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A growth-fragmentation found in the cone excursions of Brownian motion

- Take a Brownian motion in the 2D plane, consider two types of ‘cone excursion’:

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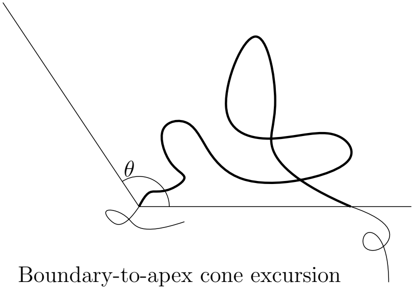
  - Boundary-to-apex cone excursions



Boundary-to-apex cone excursion

Cone excursions

- Take a Brownian motion in the 2D plane, consider two types of ‘cone excursion’:
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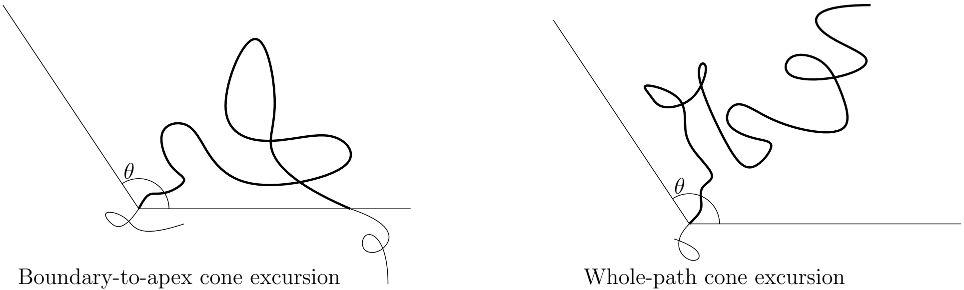
  - Whole-path cone excursions



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Whole-path cone excursion

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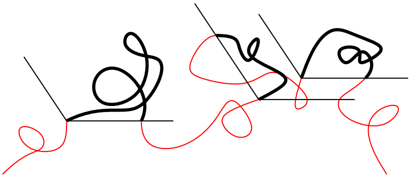
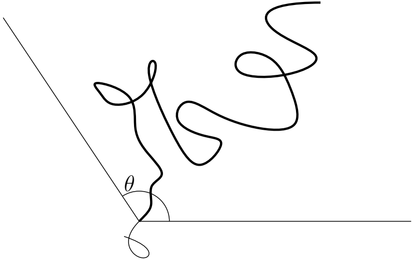
- ‘Cone-free times’ (between boundary-to-apex excursions) form a regenerative set



Boundary-to-apex cone-free times and **cone excursions**



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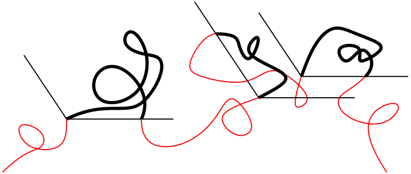

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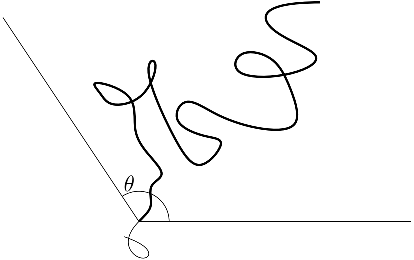

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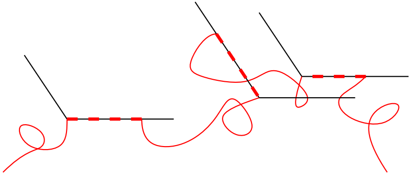
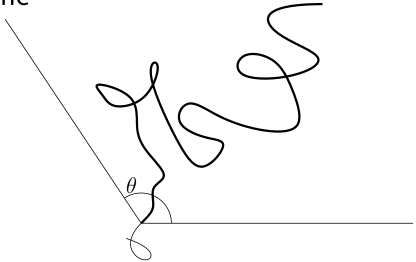
- The path is cut into cone excursions between said times
- Write $\tau^{>}$ for boundary-to-apex inverse local time



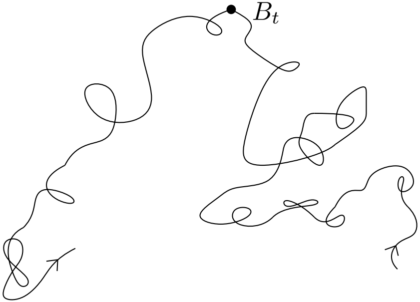
The path at boundary-to-apex cone-free times (with **jumps**)



Whole-path cone excursion

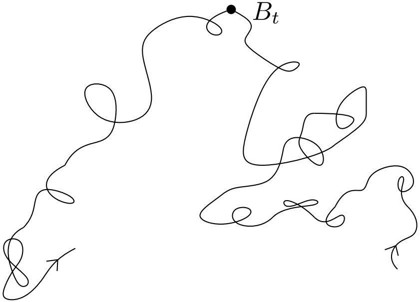
Targeting a time

- Take Brownian path $(B_s\colon 0 \le s \le \zeta)$ and single out point B_t



Targeting a time

- Take Brownian path $(B_s : 0 \le s \le \zeta)$ and single out point B_t
- An excursion targeting t is the largest cone excursion in $(B_s : u \le s \le \zeta)$



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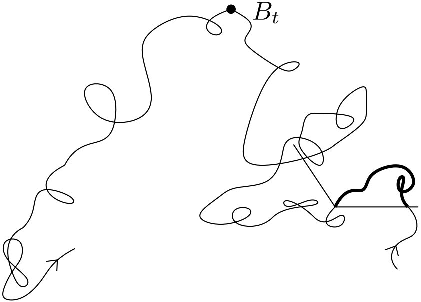

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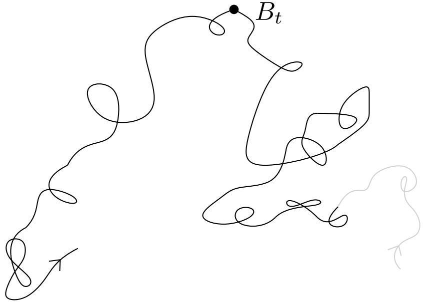

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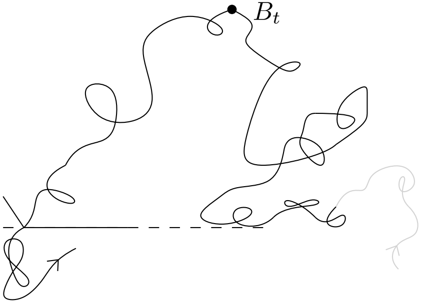

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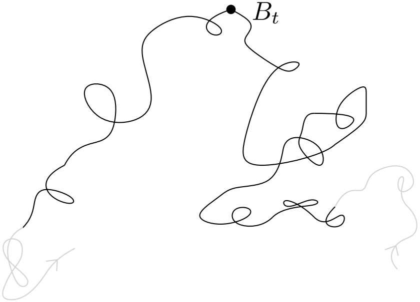

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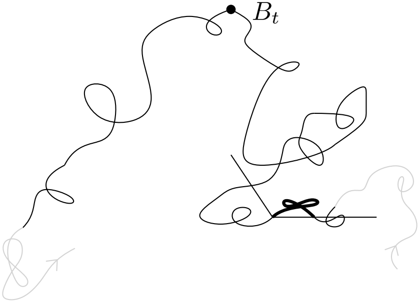

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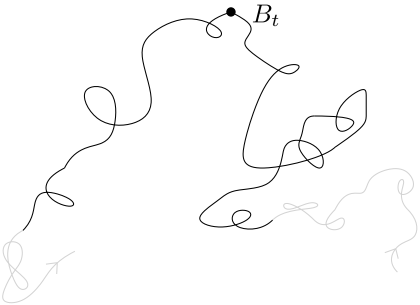

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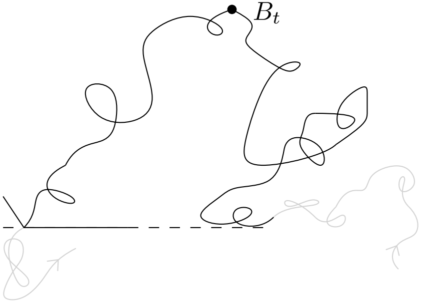

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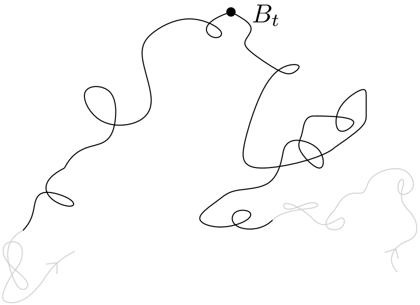

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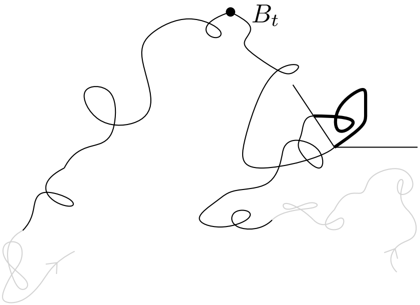

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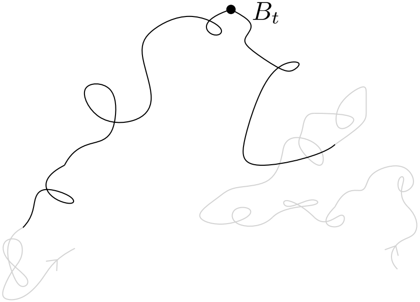

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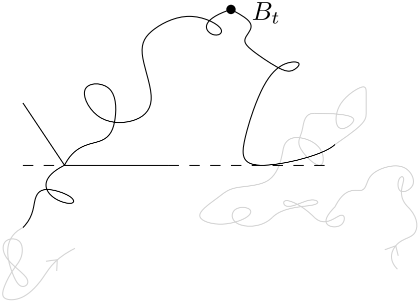

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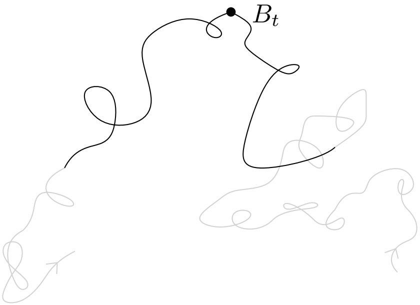
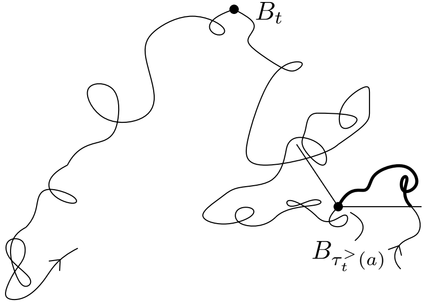


Targeting a time (more precisely)

- Fix $t > 0$, write $\tau_t^>$ for boundary-to-apex inverse local time of $(B_s : s \le t)$



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- Fix $t > 0$, write $\tau_t^>$ for boundary-to-apex inverse local time of $(B_s : s \leq t)$
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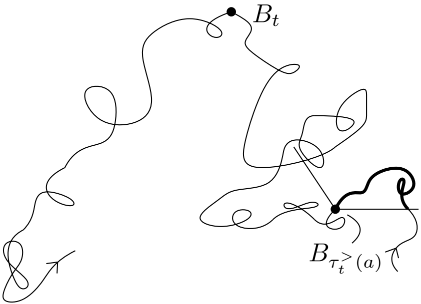

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- Fix $t > 0$, write $\tau_t^>$ for boundary-to-apex inverse local time of $(B_s : s \le t)$
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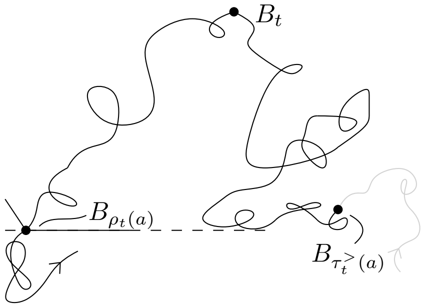

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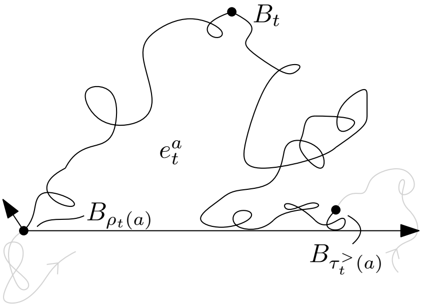

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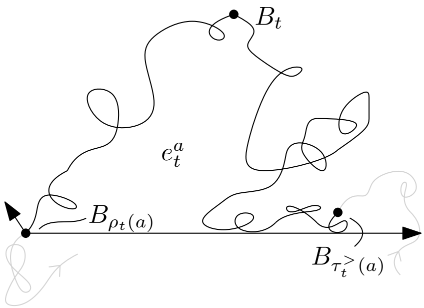

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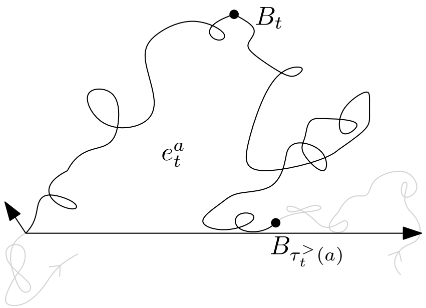

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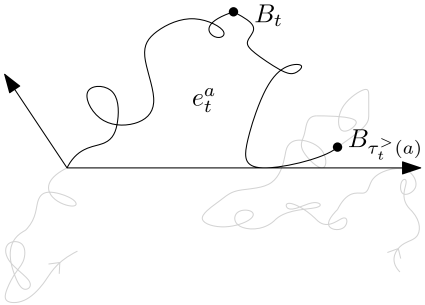
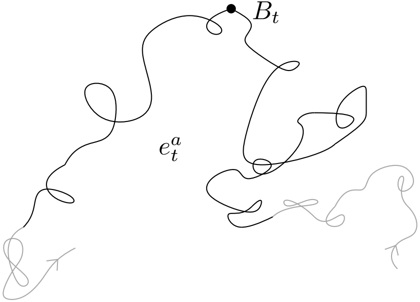


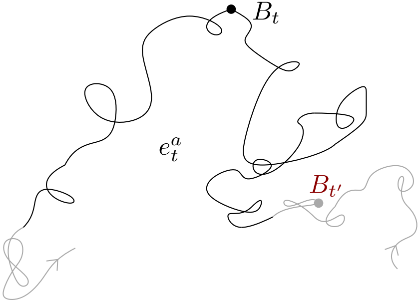
Targeting multiple times

- We have the excursion targeting t



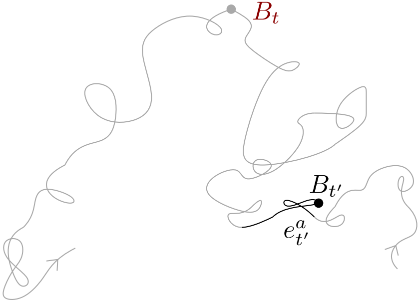
Targeting multiple times

- We have the excursion targeting t
- What if we target some other t'?



Targeting multiple times

- We have the excursion targeting t
- What if we target some other t'?
- Every piece cut out while targeting t is one which is included in the excursion targeting some other t'



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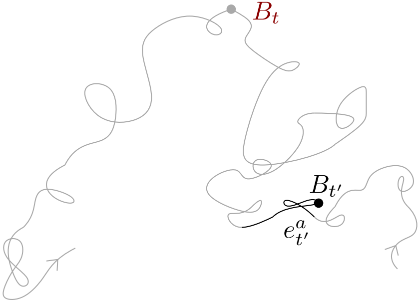
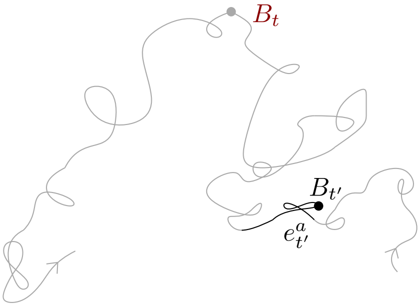


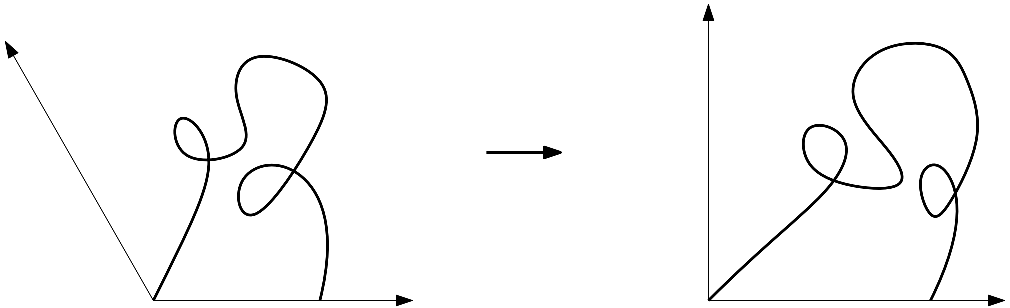
Targeting multiple times

- We have the excursion targeting t
- What if we target some other t'?
- Every piece cut out while targeting t is one which is included in the excursion targeting some other t'
- Consider targeting every time simultaneously
- There is some kind of branching process for us to capture



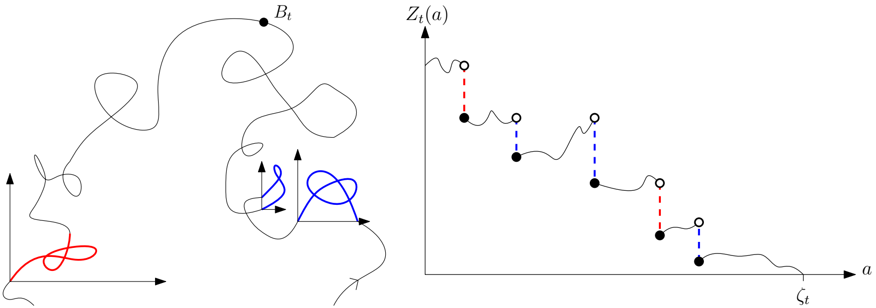
Summarising the path targeting t

- Map the cone with apex angle ϑ to the positive quadrant $\mathbb{R}^2_+$; standard Brownian motion becomes correlated



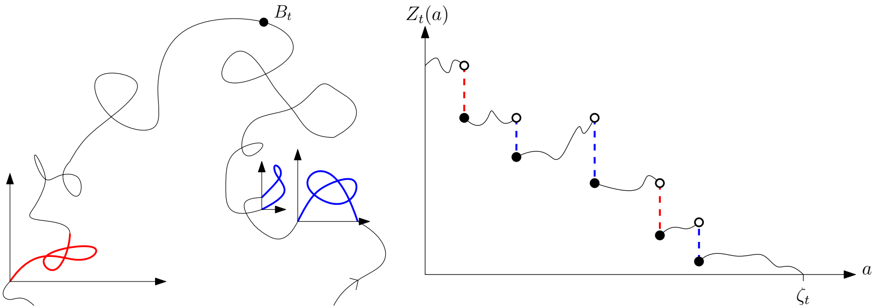
Summarising the path targeting t

- Map the cone with apex angle ϑ to the positive quadrant $\mathbb{R}_+^2$; standard Brownian motion becomes correlated
- The initial displacement of the excursion targeting t at local time a: $e_t^a(0) \in \mathbb{R}_+^2$



Summarising the path targeting t

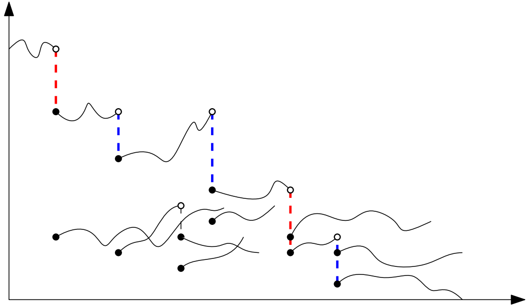
- Map the cone with apex angle ϑ to the positive quadrant $\mathbb{R}^2_+$; standard Brownian motion becomes correlated
- The initial displacement of the excursion targeting t at local time a: $e^a_t(0) \in \mathbb{R}^2_+$
- In the case $\vartheta = 2\pi/3$ look at its ℓ^1-norm: $Z_t(a) = \|e^a_t(0)\|_1$



Growth-fragmentations

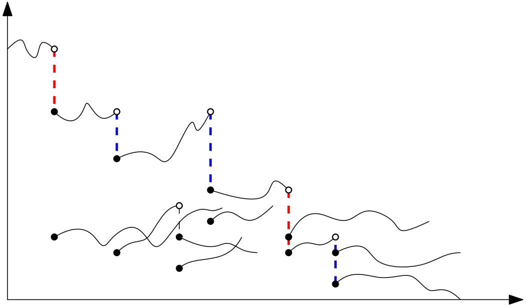
A growth-fragmentation is:

- a system of particles (excursions targeting each time t)...



A growth-fragmentation is:

- a system of particles (excursions targeting each time t)...
- ...each summarised by a trait (ℓ^1-norm of its initial displacement)...



Growth-fragmentations

A growth-fragmentation is:

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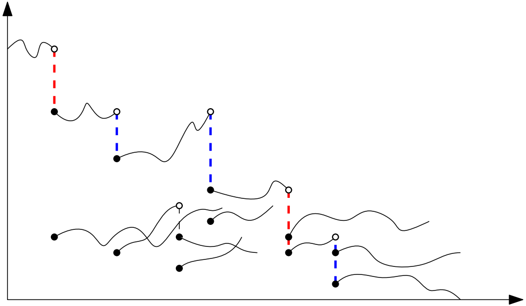

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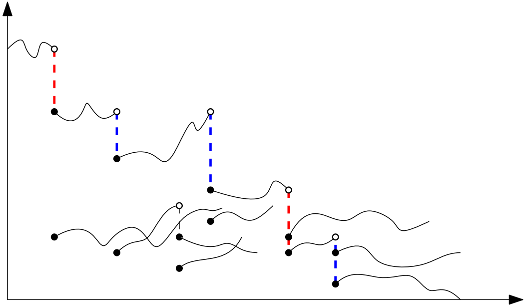

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A growth-fragmentation is:

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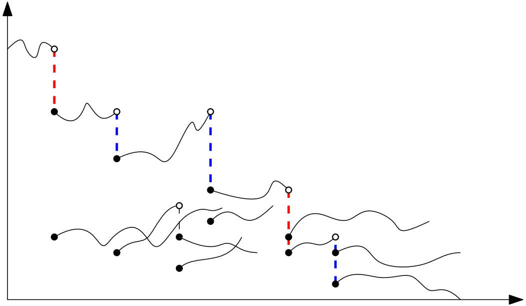
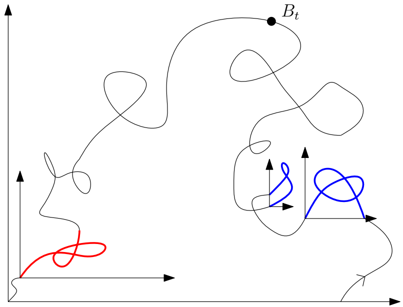
- ...whose path only jumps down...
- ...and each jump of which is accompanied by the birth of another particle, conditionally independent given initial trait value

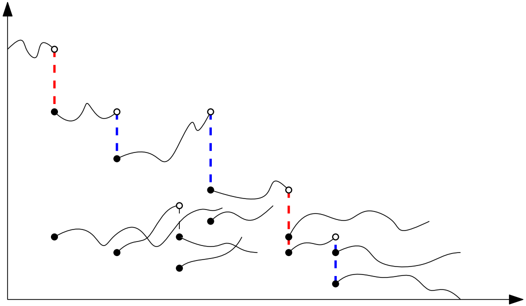


We do all this starting with B given by a boundary-to-apex excursion with fixed initial value $B_0 = z \in \partial\mathbb{R}^2_+$.

Theorem (Da Silva-Powell-W, vague version)

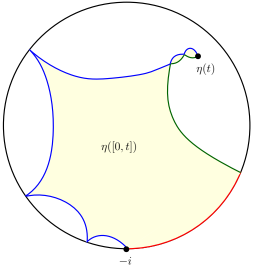
The particles t with traits $(Z_t(a)\colon 0 \le a \le \zeta_t)$ (the ℓ^1-norm summary of initial displacements of excursions targeting t) form a growth-fragmentation process whose law we can characterise.





Relationship with the quantum disc

- The quantum disc is a ball of radius 1 in the complex plane, loosely speaking endowed with a Riemannian metric $e^{\gamma h(z)}(dx^2 + dy^2)$ at $z = x + iy$, where h is a Gaussian free field



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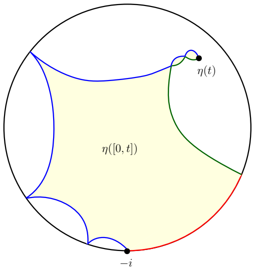

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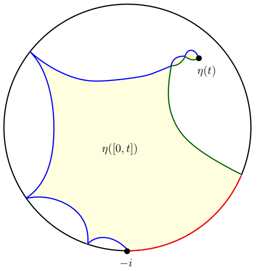

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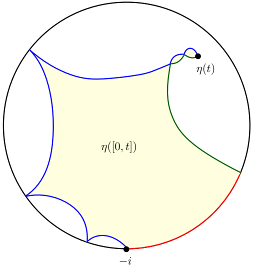

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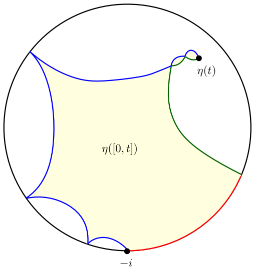
- Our growth-fragmentation describes the total (left and right) boundary length of the branches



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- Other growth-fragmentations in statistical physics models: Miller, Sheffield and Werner (2020), Le Gall and Riera (2020) and Bertoin, (Budd,) Curien and Kortchemski (2018)

Describing the growth-fragmentation

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(S is a positive stochastic process with jump kernel $J(x, x + \mathrm{d}y) = \frac{x+y}{x} y^{-5/2}\,\mathrm{d}y$; $x > 0, y > 0$.)

A beautiful construction of a conditioned 3/2-stable process

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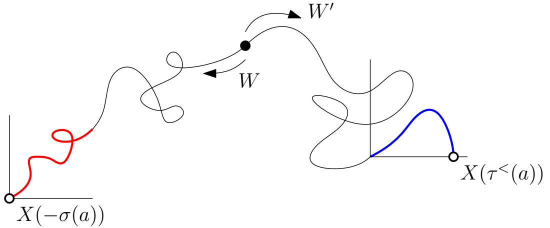

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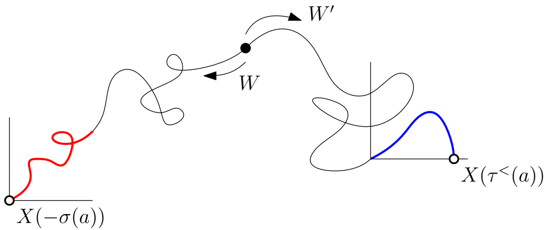

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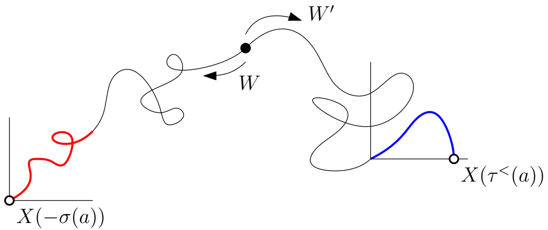

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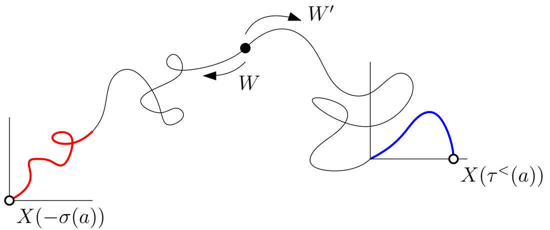

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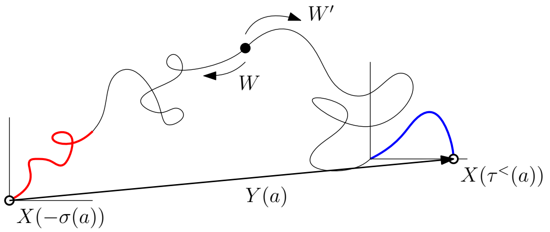

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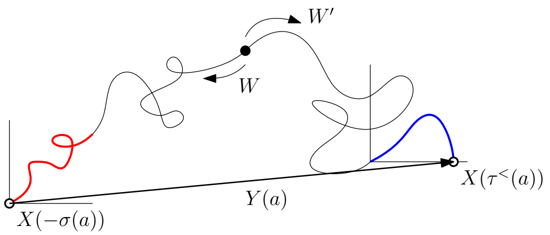
$\mathcal{S}$ is a 3/2-stable process, with only positive jumps, conditioned to stay positive.



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- ...and find a special martingale whose limit law is that of the lifetime of a typical excursion (recovering a result about the volume of Boltzmann triangulations)

Posterior art

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- It is in the same class as the one found in percolation of CLE carpets (Miller, Sheffield and Werner) and metric exploration of random planar maps (Bertoin, Budd, Curien and Kortchemski)
- It may be possible to derive our results with quantum gravity arguments, but we use nothing but an analysis of Brownian motion

Open questions

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- Can we start things at zero?

Further reading

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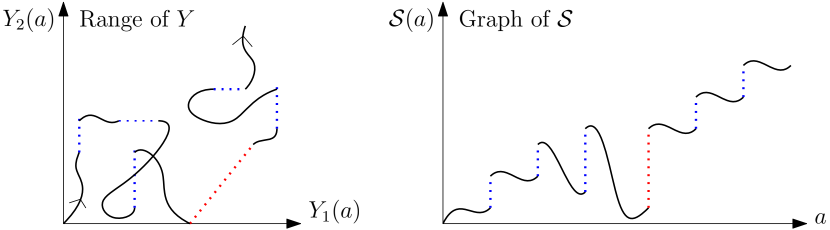

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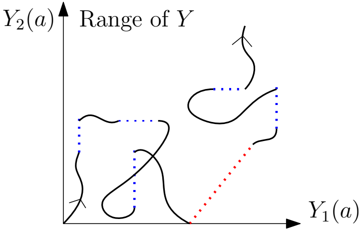

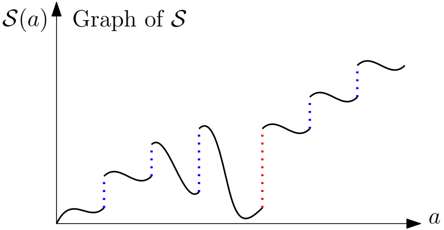

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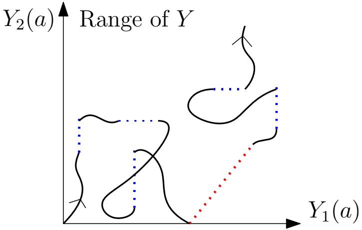

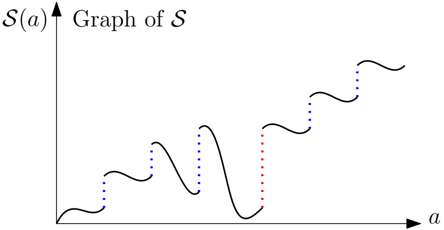

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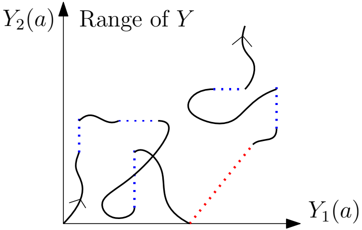

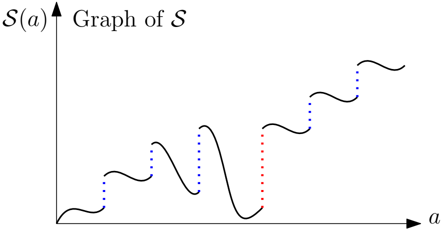

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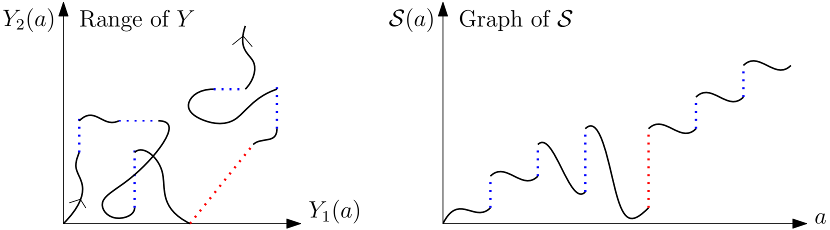

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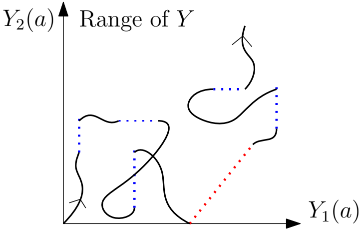

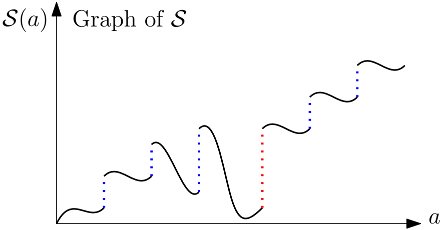

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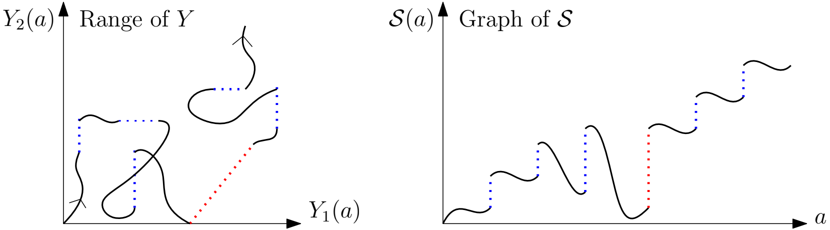

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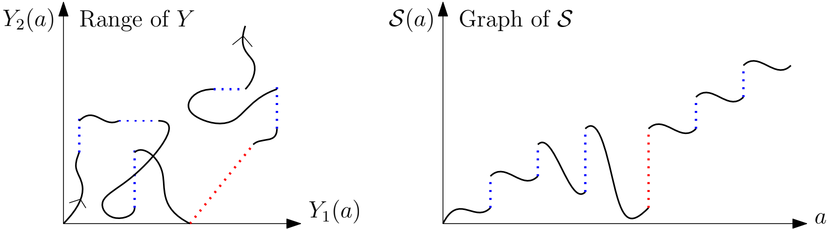

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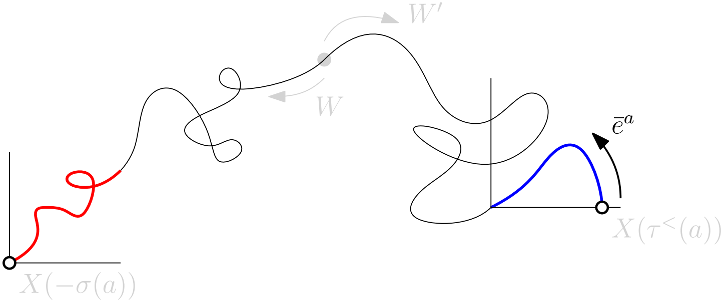

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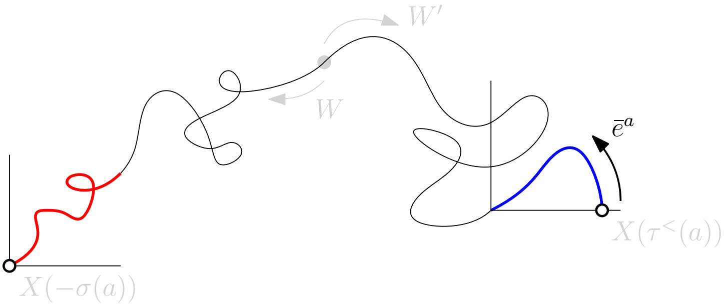

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- Then $(\bar{e}^A, A)$ has the same distribution as a generic whole-path excursion together with a time uniformly chosen within its lifetime