BULGARIAN ACADEMY OF SCIENCE
NATIONAL MATHEMATICS COLLOQIUM
SOFIA, BULGARIA

The Squeezing Function

John Erik FORNÆSS

Norwegian University of Science and Technology

Wednesday September 4, 2019

Abstract

Abstract: In complex analysis the most important domain is the unit disc. In fact all domains (at least simply connected and bounded) are biholomorphic, i.e. analytically equivalent, to the disc.
In higher dimension, the natural analogue is the unit ball. But in higher dimension, the general domain is not biholomorphic to the ball.
A basic question is then how well a general domain can be approximated by the ball.
If we have a ball B_r of radius $r < 1$ contained in the unit ball B_1, then a domain U with $B_r \subset U \subset B_1$ is said to be squeezed between the two balls.
The larger we can choose r, the closer the domain U is to the ball.

Abstract

Abstract: In complex analysis the most important domain is the unit disc. In fact all domains (at least simply connected and bounded) are biholomorphic, i.e. analytically equivalent, to the disc.
In higher dimension, the natural analogue is the unit ball. But in higher dimension, the general domain is not biholomorphic to the ball.
A basic question is then how well a general domain can be approximated by the ball.
If we have a ball B_r of radius $r < 1$ contained in the unit ball B_1, then a domain U with $B_r \subset U \subset B_1$ is said to be squeezed between the two balls.
The larger we can choose r, the closer the domain U is to the ball.

Abstract

Abstract: In complex analysis the most important domain is the unit disc. In fact all domains (at least simply connected and bounded) are biholomorphic, i.e. analytically equivalent, to the disc.
In higher dimension, the natural analogue is the unit ball. But in higher dimension, the general domain is not biholomorphic to the ball.
A basic question is then how well a general domain can be approximated by the ball.
If we have a ball B_r of radius $r < 1$ contained in the unit ball B_1, then a domain U with $B_r \subset U \subset B_1$ is said to be squeezed between the two balls.
The larger we can choose r, the closer the domain U is to the ball.

Abstract

Abstract: In complex analysis the most important domain is the unit disc. In fact all domains (at least simply connected and bounded) are biholomorphic, i.e. analytically equivalent, to the disc.
In higher dimension, the natural analogue is the unit ball. But in higher dimension, the general domain is not biholomorphic to the ball.
A basic question is then how well a general domain can be approximated by the ball.
If we have a ball B_r of radius $r < 1$ contained in the unit ball B_1, then a domain U with $B_r \subset U \subset B_1$ is said to be squeezed between the two balls.
The larger we can choose r, the closer the domain U is to the ball.

Abstract

Abstract: In complex analysis the most important domain is the unit disc. In fact all domains (at least simply connected and bounded) are biholomorphic, i.e. analytically equivalent, to the disc.
In higher dimension, the natural analogue is the unit ball. But in higher dimension, the general domain is not biholomorphic to the ball.
A basic question is then how well a general domain can be approximated by the ball.
If we have a ball B_r of radius $r < 1$ contained in the unit ball B_1, then a domain U with $B_r \subset U \subset B_1$ is said to be squeezed between the two balls.
The larger we can choose r, the closer the domain U is to the ball.

Overview of Lecture

- Introduction, 2004, 2009
- 2012
- Applications to metrics
- Strongly convex domains
- Strongly pseudoconvex domains
- Weakly pseudoconvex domains

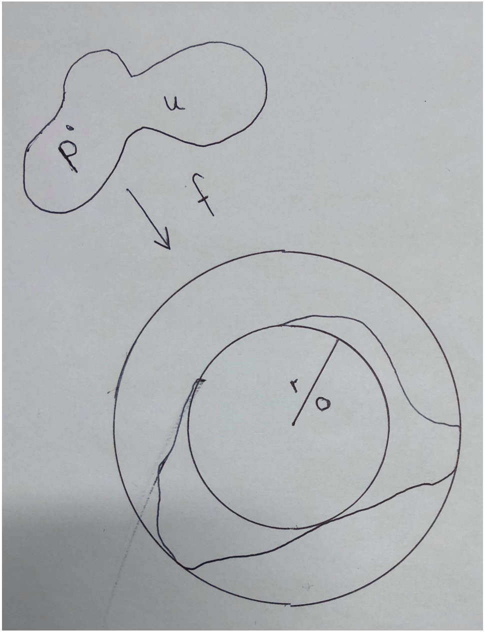

Figure: Squeezing

Liu-Sun-Yau, 2004

The topic of squeezing goes back to the paper Canonical Metrics on the Moduli Space of Riemann Surfaces I by Kefeng LIU, Xiaofeng SUN and Shing-Tung YAU in J. Differential Geometry, **68** (2004), 571-637. Definition 7.2:

Definition

A complex manifold X of dimension n is called **holomorphic homogeneous regular** (HHR) if there are positive constants $r < R$ such that for each point $p \in X$, there is a holomorphic map $f_p : X \to \mathbb{C}^n$ which satisfies

- $f_p(p) = 0$
- $f_p : X \to f_p(X)$ is a biholomorphism
- $B_r \subset f_p(X) \subset B_R$ where B_r and B_R are Euclidean balls with center 0 in $\mathbb{C}^n$.

Equivalence of metrics

Liu-Sun-Yau proved:

Theorem

Let X be an HHR manifold. Then the .Kobayashi metric, the Bergman metric and the Caratheodory metric on X are **equivalent***.*

Yeung

The next author to study this topic was Sai-Kee YEUNG, Geometry of domains with the uniform squeezing property, Advances of Math. **221** (2009), 547-569.

He renamed the HHR property as the **Uniform Squeezing Property**.

Examples by Yeung

Yeung found the following examples of domains with the uniform squeezing property:

- Bounded Homogeneous Domains
- Bounded Strongly Convex Domains
- Bounded domains which cover a compact Kähler manifold
- Teichmuller spaces $\mathcal{T}_{g,n}$ of hyperbolic Riemann surfaces of genus g with n punctures.

Theorem by Yeung

Theorem

Let M be a domain with the uniform squeezing property. Then the following conclusions are valid.

- *The Bergman metric is complete*
- *M is a pseudoconvex domain*
- *There exists a complete Kähler-Einstein metric on M.*

The squeezing function

In 2012, Fusheng DENG, Qi'an GUAN and Liyou ZHANG introduced the *squeezing function*:

Denote by $\mathbb{B}(r)$ the ball of radius $r > 0$ centered at the origin 0. Let U be a bounded domain in $\mathbb{C}^n$, and $p \in U$. For any holomorphic embedding $f : U \to \mathbb{B}(1)$, with $f(p) = 0$, set

$$s_{U,f}(p) := \sup\{r > 0 : \mathbb{B}(r) \subset f(U)\}.$$

Then, the squeezing function of U at p is defined as

$$s_U(p) := \sup_f \{s_{U,f}(p)\}.$$

They dropped the condition of uniformity, namely they **don't** require that there exists a positive constant $c_U > 0$ so that $s_U \geq c_U > 0$ for all $p \in U$.

The squeezing function

In 2012, Fusheng DENG, Qi'an GUAN and Liyou ZHANG introduced the *squeezing function*:
Denote by $\mathbb{B}(r)$ the ball of radius $r > 0$ centered at the origin 0. Let U be a bounded domain in $\mathbb{C}^n$, and $p \in U$. For any holomorphic embedding $f : U \to \mathbb{B}(1)$, with $f(p) = 0$, set

$$s_{U,f}(p) := \sup\{r > 0 : \mathbb{B}(r) \subset f(U)\}.$$

Then, the squeezing function of U at p is defined as

$$s_U(p) := \sup_f \{s_{U,f}(p)\}.$$

They dropped the condition of uniformity, namely they **don't** require that there exists a positive constant $c_U > 0$ so that $s_U \geq c_U > 0$ for all $p \in U$.

The squeezing function

In 2012, Fusheng DENG, Qi'an GUAN and Liyou ZHANG introduced the *squeezing function*:
Denote by $\mathbb{B}(r)$ the ball of radius $r > 0$ centered at the origin 0. Let U be a bounded domain in $\mathbb{C}^n$, and $p \in U$. For any holomorphic embedding $f : U \to \mathbb{B}(1)$, with $f(p) = 0$, set

$$s_{U,f}(p) := \sup\{r > 0 : \mathbb{B}(r) \subset f(U)\}.$$

Then, the squeezing function of U at p is defined as

$$s_U(p) := \sup_f \{s_{U,f}(p)\}.$$

They dropped the condition of uniformity, namely they **don't** require that there exists a positive constant $c_U > 0$ so that $s_U \geq c_U > 0$ for all $p \in U$.

Deng-Guan-Zhang 2012

They proved:

Theorem

The squeezing function is continuous.

The ball

Lemma

The squeezing function is a **biholomorphic invariant***: If $F : U \to V$ is a biholomorphic map, then for every $p \in U$, $s_V(F(p)) = s_U(p)$.*

Corollary

On the ball, $s \equiv 1$.

Proof.

We use the identity map $f(z) = z$ on the ball $\mathbb{B}(1)$. Then $f(0) = 0$ and we can choose $r = 1$. Hence $s_{\mathbb{B}(1)}(0) = 1$. Now the ball is homogeneous: For every $p \in \mathbb{B}(1)$ there exists a biholomorphic map $f : \mathbb{B}(1) \to \mathbb{B}(1)$ with $f(p) = 0$. Hence $s_{\mathbb{B}(1)}(p) = s_{\mathbb{B}(1)}(0) = 1$. □

The ball

Lemma

The squeezing function is a **biholomorphic invariant***: If $F : U \to V$ is a biholomorphic map, then for every $p \in U, s_V(F(p)) = s_U(p)$.*

Corollary

On the ball, $s \equiv 1$.

Proof.

We use the identity map $f(z) = z$ on the ball $\mathbb{B}(1)$. Then $f(0) = 0$ and we can choose $r = 1$. Hence $s_{\mathbb{B}(1)}(0) = 1$. Now the ball is homogeneous: For every $p \in \mathbb{B}(1)$ there exists a biholomorphic map $f : \mathbb{B}(1) \to \mathbb{B}(1)$ with $f(p) = 0$. Hence $s_{\mathbb{B}(1)}(p) = s_{\mathbb{B}(1)}(0) = 1$. □

The ball

Lemma

The squeezing function is a **biholomorphic invariant***: If $F : U \to V$ is a biholomorphic map, then for every $p \in U, s_V(F(p)) = s_U(p)$.*

Corollary

On the ball, $s \equiv 1$.

Proof.

We use the identity map $f(z) = z$ on the ball $\mathbb{B}(1)$. Then $f(0) = 0$ and we can choose $r = 1$. Hence $s_{\mathbb{B}(1)}(0) = 1$. Now the ball is homogeneous: For every $p \in \mathbb{B}(1)$ there exists a biholomorphic map $f : \mathbb{B}(1) \to \mathbb{B}(1)$ with $f(p) = 0$. Hence $s_{\mathbb{B}(1)}(p) = s_{\mathbb{B}(1)}(0) = 1$. □

The case $s = 1$

Conversely, Deng, Guan and Zhang showed that

Theorem

If $s(p) = 1$ for at least one point p, then the domain is (biholomorphic to) the ball.

Proof.

Assume that there is a point $p \in U$ so that $s(p) = 1$. Then there exists a sequence of 1-1 holomorphic maps $f_n : U \to \mathbb{B}(1)$ so that $f_n(p) = 0$ and $f(U)$ contains the ball $\mathbb{B}(1 - 1/n)$. We can apply the Schwartz lemma to show that the derivatives are bounded at p. The same can be done for the inverse. Hence the Jabobians are uniformly bounded above and below at p. By Montels theorem we can take converging subsequences with limits f. The inverses also converge. We get that the limit map f is a biholomorphic map from U onto $\mathbb{B}(1)$. □

In particular, if $s(p) = 1$ for some point, then $s \equiv 1$.

The case $s = 1$

Conversely, Deng, Guan and Zhang showed that

Theorem

If $s(p) = 1$ for at least one point p, then the domain is (biholomorphic to) the ball.

Proof.

Assume that there is a point $p \in U$ so that $s(p) = 1$. Then there exists a sequence of 1-1 holomorphic maps $f_n : U \to \mathbb{B}(1)$ so that $f_n(p) = 0$ and $f(U)$ contains the ball $\mathbb{B}(1 - 1/n)$. We can apply the Schwartz lemma to show that the derivatives are bounded at p. The same can be done for the inverse. Hence the Jabobians are uniformly bounded above and below at p. By Montels theorem we can take converging subsequences with limits f. The inverses also converge. We get that the limit map f is a biholomorphic map from U onto $\mathbb{B}(1)$. □

In particular, if $s(p) = 1$ for some point, then $s \equiv 1$.

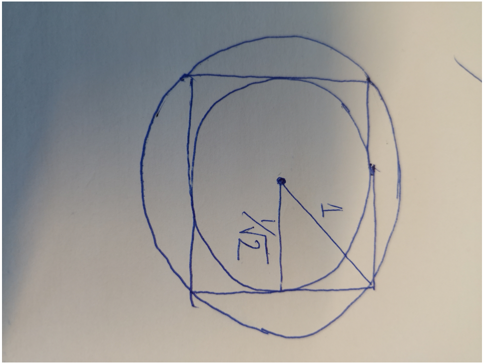

Figure: Bidisc

The bidisc

Example

Let $\Delta^2(1,1)$ be the bidisc $\Delta(1) \times \Delta(1)$. Then the squeezing function is $s \equiv \frac{1}{\sqrt{2}}$.

Proof.

We consider the squeezing function at the origin. We can let $f : \Delta^2(1) \to \mathbb{B}(1)$ be given by $f(z,w) = (\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}})$. This maps the points $(z,w); |z| = |w| = 1$ to the boundary of the ball. Inside the image we have room for the ball $\mathbb{B}(\frac{1}{\sqrt{2}})$. So the squeezing function is at least $s = \frac{1}{\sqrt{2}}$. By a theorem of Herb ALEXANDER, this cannot be improved, Extremal holomorphic embeddings between the ball and the polydisc, Proc. Amer. Math. Soc. **68** (1978), 200-202. □

Note that this result is older than the squeezing function.

The bidisc

Example

Let $\Delta^2(1,1)$ be the bidisc $\Delta(1) \times \Delta(1)$. Then the squeezing function is $s \equiv \frac{1}{\sqrt{2}}$.

Proof.

We consider the squeezing function at the origin. We can let $f : \Delta^2(1) \to \mathbb{B}(1)$ be given by $f(z,w) = (\frac{z}{\sqrt{2}}, \frac{w}{\sqrt{2}})$. This maps the points $(z,w); |z| = |w| = 1$ to the boundary of the ball. Inside the image we have room for the ball $\mathbb{B}(\frac{1}{\sqrt{2}})$. So the squeezing function is at least $s = \frac{1}{\sqrt{2}}$. By a theorem of Herb ALEXANDER, this cannot be improved, Extremal holomorphic embeddings between the ball and the polydisc, Proc. Amer. Math. Soc. **68** (1978), 200-202. □

Note that this result is older than the squeezing function.

The annulus

CLOSED! QUESTION:

Problem

Find an explicit formula for the squeezing function on an annulus $A = \{a < |z| < b\}$.

Very recently, the formula has been found by Tuen Wai NG, Chui Chak TANG and Jonathan TSAI (preprint 2019).

Monotone invariants

Let $p \in U$. We say that an invariant $I(p, U)$ is monotonically decreasing if $I(p, U) \geq I(p, V)$ for any larger domain V.

A common fact about invariants is that they have explicit formulas for a ball.

Metatheorem

Monotone invariants can be estimated.

Proof.

Let $I(p, U)$ be a monotone invariant. Using the squeezing function there is a map : $U \to \mathbb{B}(1)$ with $f(p) = 0$ and $\mathbb{B}(s_p(U)) \subset f(U)$. Hence $I(0, \mathbb{B}(s_p(U)) \geq I(p, U) \geq I(0, \mathbb{B}(1))$. □

Monotone invariants

Let $p \in U$. We say that an invariant $I(p, U)$ is monotonically decreasing if $I(p, U) \geq I(p, V)$ for any larger domain V.
A common fact about invariants is that they have explicit formulas for a ball.

Metatheorem

Monotone invariants can be estimated.

Proof.

Let $I(p, U)$ be a monotone invariant. Using the squeezing function there is a map : $U \to \mathbb{B}(1)$ with $f(p) = 0$ and $\mathbb{B}(s_p(U)) \subset f(U)$. Hence $I(0, \mathbb{B}(s_p(U))) \geq I(p, U) \geq I(0, \mathbb{B}(1))$. □

Monotone invariants

Let $p \in U$. We say that an invariant $I(p, U)$ is monotonically decreasing if $I(p, U) \geq I(p, V)$ for any larger domain V.
A common fact about invariants is that they have explicit formulas for a ball.

Metatheorem

Monotone invariants can be estimated.

Proof.

Let $I(p, U)$ be a monotone invariant. Using the squeezing function there is a map : $U \to \mathbb{B}(1)$ with $f(p) = 0$ and $\mathbb{B}(s_p(U)) \subset f(U)$. Hence $I(0, \mathbb{B}(s_p(U)) \geq I(p, U) \geq I(0, \mathbb{B}(1))$. □

Monotone invariants

Let $p \in U$. We say that an invariant $I(p, U)$ is monotonically decreasing if $I(p, U) \geq I(p, V)$ for any larger domain V.
A common fact about invariants is that they have explicit formulas for a ball.

Metatheorem

Monotone invariants can be estimated.

Proof.

Let $I(p, U)$ be a monotone invariant. Using the squeezing function there is a map : $U \to \mathbb{B}(1)$ with $f(p) = 0$ and $\mathbb{B}(s_p(U)) \subset f(U)$. Hence $I(0, \mathbb{B}(s_p(U)) \geq I(p, U) \geq I(0, \mathbb{B}(1))$. □

Bad properties of the squeezing function

Lemma

The squeezing function is an invariant, but not a monotone invariant.

Proof.

See picture of ball in bidisc in ball, with squeezing functions $1, \frac{1}{\sqrt{2}}, 1$. □

Theorem

(F-Nikolay SHCHERBINA, 2018) The squeezing function is not always plurisubharmonic, i.e. subharmonic on complex lines.

The Caratheodory metric

Let $U \subset \mathbb{C}^n$ be a domain, let $p \in U$ and let v be a tangent vector at p. We consider holomorphic maps $f : U \to \Delta$ with $f(p) = 0$. Then we want to maximize the derivative of f at p. This defines the Caratheodory metric, $d_C^U(p, v)$.

Definition

$d_C^U(p, v) = \sup\{|f'(p)(v)|; f : U \to \Delta, f(p) = 0\}.$

The unit disc

The Caratheodory metric on the unit disc coincides with the Poincare metric.

Definition

The Poincare metric at a point $z \in \Delta$ for the vector v is

$$d_P^{\Delta}(z, v) = \frac{|v|}{1 - |z|^2}.$$

Theorem

The Caratheodory metric is a monotone invariant.

Proof.

Let $U \subset V$ and $p \in U$. Suppose that $f : V \to \Delta$ is holomorphic and $f(p) = 0$. Let A be any number strictly less than $d_C^V(p, v)$. Then we can choose f so that $|f'(p)(v)| > A$. Let g be the restriction of f to U. Then still $|f'(p)(v)| > A$. Hence $d_C^U(p, v) > A$. Hence $d_C^U(p, v) \geq d_C^V(p, v)$. □

The Kobayashi metric

The Kobayashi metric measures the largest possible discs that fit into a domain.

We define the Kobayashi metric. It is somehow dual to the Caratheodory metric. Let $p \in U$ and let v be a tangent vector at p. We consider the family of all holomorphic maps $f : \Delta \to U, f(0) = p$ and $f'(0) = \lambda v$. We maximize the constant λ.

Theorem

The Kobayashi metric is a monotone invariant. On the disc and ball it coincides with the Caratheodory metric.

In general the two metrics are different. For example, they are different on an annulus.

The Kobayashi metric

The Kobayashi metric measures the largest possible discs that fit into a domain.

We define the Kobayashi metric. It is somehow dual to the Caratheodory metric. Let $p \in U$ and let v be a tangent vector at p. We consider the family of all holomorphic maps $f : \Delta \to U, f(0) = p$ and $f'(0) = \lambda v$. We maximize the constant λ.

Theorem

The Kobayashi metric is a monotone invariant. On the disc and ball it coincides with the Caratheodory metric.

In general the two metrics are different. For example, they are different on an annulus.

The Kobayashi metric

The Kobayashi metric measures the largest possible discs that fit into a domain.

We define the Kobayashi metric. It is somehow dual to the Caratheodory metric. Let $p \in U$ and let v be a tangent vector at p. We consider the family of all holomorphic maps $f : \Delta \to U, f(0) = p$ and $f'(0) = \lambda v$. We maximize the constant λ.

Theorem

The Kobayashi metric is a monotone invariant. On the disc and ball it coincides with the Caratheodory metric.

In general the two metrics are different. For example, they are different on an annulus.

The Kobayashi metric

The Kobayashi metric measures the largest possible discs that fit into a domain.
We define the Kobayashi metric. It is somehow dual to the Caratheodory metric. Let $p \in U$ and let v be a tangent vector at p. We consider the family of all holomorphic maps $f : \Delta \to U, f(0) = p$ and $f'(0) = \lambda v$. We maximize the constant λ.

Theorem

The Kobayashi metric is a monotone invariant. On the disc and ball it coincides with the Caratheodory metric.

In general the two metrics are different. For example, they are different on an annulus.

Theorem

The Kobayashi metric is a monotone invariant.

The Bergman Metric

The Bergman metric is not quite monotone. But it is the quotient of two monotone invariants.

We define the Bergman metric B on U. Let $H^2(U)$ denote the collection of functions $f : U \to \mathbb{C}$ which are holomorphic and in $L^2, \|f\|^2 = \int_U |f|^2 dV < \infty$.

We have two monotone invariants, A, C.

- $A(z) := \sup\{|f(z)|; \|f\| = 1\}$
- $C(z, v) := \sup\{|f'(z)(v)|, \|f\| = 1\}$
- $B(z, v) : \frac{C}{A}$.

The Bergman Metric

The Bergman metric is not quite monotone. But it is the quotient of two monotone invariants.

We define the Bergman metric B on U. Let $H^2(U)$ denote the collection of functions $f : U \to \mathbb{C}$ which are holomorphic and in $L^2, \|f\|^2 = \int_U |f|^2 dV < \infty$.

We have two monotone invariants, A, C.

- $A(z) := \sup\{|f(z)|; \|f\| = 1\}$
- $C(z, v) := \sup\{|f'(z)(v)|, \|f\| = 1\}$
- $B(z, v) : \frac{C}{A}$.

The Bergman Metric

The Bergman metric is not quite monotone. But it is the quotient of two monotone invariants.

We define the Bergman metric B on U. Let $H^2(U)$ denote the collection of functions $f : U \to \mathbb{C}$ which are holomorphic and in $L^2, \|f\|^2 = \int_U |f|^2 dV < \infty$.

We have two monotone invariants, A, C.

- $A(z) := \sup\{|f(z)|; \|f\| = 1\}$
- $C(z, v) := \sup\{|f'(z)(v)|, \|f\| = 1\}$
- $B(z, v) : \frac{C}{A}$.

Questions about the sqeezing function

There are two properties that are **particularly desirable** for the squeezing function. Given a domain U.

Desirable Property

Does the squeezing function $s(p)$ approach 1 as $p \to \partial U$?

The next property is called, as mentioned earlier, uniformly squeezing, (Yeung).

Desirable Property

Is there a constant $c_U > 0$ so that $s(p) > c_U$ for all $p \in U$?

Yeung's theorems

First case is when the bounded domain U is strongly convex with $\mathcal{C}^2$ boundary.
In 2009 Yeung proved the following two theorems:

Theorem

If U is a bounded strongly convex domain in $\mathbb{C}^n$, then $s(p) \to 1$ when $p \to \partial U$.

Theorem

If U is a bounded strongly convex domain in $\mathbb{C}^n$, then U is uniformly squeezing.

Kangtae KIM and Liyou ZHANG proved the following theorem. On the uniform squeezing property of bounded convex domains in $\mathbb{C}^n$, Pacific J. of Math. **282** (2016), 341-358. The key improvement from Yeungs theorem is that there is no condition of smoothness on the convex domain.

Theorem

Let U be a **bounded convex** *domain in $\mathbb{C}^n$. Then U is uniformly squeezing.*

Non convex domains

If U is strongly convex and $F : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphic map, then $F(U)$ is not necessarily convex. So convexity is not a biholomorphic invariant. We say that $F(U)$ is strongly **PSEUDOconvex**.

Definition

Let U be a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary. Then U is said to be strongly pseudoconvex if for every $p \in \partial U$ there exists a neighborhood $U(p)$ and a local biholomorphic map F so that $F(p)$ is a strongly convex boundary point of $F(U)$.

Example

An annulus $A = \{a < |z| < b\}$. For every point p with $|p| = b$ we can use $F = Id$ since the outer boundary is already convex. On the other hand, for points p with $|p| = a$, we use $F(z) = 1/z$. Then the inner boundary becomes the outer boundary of some annulus, so convex as well.

Non convex domains

If U is strongly convex and $F : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphic map, then $F(U)$ is not necessarily convex. So convexity is not a biholomorphic invariant. We say that $F(U)$ is strongly **PSEUDOconvex**.

Definition

Let U be a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary. Then U is said to be strongly pseudoconvex if for every $p \in \partial U$ there exists a neighborhood $U(p)$ and a local biholomorphic map F so that $F(p)$ is a strongly convex boundary point of $F(U)$.

Example

An annulus $A = \{a < |z| < b\}$. For every point p with $|p| = b$ we can use $F = Id$ since the outer boundary is already convex. On the other hand, for points p with $|p| = a$, we use $F(z) = 1/z$. Then the inner boundary becomes the outer boundary of some annulus, so convex as well.

Non convex domains

If U is strongly convex and $F : \mathbb{C}^n \to \mathbb{C}^n$ is a biholomorphic map, then $F(U)$ is not necessarily convex. So convexity is not a biholomorphic invariant. We say that $F(U)$ is strongly **PSEUDOconvex.**

Definition

Let U be a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary. Then U is said to be strongly pseudoconvex if for every $p \in \partial U$ there exists a neighborhood $U(p)$ and a local biholomorphic map F so that $F(p)$ is a strongly convex boundary point of $F(U)$.

Example

An annulus $A = \{a < |z| < b\}$. For every point p with $|p| = b$ we can use $F = Id$ since the outer boundary is already convex. On the other hand, for points p with $|p| = a$, we use $F(z) = 1/z$. Then the inner boundary becomes the outer boundary of some annulus, so convex as well.

Fusheng DENG, Qi'an GUAN and Liyou ZHANG proved in Properties of squeezing functions and global transformations of bounded domains, Trans. Amer. Math. Soc, **368** (2016), 2679-2696

Theorem

Let U be strongly pseudoconvex with $\mathcal{C}^2$ boundary. Then the squeezing function approaches 1 at the boundary.

More precise estimates

John Erik FORNÆSS-Erlend Fornæss WOLD, An estimate for the squeezing function and invariant metrics, Complex Analysis and Geometry, KSCV 10. Springer Proceedings in Mathematics and Statistics, **144** (2015), 135-147. Let d denote the distance function to the boundary.

Theorem

If a bounded strongly pseudoconvex domain has $\mathcal{C}^3$ boundary, then the squeezing function is larger than $1 - C\sqrt{d}$. If the boundary is $\mathcal{C}^4$, then this can be replaced by $1 - Cd$.

It is **OPEN** if these estimates are sharp. It is **OPEN** if there is an estimate in the $\mathcal{C}^2$ case.

Exposing points

The key tool in the proof is that of exposing of points:
Klas DIEDERICH, J. E. FORNÆSS, E. F. WOLD, Exposing points on the boundary of a strictly pseudoconvex domain or a locally convexifiable domain of finite 1-type, J. Geom. Anal **24** (2014), 2124-2134,

Nikolov-Trybula

Estimates for the squeezing function near strictly pseudoconvex boundary points with applications, ArXiv: 1808.07892.
Estimates on rate of growth towards 1 in strictly pseudoconvex domains: Nikolai NIKOLOV- Maria TRYBULA. This paper improved the previous one by extending to domains with $\mathcal{C}^{2,\epsilon}$, $\mathcal{C}^{3,\epsilon}$ respectively.

Theorem

$$s(p) \geq 1 - Cd^{\frac{\epsilon}{2}}, 1 - Cd^{\frac{1+\epsilon}{2}}$$

respectively.

When $\epsilon \to 0$ you get close to the $\mathcal{C}^2$-case, which as I mentioned is still open.

The converse of Yeungs theorem on $s(p) \to 1$

A natural question was whether the converse of Yeungs Theorem holds: If U is a bounded convex domain with smooth boundary, and the squeezing function goes to 1 at the boundary, must U be STRONGLY convex?

The answer depends on the meaning of smooth.

Andrew ZIMMER proved in A Gap Theorem for the Complex Geometry of convex domains, Trans. Amer. Math. Soc. **370** (2018) 7489-7509.

Theorem

There exists a function $\epsilon(n) > 0$ only depending on the dimension: Let U be a bounded convex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary. Suppose the squeezing function $s_U(p) > 1 - \epsilon(n)$ outside some compact set. Then U is strongly convex.

The converse of Yeungs theorem on $s(p) \to 1$

A natural question was whether the converse of Yeungs Theorem holds: If U is a bounded convex domain with smooth boundary, and the squeezing function goes to 1 at the boundary, must U be STRONGLY convex?

The answer depends on the meaning of smooth.

Andrew ZIMMER proved in A Gap Theorem for the Complex Geometry of convex domains, Trans. Amer. Math. Soc. **370** (2018) 7489-7509.

Theorem

There exists a function $\epsilon(n) > 0$ only depending on the dimension: Let U be a bounded convex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary. Suppose the squeezing function $s_U(p) > 1 - \epsilon(n)$ outside some compact set. Then U is strongly convex.

The converse of Yeungs theorem on $s(p) \to 1$

A natural question was whether the converse of Yeungs Theorem holds: If U is a bounded convex domain with smooth boundary, and the squeezing function goes to 1 at the boundary, must U be STRONGLY convex?
The answer depends on the meaning of smooth.
Andrew ZIMMER proved in A Gap Theorem for the Complex Geometry of convex domains, Trans. Amer. Math. Soc. **370** (2018) 7489-7509.

Theorem

There exists a function $\epsilon(n) > 0$ only depending on the dimension: Let U be a bounded convex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary. Suppose the squeezing function $s_U(p) > 1 - \epsilon(n)$ outside some compact set. Then U is strongly convex.

The Gap theorem of Zimmer has as corollary:

Corollary

Let U be a bounded convex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary. Suppose that the squeezing function $s_U(p) \to 1$ when $p \to \partial U$. Then U is strongly convex.

Counterexample

On the other hand, J. E. Fornæss- E. F. Wold 2018 showed: A non-strictly pseudoconvex domain for which the squeezing function tends to 1 towards the boundary. Pacific J. of Math **297** (2018), 79-86.

Theorem

There exists a bounded convex domain U in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary for which

- *The squeezing function $s(p) \to 1$ when $p \to \partial U$.*
- *The domain is NOT strongly convex.*

The following is an **OPEN PROBLEM**:

Question

If U is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^{2,\epsilon}$ boundary and $s(p) \to 1$ when $p \to \partial U$, is U strongly pseudoconvex?

Counterexample

On the other hand, J. E. Fornæss- E. F. Wold 2018 showed: A non-strictly pseudoconvex domain for which the squeezing function tends to 1 towards the boundary. Pacific J. of Math **297** (2018), 79-86.

Theorem

There exists a bounded convex domain U in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary for which

- *The squeezing function $s(p) \to 1$ when $p \to \partial U$.*
- *The domain is NOT strongly convex.*

The following is an **OPEN PROBLEM**:

Question

If U is a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^{2,\epsilon}$ boundary and $s(p) \to 1$ when $p \to \partial U$, is U strongly pseudoconvex?

Klas DIEDERICH-J. E. FORNÆSS-Erlend Fornæss WOLD, for the case when the squeezing function goes a little bit faster to 1. A characterization of the unit ball in $\mathbb{C}^n$, Int. J. Math. **27** (2016).

Theorem

Let U be a bounded domain in $\mathbb{C}^n$ with $\mathcal{C}^2$ boundary. Suppose that there exists a sequence of points $p_i \to \partial U$ and a sequence of numbers $\epsilon_i \searrow 0$ so that $s(p_i) \geq 1 - \epsilon_i d(p_i, \partial U)$. Then U is (biholomorphic to) the ball.

Beyond strongly pseudoconvex domains

A ball $\mathbb{B}(1)$ can be exhausted by smaller balls, $\mathbb{B}(r), r < 1, r \to 1$. A generalization is that of weakly pseudoconvex domains:

Definition

A domain $U \subset \mathbb{C}^n$ is called (**weakly**) pseudoconvex if it can be exhausted by strongly pseudoconvex domains.

Example

All convex domains and all strongly pseudoconvex domains are weakly pseudoconvex.

The Kohn-Nirenberg domain

The Kohn-Nirenberg domain is a pseudoconvex domain in $\mathbb{C}^2$ which can not be made convex in local coordinates.
The following is an **OPEN PROBLEM**

Question

Let U be a bounded pseudoconvex domain in $\mathbb{C}^2$ with real analytic boundary. Is U uniformly squeezing?

Answer to the question in dimension 3 is NO in general.
J. E. FORNÆSS-Feng RONG, Estimate of the squeezing function for a class of bounded domains, Math. Ann. **371** (2018), 1087-1094.
Gregory HERBORT has a similar result although he did not investigate the squeezing function.

Theorem

There exists a pseudoconvex bounded domain U in $\mathcal{C}^n, n \geq 3$, *with* $\mathcal{C}^\omega$ *boundary which is* **not** *uniformly squeezing.*

One Dimension

In one variable, all domains are pseudoconvex. What can we say about the squeezing function?

Example

The punctured unit disc $\Delta^* = \Delta \setminus \{0\}$. The squeezing function takes on all values in the open interval $< 0, 1 >$.

Proof.

One can show that the squeezing function goes to 1 on the unit circle. Also one can show that the squeezing function goes to 0 at the origin. By the continuity of the squeezing function proved by Deng-Guan-Zhang, all values are taken. □

Cantor sets

One can also ask about the possible behaviour of the squeezing function in the complement of a Cantor set, i.e. a compact perfect set which is completely disconnected.
This was investigated by Leandro AROSIO, J. E. FORNÆSS, N. SHCHERBINA and E. F. WOLD, Squeezing functions and Cantor sets. Accepted for Publication in Annali della Scuola Normale Superiore di Pisa, Classe di Scienze. (arXiv 1710.10305)

Theorem

A) There are Cantor sets where the squeezing function approaches 1 *when you approach the Cantor set.*
B) There are Cantor sets where the squeezing function takes on all values in $< 0, 1 >$ *arbitrarily close to any point.*
C) For $f(z) = z^2 + c$, *where* c *is outside the Mandelbrot set, so the Julia set is a Cantor set, the squeezing function cannot even be defined.*

A new beginning

The polydisc Δ_r is an alternative higher dimensional version of the unit disc.

Definition

If we have a polydisc Δ_r of radius $r < 1$ contained in the unit polydisc Δ_1, then a domain U with $\Delta_r \subset U \subset \Delta_1$ is said to be squeezed between the two polydiscs.

The larger we can choose r, the closer the domain U is to the polydisc. The following is an **OPEN PROBLEM**

Question

What is the analogous theory in the polydisc case?

A new beginning

The polydisc Δ_r is an alternative higher dimensional version of the unit disc.

Definition

If we have a polydisc Δ_r of radius $r < 1$ contained in the unit polydisc Δ_1, then a domain U with $\Delta_r \subset U \subset \Delta_1$ is said to be squeezed between the two polydiscs.

The larger we can choose r, the closer the domain U is to the polydisc.
The following is an **OPEN PROBLEM**

Question

What is the analogous theory in the polydisc case?

A new beginning

The polydisc Δ_r is an alternative higher dimensional version of the unit disc.

Definition

If we have a polydisc Δ_r of radius $r < 1$ contained in the unit polydisc Δ_1, then a domain U with $\Delta_r \subset U \subset \Delta_1$ is said to be squeezed between the two polydiscs.

The larger we can choose r, the closer the domain U is to the polydisc. The following is an **OPEN PROBLEM**

Question

What is the analogous theory in the polydisc case?

Thank you for listening!