The number of spanning trees in a graph

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on joint works with F. Petrov and P. Prozorov

The subject of the talk lies at the intersection of combinatorics, linear algebra and complex analysis.

Calculating the number of spanning trees in a graph goes back to the celebrated result of Kirchhoff (1847), who connected the number of trees and the Laplacian matrix.

However, as was shown by Cayley (1889), there is a bijection between the summands of

$$(x_1 + x_2 + \dots + x_n)^{n-2}$$

and the trees in a complete n-vertex graph K_n (in particular the number of trees is n^{n-2}). Here a tree T corresponds to a monomial $\prod x_i^{d_i(T)-1}$, where $d_i(T)$ is the degree of vertex i in T. In other words the vertex spanning enumerator polynomial P_{K_n} has a nice factorization.

We discuss all strengthening of these results that I know, in particular the following result by C., Petrov and Prozorov (2023). The statements (i)–(iii) are equivalent.

(i) A graph G is distance-hereditary (i.e. every induced connected graph preserves distances).

(ii) The polynomial P_G factors into linear terms.

(iii) The polynomial P_G is real stable (i.e., it does not vanish when substituting any variables from the open upper complex half-plane).

We finish with open questions.