Stability of Nonlinear waves

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- stability theory for solitary wave solutions start with Benjamin (T. B. Benjamin, The stability of solitary waves, Proc. Roy. Soc. London Ser. A328 (1972)) for KdV equation

$$u_t + u_x + uu_x + u_{xxx} = 0$$

Since a small perturbation of a solitary wave can yield a solitary wave with a phase shift or one with different speed, it is appropriate to study the orbital stability According to Benjamin, if $u(x,t)$ is a solution of KdV or BBM whose initial profile $u(x,0) = u_0(x)$ is sufficiently close to a solitary wave profile $\varphi(x-ct)$, then the

$$\inf_{y\in\mathbb{R}} ||u(\cdot,t) - \varphi(\cdot + y)||$$

remain small for all times $t > 0$.

- model equations for the propagation of long wave in nonlinear dispersive media J. P. Albert, J. L. Bona and D. B. Henry, Suffucient conditions for stability of solitary-wave solutions of model equations for long waves, Physica 24D(1987))

$$u_t + u_x + u^p u_x - M u_x = 0 \tag{1}$$

where M is an operator defined by $\widehat{Mv}(\xi) = \alpha(\xi)\widehat{v}(\xi)$. For broad class of symbols α they demonstrated that the solitary wave solutions of (1) are orbitally stable. To prove of stability of the solitary wave profile φ they construct the functional of the form

$$E(u) = H(u) + cV(u)$$

where $H = \frac{1}{2}\int_{-\infty}^{+\infty}(uMu - \frac{2}{(p+1)(p+2)}u^{p+2})dx$ and $V = \frac{1}{2}\int_{-\infty}^{+\infty} u^2 dx$ are conserved quantities for the equation (1).

- A set of sufficient condition for stability, which will be to satisfied by the solitary wave solutions of a class of equation of the type (1) are the following:
  If the linear operator L defined by

$$L = M + (c - \varphi^p)$$

  have the following properties:
  P1) L has a simple negative eigenvalue λ
  P2) L has no negative eigenvalue other than λ
  P3) the eigenvalue 0 of L is simple
  then the stability and instability of the solitary wave φ is determined by the sign of the quantity $\frac{d}{dc}\int_{-\infty}^{+\infty}\varphi_c^2(x)dx$.

- A general Hamiltonian system was considered by (M. Grillakis, J. Shatah and W.S trauss, Stability theory of solitary waves in the presence of symmetry I, Journal of Functional Analysis 74(1987))

$$u_t = JE'(u(t)) \tag{2}$$

where E is the energy functional and J is a skew-symmetric linear operator. Equation (2) is assumed to be invariant under a representation $T(\cdot)$ of the group $U(1)$. Under assumptions on the spectrum of the linearized Hamiltonian

$$H_w = E^{''}(\phi_w) - wQ^{''}(\phi_w)$$

is proved that the solitary wave $T(wt)\phi_w$ is orbitally stable if $d^{''}(w) > 0$, where $d(w) = E(\phi_w) - wQ(\phi_w)$.

- They also considered the orbital stability of solitary waves in the Hamiltonian systems symmetric with respect to the action of a general group (M. Grillakis, J. Shatah and W. Strauss, Stability theory of solitary waves in the presence of symmetry II, Journal of Functional Analysis 94(1990)). In this case the stability depends on the relations of the numbers of negative eigenvalue of the linearized Hamiltonian and the numbers of positive eigenvalues of the Hessian of d''.

- For the higher order water wave models, the techniques developed in the above cited works is not applicable, since the assumptions on the spectrum of the linear operator obtained by linearizing the solitary wave equations in this case is difficult to verify. For the fifth order Korteweg-de Vries equation

$$u_t + u_{xxxxx} + bu_{xxx} = (f(u, u_x, u_{xx}))_x, \tag{3}$$

where the nonlinear term has the variational structure

$$f(q, r, s) = F_q(q, r) - rF_{qr}(q, r) - sF_{rr}(q, r)$$

for some $F(q, r) \in C^3(\mathbb{R}^3)$ which is homogeneous of degree $p + 1$, the existence and stability of solitary wave solutions are considered by Levandosky (S. Levandosky, A stability analysis of fifth-order water wave models, Physica D, 125(1999)).

- The solutions are obtained by solving a constrained minimization problem

$$\inf\{I_c(u), K(u) = \lambda\}$$

where

$$I_c(u) = \frac{1}{2}\int_{-\infty}^{+\infty} |u_{xx}|^2 - b|u_x|^2 + c|u|^2 dx$$

and

$$K(u) = \int_{-\infty}^{+\infty} F(u, u_x) dx$$

The concentration compactness principle Lions, (P.L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, part I, Annales de . H. P., section C, tome 1, no 2 (1984), p. 109-145.) is used to show that all minimizing sequences for this problem are relatively compact in the Sobolev

- The method of proving the stability is based on the variational characterization of the solitary wave solutions and the theory of Grillakis-Shatah-Strauss.So the use of the function

$$d(c) = E(\varphi) + cQ(\varphi),$$

where E and Q are conserved quantities for equation (**??**), defined by

$$E(u) = \frac{1}{2}\int_{-\infty}^{+\infty}(u_{xx}^2 - bu_x^2 - 2F(u, u_x))dx$$

$$Q(u) = \frac{1}{2}\int_{-\infty}^{+\infty}u^2 dx,$$

was a main ingredient in the analysis of stability. In fact, as well-known the solitary waves with speed c will be stable if and only if d is convex at c.

- eigenvalue problem in the form

$$\mathcal{J}\mathcal{H}V = \lambda V \tag{4}$$

$$\mathcal{H} = \mathcal{H}^*, dim(Ker(\mathcal{H}) < \infty, \mathcal{J}^* = -\mathcal{J}$$

Let k_r be the number of positive eigenvalues of the spectral problem (3) (i.e. the number of real instabilities or real modes),
k_c be the number of quadruplets of eigenvalues with non-zero real and imaginary parts,
k_i^-, the number of pairs of purely imaginary eigenvalues with negative Krein-signature.
For a simple pair of imaginary eigenvalues $\pm i\mu, \mu \neq 0$, and the corresponding eigenvector $\vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$, the Krein index is $sgn(\langle \mathcal{H}\vec{z}, \vec{z}\rangle$

- Assume that

$$dim(gKer(\mathcal{J}\mathcal{H})) < \infty$$

basis in

$$gKer(\mathcal{J}\mathcal{H}) \ominus Ker(\mathcal{H}), \eta_j, j = 1, \ldots, N$$

Then $\mathcal{D} \in M_{N\times N}$ is defined via

$$\mathcal{D} := \{\mathcal{D}_{ij}\}_{i,j=1}^{N} : \mathcal{D}_{ij} = \langle \mathcal{H}\eta_i, \eta_j \rangle.$$

$$k_{Ham} := k_r + 2k_c + 2k_i^- = n(\mathcal{H}) - n(\mathcal{D}). \quad (5)$$

$n(\mathcal{H})$ - number of negative eigenvalues

- Boussinesq equation

$$u_{tt} + u_{xxxx} - u_{xx} + (u^3)xx = 0. \tag{6}$$

traveling wave solutions for (5) in the form $u(t,x) = \varphi(x + ct)$

$$\varphi(x) = \varphi_0 dn(\alpha x) \tag{7}$$

$$\kappa^2 = \frac{\varphi_0^2 - \varphi_1^2}{\varphi_1^2} = \frac{2\varphi_0^2 - 2w}{\varphi_1^2}, \quad \alpha = \frac{\varphi_0}{\sqrt{2}}.$$

- We now set up the linear stability/instability problem for (5). Set the ansatz $u = \varphi(x + ct) + v(t, x + ct)$ and ignore all terms $O(v^2)$. We get $v_{tt} + 2cvtx + Mv = 0$, where

$$Mv = \partial_x^4 v - (1 - c^2)\partial_x^2 v + (3\varphi^2 v)_{xx}.$$

Note that this operator M is not self-adjoint. However, if we introduce the variable $z : z_x = v$, we get the following linearized equation in terms of z:

$$z_{ttx} + 2cz_{txx} + M[z_{tx}] = 0.$$

- The question of linear stability of equations in the form

$$z_{tt} + 2cz_{tx} + \mathcal{H}z = 0$$

or what is equivalent (at least in this case) to the solvability of

$$\lambda^2\psi + 2w\lambda\psi + \mathcal{H}\psi = 0,$$

where $\mathcal{H}$ is self-adjoint and is in the form

$$\mathcal{H} = -\partial_x \mathcal{L} \partial_x \quad \mathcal{L} = -\partial_x^2 + (1 - c^2) - 3\varphi^2.$$

Definition: We say that the travelling wave φ is spectrally/linearly unstable if there exists a T-periodic mean value zero function $\psi \in D(\mathcal{H})$ and $\lambda : \Re\lambda > 0$

$$\lambda^2\psi + 2c\lambda\psi' + H\psi = 0.$$

- we assume the the following for the spectrum of $\mathcal{H}$

$$\begin{cases} \mathcal{H}\phi = -\delta^2\phi, \mathcal{H}\mid_{\{\phi\}^\perp} \geq 0 \\ Ker[\mathcal{H}] = span[\vec{\psi_0}]. \end{cases} \tag{8}$$

index for estimates $\langle \mathcal{H}^{-1}[\psi_0'], [\psi_0'] \rangle$
Theorem.(S. Hakkaev, M. Stanislavova, A. Stefanov, PRSE, 2014) The two parameter family of dnoidal solutions, described in (6), is then linearly stable, if and only if

$$|c| \geq \sqrt{\frac{M(\kappa)}{4 + M(\kappa)}},$$

where

$$M(\kappa) = \frac{[4E(\kappa) - \pi^2/K(\kappa)][(2-\kappa^2)E(\kappa) - 2(1-\kappa^2)K(\kappa)]}{(2-\kappa^2)[E^2(\kappa) - (1-\kappa^2)K^2(\kappa)]}.$$

- quadratic pencil in the form

$$\lambda^2\psi + \lambda J\psi + \mathcal{H}\psi = 0. \tag{9}$$

Let $\mathcal{H} = \mathcal{H}^*$, so that $n(\mathcal{H}) < \infty$ and $dim(Ker(\mathcal{H})) < \infty$
index

$$k_r + k_c + k_- = n(H) - n((Id - J\mathcal{H}^{-1}J)|_{Ker(\mathcal{H})}). \tag{10}$$

- short pulse model

$$(u_t + (f(u))_x)_x = u. \tag{11}$$

traveling waves of the form $u(t, x) = \varphi(x - ct)$ profile equation for φ is

$$((\varphi^{p-1} - c)\varphi_\xi)_\xi = \varphi, \quad -L \leq \xi \leq L. \tag{12}$$

is not very nice object. Change of variables

$$\xi = \Xi(\eta) := \eta - \frac{\Psi(\eta)}{c}, \quad \varphi(\xi) = \Phi(\eta) = \Psi'(\eta). \tag{13}$$

which leads to the profile equation

$$-c^2\Phi'' - c\Phi + \Phi^p = 0. \tag{14}$$

- For $p = 2$ and $p = 3$ we construct the explicit expression for Φ,

$$\Phi = \Phi_0 + (\Phi_1 - \Phi_0) sn^2(\alpha x, \kappa), \tag{15}$$

and

$$\Phi = \Phi_2 sn(\alpha x, \kappa), \tag{16}$$

respectively.

- eigenvalue problem, take the ansatz $u(t,x) = \varphi(x - ct) + v(t, x - ct)$ by letting $v(t,\xi) = e^{\lambda t} w(\xi), w \in H^2[-L, L]$. This results in

$$(\lambda w + ((\varphi^{p-1} - c)w)_\xi)_\xi = w \quad -L < \xi < L. \tag{17}$$

Again we perform a change of variables

$$-c^2 Z_{\eta\eta} - cZ + \Phi^{p-1} Z = -\lambda c Z_\eta, Z \in L^2(-M, M). \tag{18}$$

Definition

Assume that there exists a one-to-one mapping $\Xi : (-M, M) \to (-L, L), M \in (0, \infty]$ satisfying (12). We say the the wave is spectrally unstable, if there exists $\mu : \Re\mu > 0$ and a function $Z \in H^2[-M, M] \cap C^2(-M, M)$, so that

$$\mathcal{L}[Z] := -c^2 Z_{\eta\eta} - cZ + \Phi^p Z = \mu Z'. \qquad (19)$$

Theorem.(S. Hakkaev, M. Stanislavova, A. Stefanov, SAM,2017) The waves described in (14) and (15)are spectrally stable for all wave speeds $c > 0$.

Thank you for Attention