building a function f with a proper sign on $[-1,1] \setminus T$ and proper coefficients in the Gegenbauer expansion

$$f(t) = \sum_{i=0}^{\deg f} f_i P_i^{(n)}(t). \tag{3}$$

The Fourier coefficients in (3) can be computed in a standard way.

The following key identity (see [11, 15]) is based on a double-counting for $f(\mathbf{x} \cdot \mathbf{y})$ over $C \times C$:

$$f(1) \cdot |C| + \sum_{\mathbf{x},\mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} f(\mathbf{x} \cdot \mathbf{y}) = f_0 \cdot |C|^2 + \sum_{i=1}^{\deg f} f_i M_i(C), \tag{4}$$

where $f(t)$ is a polynomial and $C \subset \mathbb{S}^{n-1}$ is a code. Additional assumptions on f, $T \subset [-1,1]$, and moments $M_i(C)$ give bounds on the cardinalities of maximal T-avoiding codes, minimal T-avoiding designs and on h-energy of C.

For any point $\mathbf{x} \in C$ and inner product $t \in I(C)$, we denote by $A_t(\mathbf{x})$ the number of points of C with inner product t with $\mathbf{x}$. The system of nonnegative integers $(A_t(\mathbf{x}) : t \in I(C))$ is called the *distance distribution of* C *with respect to the point* $\mathbf{x}$. If the numbers $A_t(\mathbf{x})$ do not depend on the choice of $\mathbf{x}$ for any $t \in I(C)$, then the code C is called *distance invariant*. If the code is distance invariant, we shall omit $\mathbf{x}$ in the notation $A_t(\mathbf{x})$. Then the distance distribution of a distance-invariant code C is naturally defined as the system of positive integers

$$F(C) := \{A_{t_i} : t_i \in I(C) \cup \{1\}\},$$

where $I(C) = \{t_1, \ldots, t_d\}$ and $t_1 < \cdots < t_d < t_{d+1} := 1$. The distance distribution of a distance-invariant τ-design with $d \leq \tau - 1$ can be found from a Vandermonde-type system, see [11, Theorem 7.4]. Furthermore, the definition of design implies the following quadrature formula for any polynomial f of degree at most τ

$$Nf_0 = \sum_{i=1}^{d+1} A_{t_i} f(t_i). \tag{5}$$

Note that the above-mentioned Vandermonde system can be obtained from (5) for $f = 1, t, \ldots, t^d$.

3 Extremal even unimodular lattices in $\mathbb{R}^{32}$ and their codes.

3.1 Lattices. The codes we are interested in are found as the sets of minimum length vectors in *extremal even unimodular lattices* as defined below. We first provide general definitions and then focus on dimension 32.

A lattice $L \in \mathbb{R}^n$ is called *even* if all norms (squared lengths) of vectors of L are even. It is called *unimodular* if it coincides with its dual lattice

$$L^* := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{y} \in \mathbb{Z} \ \forall \mathbf{y} \ \in L\}.$$

If L is even unimodular, then 8 divides n. It is known that the minimum norm of an even unimodular lattice L satisfies

$$\min\{\mathbf{x} \cdot \mathbf{x} : \mathbf{x} \in L, x \neq 0\} \leq 2\left[\frac{n}{24}\right] + 2. \tag{6}$$

The vectors of L can be partitioned into *layers*

$$L_m := \{\mathbf{x} \in L : \mathbf{x} \cdot \mathbf{x} = m\}.$$

An even unimodular lattice is called *extremal* if the bound (6) is attained; in other words, in the extremal lattices the layers L_m with $m < 2[n/24] + 2$ are empty.

In $\mathbb{R}^{32}$ the extremal even unimodular lattices have not been classified yet. King [14] proved that there are more than 10^7 non-isometric extremal even unimodular lattices without roots. All these lattices attain the bound (6) and are extreme; i.e., they realize a local maximum of the density function [17]. Recently these lattices were examined in terms of optimality for the Gaussian potential function

$$E(\alpha, L) = \sum_{\mathbf{x} \in L} e^{-\alpha(\mathbf{x} \cdot \mathbf{x})},$$

see Heimendahl–Marafioti–Thiemeyer–Vallentin–Zimmermann [13]. In particular, they show that for $\alpha = \pi$ such a lattice is a local maximum.

We consider the set $\mathcal{C}$ of spherical codes obtained by rescaling to $\mathbb{S}^{31}$ the first non-empty layer L_4 of an extremal even unimodular lattice L in $\mathbb{R}^{32}$.

3.2 The set $\mathcal{C}$ of codes on $\mathbb{S}^{31}$ from minimum norm 4 vectors. The first nontrivial layer L_4 of an extremal even unimodular lattice in $\mathbb{R}^{32}$ contains 146880 vectors. After rescaling to $\mathbb{S}^{31}$, we obtain a spherical code $C \in \mathcal{C}$ with cardinality $|C| = 146880$, set of inner products

$$I(C) = \left\{-1, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}\right\}. \tag{7}$$

Furthermore, C is a spherical $7\frac{1}{2}$-design; i.e., a 7-design with $M_{10}(C) = 0$. Since $|I(C)| = 6 < \tau = 7$, C is distance regular [11, Theorem 7.4]. Its distance distribution is given by

$$\mathcal{F}(C) = \{1, 1240, 31744, 80910, 31744, 1240, 1\}. \tag{8}$$

See, for example, [17, 20]. The quadrature rule (5) takes the form

$$\begin{aligned} 146880 f_0 = f(-1) + f(1) + 1240 &\left(f\left(-\frac{1}{2}\right) + f\left(\frac{1}{2}\right)\right) \\ + 31744 &\left(f\left(-\frac{1}{4}\right) + f\left(\frac{1}{4}\right)\right) + 80910 f(0), \end{aligned} \tag{9}$$

holding true for every polynomial f of degree at most 7.

For certain specific choices of T, we shall prove that all these codes are simultaneously maximal spherical T-avoiding codes among the codes that are 3-designs (an additional condition of such a type is inevitable, see Remark 4.1). Also, these codes are minimal (tight) T-avoiding spherical 7-designs.

3.3 Lattices in $\mathbb{R}^{32}$ obtained from self-dual binary codes. Of the more than 10^7 extremal even unimodular lattices in dimension 32, there are few that have been studied in detail [21]. Five of them can be called special since they can be obtained from the five (see Conway–Pless [9] for the classification) extremal doubly-even binary self-dual codes of length 32, denoted by CP_i, $i = 1, 2, \ldots, 5$. As follows from their self-duality and extremality, all these five codes have dimension 16 and minimum distance 8.

Let C be one of those five codes. We define a lattice $L(C)$ using construction B (see [10, Chapter 7]) followed by an extension (doubling). One first maps C into $\mathbb{R}^{32}$ by

$$G(C) = \big\{ \mathbf{x} = (x_1, \ldots, x_{32}) \in \mathbb{Z}^{32} \;\big|\; x_1 + \cdots + x_{32} \equiv 0 \pmod 4, \\ \mathbf{x} \pmod 2 \in C \big\}.$$

Then a “doubling” is performed via

$$L(C) = \frac{1}{\sqrt{2}} G(C) \cup \frac{1}{\sqrt{2}} \left(\left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2} \right) + G(C) \right).$$

The resulting lattice is extremal, even, and unimodular. We are mostly interested in $L(C)_4$, the first non-empty layer of $L(C)$.

In search of classifying patterns, Venkov [20] introduced and investigated the parameter

$$D(L) = \big\{ e_{2,2}(\mathbf{x}, \mathbf{z}) = \big|\{ \mathbf{y} \in L(C)_4 \;\big|\; \mathbf{x} \cdot \mathbf{y} = 2,\ \mathbf{z} \cdot \mathbf{y} = 2 \}\big|, \\ \mathbf{x}, \mathbf{z} \in L(C)_4 \text{ with } (\mathbf{x}, \mathbf{z}) = 0 \big\}.$$

He showed that $D(L)$ consists of even integers between 0 and 60. We see that $D(L(CP_i))$ contains the maximum value 60 as follows. The vectors

$$\mathbf{y}_1 = \frac{1}{\sqrt{2}}(2, 0, \ldots, 0, 2),\ \mathbf{y}_2 = \frac{1}{\sqrt{2}}(0, 2, \ldots, 0, 0, 2), \ldots$$

$$\mathbf{y}_{30} = \frac{1}{\sqrt{2}}(0, 0, \ldots, 2, 0, 2),\ \mathbf{y}_{31} = \frac{1}{\sqrt{2}}(-2, 0, \ldots, 0, 2),$$

$$\mathbf{y}_{32} = \frac{1}{\sqrt{2}}(0, -2, \ldots, 0, 0, 2), \ldots, \mathbf{y}_{60} = \frac{1}{\sqrt{2}}(0, 0, \ldots, -2, 0, 2)$$

belong to the norm 4 layer of $L(C)$ and $\mathbf{x} \cdot \mathbf{y}_i = \mathbf{z} \cdot \mathbf{y}_i = 2$ for $i = 1, 2, \ldots, 60$.

Venkov showed that every lattice L with $60 \in D(L)$ can be constructed as above, and so such L coincides with one of the $L(CP_i)$. In order to show that the five lattices $L(CP_i)$ are not isometric Koch and Venkov [21] classified $D(L(CP_i))$ for $1 \leq i \leq 5$ and noted that these sets are distinct.

We confirm this result using the computational algebra system Magma [2], by showing that the five lattices have automorphism groups of different orders, namely:

$$\begin{aligned}
975175680 &= 2^{21}\cdot 3\cdot 5\cdot 31,\\
48126558103142400 &= 2^{31}\cdot 3^5\cdot 5^2\cdot 7\cdot 17\cdot 31,\\
91321742131200 &= 2^{31}\cdot 3^5\cdot 5^2\cdot 7,\\
16911433728 &= 2^{28}\cdot 3^2\cdot 7,\\
1509949440 &= 2^{25}\cdot 3^2\cdot 5.
\end{aligned}$$

The current Magma lattice database contains 19 non-isometric extremal even unimodular lattices of dimension 32. All of them have different automorphism group orders with the least one being $12288 = 2^{12}\cdot 3$. The lattices $L(CP_i)$, $i = 2,3,4,5$, have the largest automorphism groups. Among the remaining lattices in the Magma database, just one is with an automorphism group order greater than $2^{21}\cdot 3\cdot 5\cdot 31$ and that number is $1244160000 = 2^{13}\cdot 3^5\cdot 5^4$.

4 Maximal T-avoiding (1/2)-codes in 32 dimensions. In this section we prove that every code $C \in \mathcal{C}$ is a maximal T-avoiding (1/2)-code for $T = (0, 1/4)$ under the additional assumption that the code is at least a 3-design. As noted above, all such codes have cardinality 146880 and are, in fact, spherical $7\frac{1}{2}$-designs.

Remark 4.1. *One may be tempted to favor the embedding of the kissing configuration of the Leech lattice in a maximal T-avoiding (1/2)-code in 32 dimensions for some T. However, an easy calculation shows that the second moment of that embedding is positive; i.e., it is not even a 2-design. This prevents a contradiction when maximality is proven via polynomials in (4) with negative coefficient f_2 for codes which are at least 2-designs. Therefore, we can possibly have maximal codes with less than 196560 points if we additionally require that our codes are at least spherical 2-designs*[3].

Theorem 4.2. *Let $C \subset \mathbb{S}^{31}$ be a $(0, 1/4)$-avoiding (1/2)-code that is (at least) a spherical 3-design. Then $|C| \le 146880$. This bound is attained by every code $C \in \mathcal{C}$.*

Proof. Assume that $C \subset \mathbb{S}^{31}$ is a $(0, 1/4)$-avoiding (1/2)-code and a spherical 3-design. Consider (4) with C and the degree 10 polynomial

$$f(t) = (t+1)\left(t+\frac{1}{2}\right)^2\left(t+\frac{1}{4}\right)^2 t\left(t-\frac{1}{4}\right)\left(t-\frac{1}{2}\right)^3.$$

Note that $f(1) = 675/1024$ and $f_0 = 5/1114112$ in the Gegenbauer expansion (3) of f.

Since $f(t) \le 0$ for every $t \in [-1, 1/2] \setminus (0, 1/4)$, the left-hand side of (4) does not exceed $f(1)|C| = 675|C|/1024$. For the right-hand side we first see that since $M_i(C) = 0$ for $i = 1,2,3$, the signs of the coefficients f_i, $i = 1,2,3$, do not matter (in fact, we have $f_2 < 0$ and $f_3 < 0$; see the Leech lattice obstruction Remark 4.1). Next, we observe that $f_i > 0$ for $i \in \{4,\dots,10\} \setminus \{8\}$, while $f_8 = 0$. We conclude that the right-hand side of (4) is at least $f_0|C|^2 = 5|C|^2/1114112$. This gives

$$|C| \le \frac{f(1)}{f_0} = 146880.$$

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[3] In fact, our proofs will require strength 3.

It follows from the above argument that any attaining code C must be a spherical 7-design with vanishing ninth and tenth moment (in particular, a spherical $7\frac{1}{2}$-design). Moreover, the inner products of C must belong to $\{-1, \pm 1/2, \pm 1/4, 0\}$. Every code $C \in \mathcal{C}$ possesses these properties and, therefore, attains the bound. $\square$

The polynomial for Theorem 4.2 was constructed based on the properties of the attaining codes. Indeed, the fact that the codes are antipodal spherical 7-designs with $M_{10} = 0$ prompts us to use a degree-10 polynomial with $f_8 = 0$. Further, the necessary vanishing at the inner products and the endpoints of the interval T determine the roots of the polynomial to be as in the set $I(C)$ from (7) with the corresponding multiplicities. Lucky or not, the resulting polynomial is good as shown in the proof.

We conclude this section with the explicit Gegenbauer expansion of the polynomial f used in the above proof. It is

$$f(t) = \sum_{i=0}^{10} f_i P_i^{(32)}(t),$$

with

$$f_0 = \frac{5}{1114112},\ f_1 = \frac{65}{992256},\ f_2 = -\frac{31}{196608},\ f_3 = -\frac{93}{165376},\ f_4 = \frac{217}{417792},$$
$$f_5 = \frac{899}{58368},\ f_6 = \frac{2387}{188416},\ f_7 = \frac{20119}{894976},\ f_8 = 0,\ f_9 = \frac{3441}{11776},\ f_{10} = \frac{14911}{47104}.$$

5 Tight T-avoiding spherical 7-designs in 32 dimensions. In this section we show how minimality results for T-avoiding spherical designs on $\mathbb{S}^{31}$ with some T can be derived. Such designs were called *tight T-avoiding* in [4]. We shall prove that every code $C \in \mathcal{C}$ is a tight T-avoiding spherical 7-design with $T = (-1/4, 0) \cup (1/4, 1/2)$.

Theorem 5.1. *Let $C \subset \mathbb{S}^{31}$ be a T-avoiding spherical 7-design, where $T = (-1/4, 0) \cup (1/4, 1/2)$. Then $|C| \geq 146880$. This bound is attained by every code $C \in \mathcal{C}$.*

Proof. We apply (4) for C and with the degree 7 polynomial

$$f(t) = t(t+1)\left(t+\frac{1}{2}\right)^2\left(t+\frac{1}{4}\right)\left(t-\frac{1}{4}\right)\left(t-\frac{1}{2}\right).$$

It is easy to see that $f(t) \geq 0$ for every $t \in [-1, 1] \setminus T$ for the left-hand side and that the right-hand side is equal to $f_0|C|^2$. Since $f(1) = 135/64$ and $f_0 = 1/69632$, we obtain $|C| \geq f(1)/f_0 = 146880$. $\square$

Again, the properties of the attaining designs suggest the right polynomial for use in (4). It has degree 7 to provide $\sum_{i=1}^{7} f_i M_i(C) = 0$ in the right-hand side and zeros in $I(C)$ from (7) with appropriate multiplicities to make the sum in the left-hand side nonnegative.

6 Universally optimal T-avoiding codes in 32 dimensions. Let L be any fixed even unimodular extremal lattice in $\mathbb{R}^{32}$ and L_4 be the collection of 146880 vectors of minimal norm. Among several possible choices of T we consider the symmetric case, where

$$T := \left(-\frac{1}{2}, -\frac{1}{4}\right) \cup \left(\frac{1}{4}, \frac{1}{2}\right).$$

With this choice of T, we denote the class of T-avoiding codes on $\mathbb{S}^{31}$ by

$$\mathcal{C}_T := \{C \subset \mathbb{S}^{31} : |C| = 146880, \quad I(C) \cap T = \varnothing\}. \tag{10}$$

The following theorem establishes that for any absolutely monotone potential function h a universal lower bound on the h-energy of codes in the class $\mathcal{C}_T$ holds. Clearly, $\mathcal{C} \subset \mathcal{C}_T$.

Theorem 6.1. *Let h be absolutely monotone with $h^{(8)} > 0$ in $(-1, 1)$ and $C \in \mathcal{C}_T$. Then*

$$\begin{aligned} E_h(C) \geq 146880 \Bigg[h(-1) + 1240 h\left(-\frac{1}{2}\right) + 31744 h\left(-\frac{1}{4}\right) \\ + 80910 h(0) + 31744 h\left(\frac{1}{4}\right) + 1240 h\left(\frac{1}{2}\right) \Bigg]. \end{aligned} \tag{11}$$

This bound is attained by every code $C \in \mathcal{C}$.

Proof. First, we note that every code $C \in \mathcal{C}$ has the same energy as given by the right-hand side of (11). Indeed, any such code has the same inner products and distance distribution as explained in Section 3.2 and the calculation of its h-energy from (1) with the quadrature rule (9) gives the right-hand side of (11).

So, our goal is to show that (11) holds for any T-avoiding code C. Consider the multiset

$$\mathcal{I} := \{t_1, \ldots, t_8\} = \left\{-1, -1, -\frac{1}{2}, -\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{2}\right\}.$$

Denote by $H_7(t; h, \mathcal{I})$ the polynomial, which interpolates the potential function h at the points of $\mathcal{I}$ (a repeated node indicates derivative interpolation). Newton's interpolation formula implies that

$$H_7(t; h, \mathcal{I}) = h(t_1) + \sum_{i=1}^{7} h[t_1, \ldots, t_{i+1}](t - t_1) \ldots (t - t_i), \tag{12}$$

where $h[t_1, \ldots, t_{i+1}]$, $i = 1, \ldots, 7$, are the corresponding divided differences and

$$P_i(t) := (t - t_1) \ldots (t - t_i), \quad i = 1, \ldots, 7$$

are the associated partial products. The absolute monotonicity of h implies that all divided differences are non-negative.

The positive definiteness of the partial products $P_i(t)$ can be verified directly, with which we now proceed. Since $t + c$ has positive Gegenbauer coefficients for any $c \geq 0$ and positive definiteness is preserved under multiplication of polynomials (this is the so-called Krein condition in [16]), we conclude that the partial products P_i, $i = 1, \ldots, 6$, are positive definite. The last partial product

$$P_7(t) = (t+1)^2 \left(t + \frac{1}{2}\right)\left(t + \frac{1}{4}\right) t^2 \left(t - \frac{1}{4}\right) = \sum_{i=0}^{7} f_i P_i^{(32)}(t),$$

with

$$f_0 = \frac{97}{104448}, \ f_1 = \frac{619}{41344}, \ f_2 = \frac{12245}{116736}, \ f_3 = \frac{2139}{5168}, f_4 = \frac{13981}{13056},$$

$$f_5 = \frac{4433}{2432},\ f_6 = \frac{11935}{7296},\ f_7 = \frac{341}{608},$$

is explicitly seen to be positive definite. We next utilize the Hermite error formula, which yields (recall that $T = (-1/2, -1/4) \cup (1/4, 1/2)$)

$$h(t) - H_7(t; h, \mathcal{I}) = \frac{h^{(8)}(\xi)}{8!}(t+1)^2 \left(t + \frac{1}{2}\right)\left(t + \frac{1}{4}\right) t^2 \left(t - \frac{1}{4}\right)\left(t - \frac{1}{2}\right) \geq 0$$

for every $t \in [-1, 1] \setminus T$.

To complete the argument, we combine the h- and f-energies using (4) with C, $H_7(t; h, \mathcal{I})$, and the above conclusions. In the left-hand side of (4) we estimate

$$E_h(C) = \sum_{\mathbf{x},\mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} h(\mathbf{x} \cdot \mathbf{y}) \geq \sum_{\mathbf{x},\mathbf{y} \in C, \mathbf{x} \neq \mathbf{y}} f(\mathbf{x} \cdot \mathbf{y})$$

since $h(t) \geq H_7(t; h, \mathcal{I})$ for all inner products of C. The right-hand side is at least $(H_7)_0 |C|^2$ since all Gegenbauer coefficients of $H_7(t; h, \mathcal{I})$ are non-negative. Therefore, we have

$$E_h(C) \geq 146880^2 \left((H_7)_0 - \frac{H_7(1)}{146880}\right).$$

It remains to see that the right-hand side of the last inequality is equal to the right-hand side of the bound (11). This fact follows from the quadrature formula (5) written for any of the 7-designs in $\mathcal{C}$.

If equality holds for some code C, then all moments $M_i(C)$, $i = 1, \dots, 7$, vanish, so C has to be a 7-design. Moreover, the interpolation conditions imply that the inner products of C belong to the set $\{0, -1, \pm 1/2, \pm 1/4\}$ (see (7)). From [11, Theorem 7.4] such a code is distance-invariant. Utilizing the quadrature formula (5) associated with the design property, we conclude that such a code has the same distance distribution as (8), which concludes the proof. ☐

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Peter Boyvalenkov, Danila Cherkashin
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
e-mail: `peter@math.bas.bg, jiocb@math.bas.bg`

Peter Dragnev, Daniel Yorgov
Department of Mathematical Sciences
Purdue University Fort Wayne
Fort Wayne, IN 46805, USA
e-mail: `dragnevp@purdue.edu, yorgovd@purdue.edu`

Vassil Yorgov
Department of Mathematics and Computer Science
Fayetteville State University
Fayetteville, NC 28301, USA
e-mail: `vyorgov@gmail.com`

ВЪРХУ T-ИЗБЯГВАЩИ СФЕРИЧНИ КОДОВЕ И ДИЗАЙНИ В 32-МЕРНОТО ЕВКЛИДОВО ПРОСТРАНСТВО

П. Бойваленков, Д. Черкашин, П. Драгнев,
Д. Йоргов и В. Йоргов

Абстракт

В тази статия показваме, че минималните вектори на екстремалните четни унимодуларни решетки в $\mathbb{R}^{32}$ образуват универсално оптимални T-избягващи сферични кодове за подходящи множества T. Освен това тези кодове са минимални T-избягващи сферични дизайни и максимални T-избягващи кодове при подходящ избор на T.

Ключови думи: сферични кодове, сферични дизайни, линейно програмиране.