INSTITUT de MATHEMATIQUES de TOULOUSE

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Reinforced Galton-Watson processes: Malthusian growth, survival and distribution of the population

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Galton-Watson trees and processes

Let ν be a probability distribution on $\mathbb{Z}_+$.

Definition

A ν-Galton-Watson tree (or $\mathrm{GW}(\nu)$) is a population model in which each individual, independently from every other, gives birth to a random number of children distributed according to the law ν. The tree starts from an initial individual called the root.

Notation

For all $n \in \mathbb{N}$, we write Z_n the number of individuals alive at generation n. The process $(Z_n, n \geq 1)$ is a Markov process called the ν-Galton-Watson process.

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A Galton-Watson process

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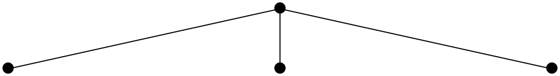

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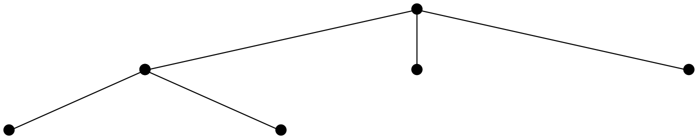

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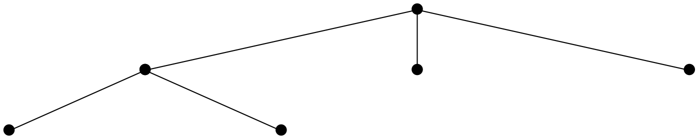

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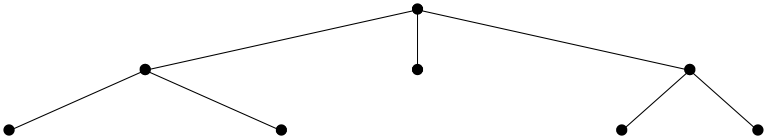

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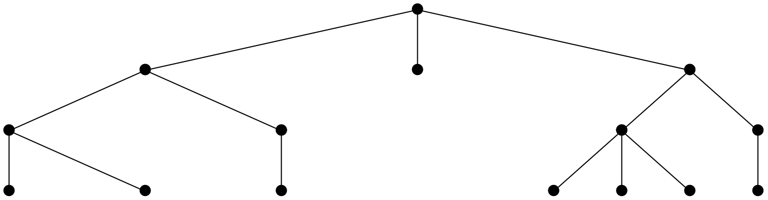

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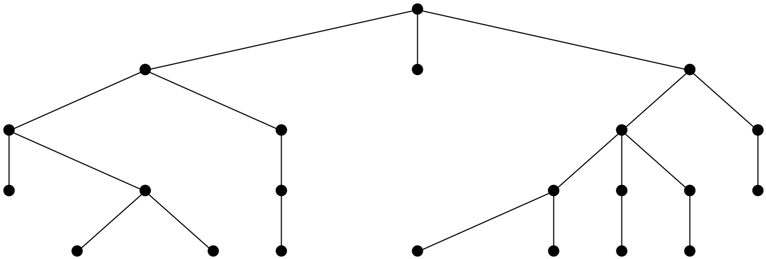

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Malthusian growth and survival of the Galton-Watson process

Mean number of children

We write $m = \mathbf{E}(Z_1) = \sum_{j=0}^{n} j\nu(j)$.

Theorem (Bienaymé 1845, Galton–Watson 1870)

We have $\mathbf{E}(Z_n) = m^n$. Moreover,

$$\mathbb{P}(\forall n \in \mathbb{N}, Z_n > 0) > 0 \iff m > 1 \quad \text{or} \quad \nu = \delta_1.$$

Proposition

The martingale (Z_n/m^n) converges a.s. to a non-negative limit W. Moreover,

$$\mathbb{P}(W > 0) = \mathbb{P}(\forall n \in \mathbb{N}, Z_n > 0).$$

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Ancestral distribution of the population

Some definitions

Definition

For u an individual of a tree T, we denote by $L_u = \frac{1}{|u|} \sum_{i=0}^{|u|-1} \delta_{N(u_i)}$ the empirical distribution of the number of children along the ancestral line of this individual.

Definition

Let T be a (deterministic or random) tree. A probability distribution ρ is called

1. *evanescent* if there exists a neighbourhood G of ρ such that $\#\{u \in T : L_u \in G\} < \infty$;
2. *weakly persistent* if for all neighbourhood G of ρ, we have $\#\{u \in T : L_u \in G\} = \infty$;
3. *strongly persistent* if there exists an infinite spine (v_n) in T such that $\lim_{n\to\infty} L_{v_n} = \rho$.

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Illustrations

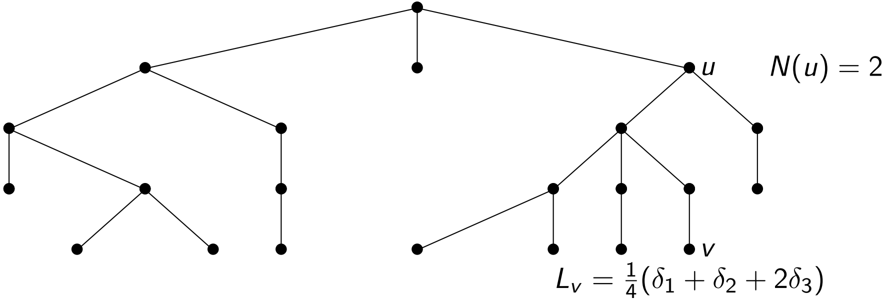

Figure – Notation for Galton-Watson tree

We write $\langle \rho, f \rangle = \sum_{k \geq 0} \rho(k) f(k)$. In particular, $\langle \rho, \ln \rangle = \sum_{k \geq 0} \rho(k) \ln(k)$ is $-\infty$ if $\rho(0) > 0$, non-negative otherwise. We also write $H(\mu|\rho) = \sum \mu(j) \log \frac{\mu(j)}{\rho(j)}$.

Theorem (..., Azaïs–Henry ('25), Bertoin–M. ('25+))

Let ν be a probability measure, that we assume to have finite support. Set $\bar{\nu}(k) = \frac{k\nu(k)}{m}$ the size-biased distribution of ν. Let T be a ν-GW. We have :

- *for all neighbourhood G of $\bar{\nu}$, there exists $\varepsilon > 0$ such that*

$$\mathbf{E}\left(\#\{|u| = n : L_u \notin G\}\right) \leq e^{-\varepsilon n} \mathbf{E}\left(Z_n\right);$$

- *a law ρ is evanescent almost surely if $\langle \rho, \ln \rangle < H(\rho|\bar{\nu})$;*
- *a law ρ is strongly persistent with positive probability if $\langle \rho, \ln \rangle \geq H(\rho|\bar{\nu})$.*

An important lemma

The many-to-one lemma

Lemma

Let T be a (deterministic) rooted tree, we write $h = (h_0, h_1, \dots)$ an *harmonic line* of descent so that $h_0 = \emptyset$ and for all $k \geq 0$, h_{k+1} is a uniformly sampled child of h_k. Let $(x_0, \dots, x_{n-1}) \in \mathbb{N}^n$, setting $\mu = \frac{1}{n} \sum_{j=0}^{n-1} \delta_{x_j}$ we have

$$\#\{|v| = n : N(v_j) = x_j, 0 \leq j \leq n-1\} = \mathbb{P}(N(h_j) = x_j, 0 \leq j \leq n-1) \exp\left(n \langle \mu, \ln \rangle\right).$$

Corollary

For a GW(ν) tree T, we have

$$\mathbf{E}(\#\{|v| = n : L_v = \mu\}) = \mathbb{P}\left(\tfrac{1}{n} \sum_{j=0}^{n-1} \delta_{X_j} = \mu\right) \exp\left(n \langle \mu, \ln \rangle\right)$$

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Almost sure growth of the population

As a summary :

- The GW process can survive with positive probability iff $m > 1$.
- We have $\mathbf{E}(Z_n) \sim_{n\to\infty} m^n$.
- We have $Z_n = W\mathbf{E}(Z_n)(1 + o(1))$ a.s. as $n \to \infty$.
- Most individuals have an ancestral lineage close to $\bar{\nu}$.
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Let $q \in (0,1)$ and ν a probability distribution on $\mathbb{Z}_+$.

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A (q,ν)-reinforced Galton-Watson tree (or rGW(q,ν)) is a population model evolving as follows. At each generation, each individual reproduces as follows :

- with probability $1-q$, they give birth to a random number of children distributed according to the law ν.
- with probability q, they create the same number of children as one of their ancestors, selected uniformly at random.

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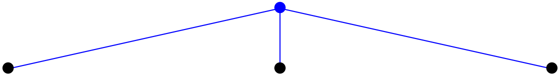

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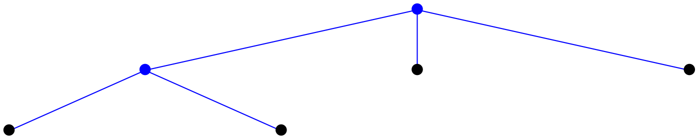

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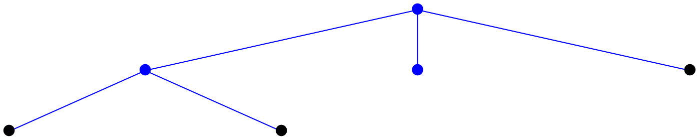

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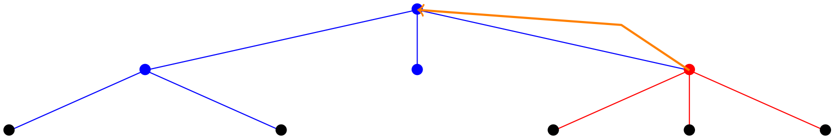

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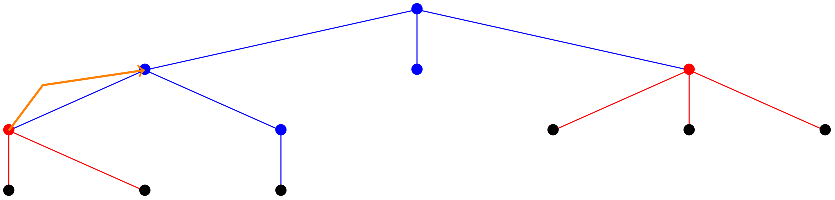

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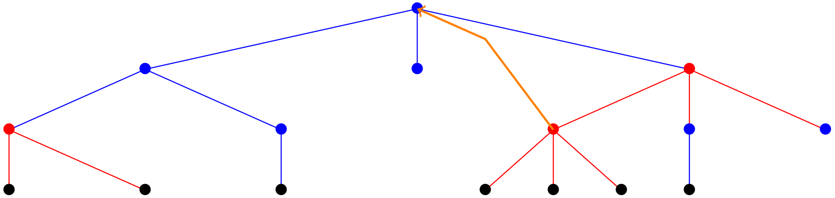

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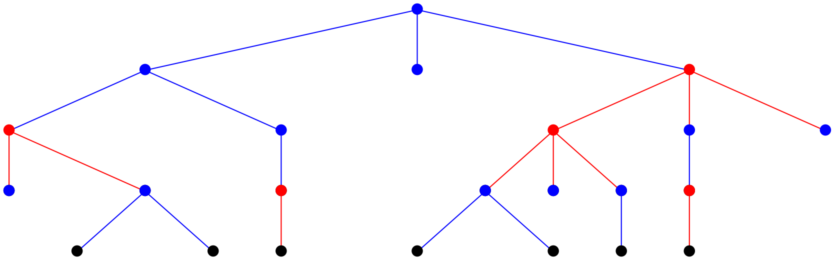

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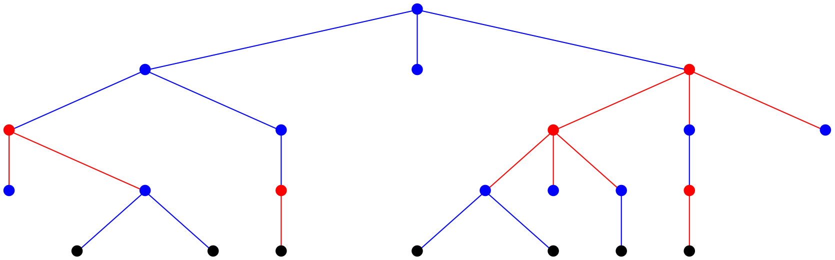

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Preliminary remarks

- Natural model for inheritance of fitness traits.
- A rGW$(\nu, 0)$ is a GW(ν). A rGW$(\nu, 1)$ is with probability $\nu(k)$ a k-ary tree.
- An individual reproducing the offspring of an ancestor always make at least one child.
- All the usual properties of GW trees, such as branching, or the Markov property of (Z_n), are lost.

Therefore, we write $\mathbb{P}_q$ for the law of the rGW(ν, q), with the convention $\mathbb{P}_0 = \mathbb{P}$.

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A simple case

Regular tree

Write $\nu = p\delta_k + (1-p)\delta_0$. Children in the rGW$(\nu, q)$ have k children with probability $(1-q)p + q$.

$$\mathbb{P}_q(\forall n \in \mathbb{N}, Z_n > 0) > 0 \iff ((1-q)p+q)k > 1.$$

- In this situation, the rGW(q, ν) can survive even if the GW(ν) does not.
- We expect the reinforcement to help the process to survive.

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Another simple case

Large support

Assume that there exists $k \in \mathbb{Z}_+$ such that $\nu(k) > 0$ and $((1-q)\nu(k) + q)k > 1$. Then the subtree consisting of individuals with exactly k children survives with positive probability.

$$\exists k \in \mathbb{Z}_+ : \nu(k) > 0 \text{ and } ((1-q)\nu(k) + q)k > 1 \Rightarrow \mathbb{P}(\forall n \in \mathbb{N}, Z_n > 0) > 0.$$

In particular, if $q > 0$ and ν has unbounded support, then the reinforced Galton-Watson process survives with positive probability.

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From now on, we always assume that ν has finite support, and we denote by $k_\star$ the largest integer such that $\nu(k_\star) > 0$.

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Similar questions to ask

1. What is the asymptotic growth rate of $\mathbf{E}_q(Z_n)$, i.e. $\lim_{n\to\infty} \mathbf{E}_q(Z_n)^{1/n}$?
2. Under which conditions on (ν, q) does (Z_n) survives with positive probability ?
3. What is the a.s. growth rate of (Z_n), conditionally on survival ?
4. What is the typical ancestral line of an individual at a large generation ?

We only have a partial picture to all these results.

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4. What is the typical ancestral line of an individual at a large generation ?

We only have a partial picture to all these results.

Similar questions to ask

1. What is the asymptotic growth rate of $\mathbf{E}_q(Z_n)$, i.e. $\lim_{n\to\infty} \mathbf{E}_q(Z_n)^{1/n}$?
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We only have a partial picture to all these results.

Growth rate of the mean of a reinforced Galton-Watson

For $n \in \mathbb{N}$, denote by Z_n the number of individuals at generation n in a rGW(ν, q).

For $t < 1/k_\star$, set $\Pi(t) = \prod_{j=1}^{k_\star} (1 - tj)^{\frac{(1-q)\nu(j)}{q}}$ and $m_{\nu,q} := \dfrac{q}{\int_0^{1/k_\star} \Pi(t) \mathrm{d}t}$.

Theorem (Bertoin–M. '24)

For all $q \in (0,1)$ and ν probability measure on $\mathbb{Z}_+$ with finite support, there exists $m_{\nu,q} > 0$ such that

$$\lim_{n \to \infty} \mathbf{E}_q(Z_n)^{1/n} = m_{\nu,q}.$$

Much more precisely, we have $\mathbf{E}_q(Z_n) \sim \frac{\nu(k_\star)}{q + \nu(k_\star)(1-q)} m_{\nu,q}^n$ as $n \to \infty$.

Growth rate of the mean of a reinforced Galton-Watson process

Some example values

Average growth rate

The formula for $m_{\nu,q}$ appears intricate and is quite difficult to compute in general. However, for some examples these can be computed explicitly.

- If $\nu = (\delta_1 + \delta_2)/2$ and $q = 1/3$, then $m_{\nu,q} = \frac{8}{5}$.
- If $\nu = (\delta_1 + \delta_2)/2$ and $q = 1/5$, then $m_{\nu,q} = \frac{48}{31}$.
- If $\nu = (\delta_0 + \delta_1 + \delta_2 + \delta_3)/4$ and $q = 1/5$, then $m_{\nu,q} = \frac{162}{95}$.

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Sufficient conditions for the survival of the process

Theorem (Bertoin–M. '25+)

- *If* $qk_\star \geq 1$ *or*

$$\sum_{j=0}^{k_\star} \frac{(1-q)j\nu(j)}{1-qj} > 1$$

then $\mathbb{P}(\forall n \in \mathbb{N}, Z_n > 0) > 0$.

- *If moreover* $m_{*,q} = ((1-q)\nu(k_\star) + q)k_\star > 1$ *, then*

$$\lim_{n\to\infty} \frac{Z_n}{m_{\nu,q}^n} = \lim_{n\to\infty} \frac{Z_n^*}{m_{*,q}^n} \quad a.s.$$

where (Z_n^*) *is the largest* $k_\star$*-ary subtree of the reinforced Galton-Watson process.*

Gap in the statements

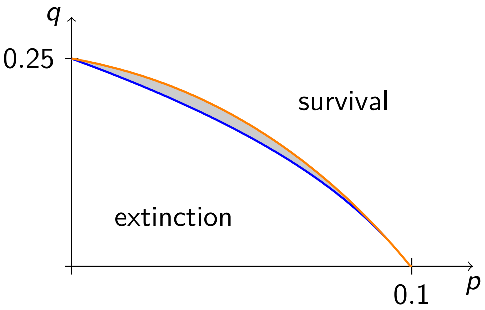

Figure – Phase diagram of a reinforced Galton-Watson process with parameter (ν_p, q) with $\nu_p = (1-4p)\delta_0 + p(\delta_1 + \delta_2 + \delta_3 + \delta_4)$, for $q \in [0, 0.25]$ and $p \in [0, 0.1]$. The blue line corresponds to (p, q) such that $m_{\nu_p, q} = 1$, the orange one such that $\sum \frac{(1-q)j\nu_p(j)}{(1-qj)} = 1$.

Population distribution

Definition

We define the *pressure function* of the rGW(ν,q) as

$$\Lambda_q : \lambda \in \mathbb{R}^{k^*} \mapsto \log q - \log \left(\int_0^\infty \prod_{j=1}^{k^*} (1 - te^{\lambda(k)})_+^{\nu(k)(1-q)/q} \mathrm{d}t \right).$$

Lemma

We have

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbf{E}_q \left(\sum_{|u|=n} \exp\left(\langle L_u, \lambda\rangle\right) \right) = \log m_{\nu,q} + \Lambda_q(\lambda).$$

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Population distribution

Theorem (Bertoin-M. 25+)

1. *With* $\bar{\nu}_q := \nabla\Lambda_q(\ln)$, *for all neighbourhood* G *of* $\bar{\nu}_q$, *there exists* $\varepsilon > 0$ *such that*
$$\mathbf{E}_q\left(\#\{|u| = n : L_u \notin G\}\right) \leq e^{-\varepsilon n}\mathbf{E}_q\left(Z_n\right);$$
2. *Any law that satisfies* $\langle\rho, \ln\rangle < \Lambda_q^*(\rho)$ *is evanescent* $\mathbb{P}_q$*-a.s.*
3. *Any law that satisfies* $\langle\rho, \ln\rangle > H(\rho|q\rho + (1-q)\nu)$ *is strongly persistent with positive probability.*

Remaining open questions

1. State a necessary and sufficient condition, in terms of ν and q for the survival of the reinforced Galton-Watson process.
2. Determine the asymptotic almost sure growth rate of Z_n, defined as $\lim_{n\to\infty} Z_n^{1/n}$.
3. Find a martingale allowing to estimate the size of the population at large times.
4. Give a probabilistic interpretation of the growth rate of $\mathbf{E}(Z_n)$.
5. Characterize the laws that are evanescent, weakly persistent or strongly persistent.

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Thank you for your attention !