Asymptotic behavior of some critical decomposable multi-type Galton–Watson processes with immigration

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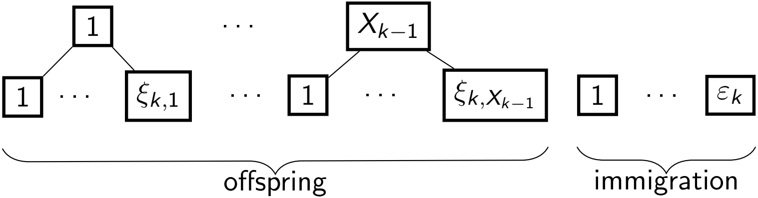
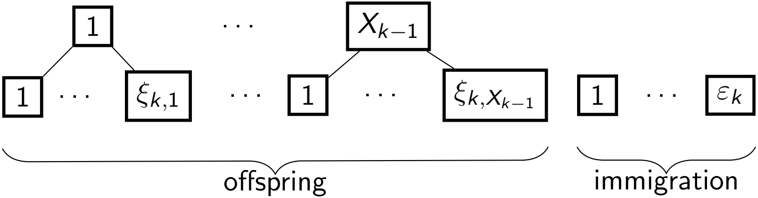
Outline of the talk

1. Multi-type Galton–Watson processes with immigration.
2. Classification and some existing convergence results.
3. Main results.
4. If time permits, some ingredients of the proofs.

Multi-type Galton–Watson processes with immigration



Multi-type Galton–Watson processes with immigration



$$X_k = \sum_{j=1}^{X_{k-1}} \xi_{k,j} + \varepsilon_k, \qquad k \in \mathbb{N} := \{1, 2, \ldots\}, \qquad X_0 = 0,$$

where

- $\{\xi_{k,j}, \varepsilon_k : k, j \in \mathbb{N}\}$ are independent $\mathbb{Z}_+$-valued random variables,
- $\{\xi_{k,j} : k, j \in \mathbb{N}\}$ are identically distributed,
- $\{\varepsilon_k : k \in \mathbb{N}\}$ are identically distributed.

Multi-type Galton–Watson process with immigration (GWI process)

Let $p \in \mathbb{N}$, $\boldsymbol{X}_0 = \boldsymbol{0} \in \mathbb{R}^p$, and

$$\boldsymbol{X}_k = \sum_{i=1}^{p} \sum_{j=1}^{X_{k-1,i}} \boldsymbol{\xi}_{k,j,i} + \boldsymbol{\varepsilon}_k, \qquad k \in \mathbb{N},$$

where

$$\boldsymbol{X}_k := \begin{bmatrix} X_{k,1} \\ \vdots \\ X_{k,p} \end{bmatrix}, \qquad \boldsymbol{\xi}_{k,j,i} := \begin{bmatrix} \xi_{k,j,i,1} \\ \vdots \\ \xi_{k,j,i,p} \end{bmatrix}, \qquad \boldsymbol{\varepsilon}_k := \begin{bmatrix} \varepsilon_{k,1} \\ \vdots \\ \varepsilon_{k,p} \end{bmatrix},$$

and

- $\{\boldsymbol{\xi}_{k,j,i}, \varepsilon_k : k, j \in \mathbb{N}, i \in \{1, \ldots, p\}\}$ are independent $\mathbb{Z}_+^p$-valued random variables,
- $\{\boldsymbol{\xi}_{k,j,i} : k, j \in \mathbb{N}\}$ are identically distributed for $i \in \{1, \ldots, p\}$,
- $\{\boldsymbol{\varepsilon}_k : k \in \mathbb{N}\}$ are identically distributed.

For notational convenience, let $\boldsymbol{\varepsilon}$ and $\boldsymbol{\xi}_i$, $i \in \{1, \ldots, p\}$ be random vectors with $\boldsymbol{\varepsilon} \stackrel{\mathcal{D}}{=} \boldsymbol{\varepsilon}_1$ and $\boldsymbol{\xi}_i \stackrel{\mathcal{D}}{=} \boldsymbol{\xi}_{1,1,i}$, $i \in \{1, \ldots, p\}$.

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Introduce the notation

$$\boldsymbol{A} := \begin{bmatrix} \mathbb{E}(\boldsymbol{\xi}_1) & \dots & \mathbb{E}(\boldsymbol{\xi}_p) \end{bmatrix} \in \mathbb{R}_+^{p \times p}, \qquad \boldsymbol{b} = \mathbb{E}(\boldsymbol{\varepsilon}) \in \mathbb{R}_+^p,$$

$$v_i := \operatorname{Var}(\xi_{i,i}) \in \mathbb{R}_+, \qquad i \in \{1, \dots, p\}.$$

$\boldsymbol{A}$ and $\boldsymbol{b}$ are called the offspring mean matrix and immigration mean vector, respectively.

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One may show that

$$\mathbb{E}(\boldsymbol{X}_k \,|\, \boldsymbol{X}_{k-1}) = \boldsymbol{A}\boldsymbol{X}_{k-1} + \boldsymbol{b}, \qquad \mathbb{E}(\boldsymbol{X}_k) = \sum_{j=1}^{k} \boldsymbol{A}^j \boldsymbol{b}, \qquad k \in \mathbb{N}.$$

Classification and some existing results

A GWI process $(\boldsymbol{X}_k)_{k \in \mathbb{Z}_+}$ with offspring mean matrix $\boldsymbol{A}$ is

- subcritical if $\varrho(\boldsymbol{A}) < 1$,
- critical if $\varrho(\boldsymbol{A}) = 1$,
- supercritical if $\varrho(\boldsymbol{A}) > 1$.

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Wei and Winnicki (1989)

Let $(X_k)_{k\in\mathbb{Z}_+}$ be a critical ($\mathbb{E}(\xi) = 1$) single-type GWI process such that $\mathbb{E}(\xi^2) < \infty$ and $\mathbb{E}(\varepsilon^2) < \infty$. Then

$$(n^{-1}X_{\lfloor nt\rfloor})_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} (\mathcal{X}_t)_{t\in\mathbb{R}_+} \qquad \text{as} \quad n \to \infty,$$

where the limit process $(\mathcal{X}_t)_{t\in\mathbb{R}_+}$ is a squared Bessel process, the pathwise unique strong solution of the SDE

$$\mathrm{d}\mathcal{X}_t = \mathbb{E}(\varepsilon)\,\mathrm{d}t + \sqrt{\mathrm{Var}(\xi)\,\mathcal{X}_t^+}\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+, \qquad \mathcal{X}_0 = 0,$$

where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is a standard Wiener process.

Reducible matrices

A matrix $\boldsymbol{A} \in \mathbb{R}_+^{p \times p}$ is reducible if there exists a permutation matrix $\boldsymbol{P} \in \mathbb{R}_+^{p \times p}$ such that

$$\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{S} & \boldsymbol{T} \end{bmatrix}.$$

A GWI process $(\boldsymbol{X}_k)_{k \in \mathbb{Z}_+}$ with offspring mean matrix $\boldsymbol{A}$ is

- decomposable if $\boldsymbol{A}$ is reducible,
- indecomposable if $\boldsymbol{A}$ is irreducible.

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- decomposable if $\boldsymbol{A}$ is reducible,
- indecomposable if $\boldsymbol{A}$ is irreducible.

Functional limit theorems are known in the critical indecomposable case (see Ispány and Pap (2014), Danka and Pap (2016)).

For all (reducible) $\boldsymbol{A} \in \mathbb{R}_{+}^{p \times p}$, there exist $N \in \{1, \ldots, p\}$ and a permutation matrix $\boldsymbol{P} \in \mathbb{R}_{+}^{p \times p}$ such that $\boldsymbol{PAP}^{-1}$ is block triangular, and all submatrices $\boldsymbol{A}_{i,i}$, $i \in \{1, \ldots, N\}$ on its main diagonal are irreducible.

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$$\boldsymbol{PAP}^{-1} = \begin{bmatrix} \boldsymbol{A}_{1,1} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{A}_{2,1} & \boldsymbol{A}_{2,2} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{A}_{N,1} & \boldsymbol{A}_{N,2} & \dots & \boldsymbol{A}_{N,N} \end{bmatrix}, \qquad (*)$$

where $\boldsymbol{A}_{i,j} \in \mathbb{R}_+^{p_i \times p_j}$, $i, j \in \{1, \dots, N\}$, with $p_i \in \{1, \dots, p\}$, $i \in \{1, \dots, N\}$ such that $p_1 + \dots + p_N = p$.

Foster and Ney (1978)

Let $(\boldsymbol{X}_k)_{k \in \mathbb{Z}_+}$ be a critical decomposable p-type GWI process whose offspring mean matrix has the form $(*)$ such that $\varrho(\boldsymbol{A}_{i,i}) = 1$, $i \in \{1, \ldots, N\}$ (strongly critical), and $\boldsymbol{A}_{i+1,i} \neq \boldsymbol{0}$, $i \in \{1, \ldots, N-1\}$. If certain second order moment conditions are satisfied, then there exists a random vector $\boldsymbol{\mathcal{X}}$ such that

$$\begin{bmatrix} n^{-1}\boldsymbol{X}_{n,1} \\ \vdots \\ n^{-N}\boldsymbol{X}_{n,N} \end{bmatrix} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{X}} \qquad \text{as } n \to \infty.$$

Main results

Barczy and Bezdány (2025+)

Let $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$ be a critical decomposable 3-type GWI process, with $\mathbb{E}(\|\boldsymbol{\varepsilon}\|^2) < \infty$, $\mathbb{E}(\|\boldsymbol{\xi}_i\|^2) < \infty$, $i \in \{1, 2, 3\}$, and with offspring mean matrix

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 0 \\ a_{2,1} & 1 & 0 \\ a_{3,1} & a_{3,2} & 1 \end{bmatrix} \in \mathbb{R}_+^{3\times 3}.$$

We can distinguish four cases, depending on the form of $\boldsymbol{A}$.

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We can distinguish four cases, depending on the form of $\boldsymbol{A}$.

These four cases may be represented in matrix form as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_1, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ ++ & + & 1 \end{bmatrix}_2, \quad \begin{bmatrix} 1 & 0 & 0 \\ ++ & 1 & 0 \\ ++ & 0 & 1 \end{bmatrix}_3, \quad \begin{bmatrix} 1 & 0 & 0 \\ ++ & 1 & 0 \\ + & ++ & 1 \end{bmatrix}_4.$$

1. If $a_{2,1} = a_{3,1} = a_{3,2} = 0$, $\mathbb{E}(\|\boldsymbol{\varepsilon}\|^4) < \infty$ and $\mathbb{E}(\|\boldsymbol{\xi}_i\|^4) < \infty$, $i \in \{1, 2, 3\}$, then

$$\left(n^{-1}\boldsymbol{X}_{\lfloor nt \rfloor}\right)_{t\in\mathbb{R}_+} \stackrel{\mathcal{D}}{\longrightarrow} (\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

where $(\boldsymbol{\mathcal{X}}_t)_{t\in\mathbb{R}_+} = \left([\mathcal{X}_{t,1}, \mathcal{X}_{t,2}, \mathcal{X}_{t,3}]^\top\right)_{t\in\mathbb{R}_+}$ is the pathwise unique strong solution of the SDE

$$\begin{cases} \mathrm{d}\mathcal{X}_{t,1} = b_1\,\mathrm{d}t + \sqrt{v_1\mathcal{X}_{t,1}^+}\,\mathrm{d}\mathcal{W}_{t,1}, \\ \mathrm{d}\mathcal{X}_{t,2} = b_2\,\mathrm{d}t + \sqrt{v_2\mathcal{X}_{t,2}^+}\,\mathrm{d}\mathcal{W}_{t,2}, \\ \mathrm{d}\mathcal{X}_{t,3} = b_3\,\mathrm{d}t + \sqrt{v_3\mathcal{X}_{t,3}^+}\,\mathrm{d}\mathcal{W}_{t,3}, \end{cases} \qquad t \in \mathbb{R}_+,$$

with initial value $[\mathcal{X}_{t,1}, \mathcal{X}_{t,2}, \mathcal{X}_{t,3}]^\top = \mathbf{0} \in \mathbb{R}_+^3$, where $(\mathcal{W}_{t,i})_{t\in\mathbb{R}_+}$, $i \in \{1, 2, 3\}$ are independent standard Wiener processes.

2. If $a_{2,1} = 0 < a_{3,1}$, $\mathbb{E}(\|\varepsilon\|^4) < \infty$, $\mathbb{E}(\|\boldsymbol{\xi}_i\|^4) < \infty$, $i \in \{1, 2\}$, and $\mathbb{E}(\|\boldsymbol{\xi}_3\|^2) < \infty$, then

$$\left(\begin{bmatrix} n^{-1} X_{\lfloor nt \rfloor,1} \\ n^{-1} X_{\lfloor nt \rfloor,2} \\ n^{-2} X_{\lfloor nt \rfloor,3} \end{bmatrix} \right)_{t \in \mathbb{R}_+} \xrightarrow{\mathcal{D}} \left(\begin{bmatrix} \mathcal{X}_{t,1} \\ \mathcal{X}_{t,2} \\ \mathcal{X}_{t,3} \end{bmatrix} \right)_{t \in \mathbb{R}_+} \qquad \text{as } n \to \infty,$$

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with initial value $[\mathcal{X}_{0,1}, \mathcal{X}_{0,2}, \mathcal{X}_{0,3}]^\top = \mathbf{0} \in \mathbb{R}_+^3$, where $(\mathcal{W}_{t,i})_{t \in \mathbb{R}_+}$, $i \in \{1, 2\}$ are independent standard Wiener processes.

3. If $a_{3,2} = 0 < \min\{a_{2,1}, a_{3,1}\}$, $\mathbb{E}(\|\varepsilon\|^2) < \infty$ and $\mathbb{E}(\|\boldsymbol{\xi}_i\|^2) < \infty$, $i \in \{1, 2, 3\}$, then

$$\left(\begin{bmatrix} n^{-1} X_{\lfloor nt \rfloor,1} \\ n^{-2} X_{\lfloor nt \rfloor,2} \\ n^{-2} X_{\lfloor nt \rfloor,3} \end{bmatrix} \right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \left(\begin{bmatrix} \mathcal{X}_{t,1} \\ \mathcal{X}_{t,2} \\ \mathcal{X}_{t,3} \end{bmatrix} \right)_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

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with initial value $[\mathcal{X}_{0,1}, \mathcal{X}_{0,2}, \mathcal{X}_{0,3}]^\top = \mathbf{0} \in \mathbb{R}_+^3$, where $(\mathcal{W}_{t,1})_{t\in\mathbb{R}_+}$, is a standard Wiener processes.

4. If $0 < \min\{a_{2,1}, a_{3,2}\}$, $\mathbb{E}(\|\boldsymbol{\varepsilon}\|^2) < \infty$ and $\mathbb{E}(\|\boldsymbol{\xi}_i\|^2) < \infty$, $i \in \{1, 2, 3\}$, then

$$\left(\begin{bmatrix} n^{-1} X_{\lfloor nt \rfloor,1} \\ n^{-2} X_{\lfloor nt \rfloor,2} \\ n^{-3} X_{\lfloor nt \rfloor,3} \end{bmatrix}\right)_{t\in\mathbb{R}_+} \stackrel{\mathcal{D}}{\longrightarrow} \left(\begin{bmatrix} \mathcal{X}_{t,1} \\ \mathcal{X}_{t,2} \\ \mathcal{X}_{t,3} \end{bmatrix}\right)_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty,$$

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with initial value $[\mathcal{X}_{0,1}, \mathcal{X}_{0,2}, \mathcal{X}_{0,3}]^\top = \mathbf{0} \in \mathbb{R}_+^3$, where $(\mathcal{W}_{t,1})_{t\in\mathbb{R}_+}$, is a standard Wiener processes.

Remark

- In case (2), for all $t \in \mathbb{R}_+$, we have

$$\mathcal{X}_{t,3} = \int_0^t (a_{3,1}\mathcal{X}_{s,1} + a_{3,2}\mathcal{X}_{s,2})\,\mathrm{d}s.$$

- In case (3), for all $t \in \mathbb{R}_+$, we have

$$\mathcal{X}_{t,2} = a_{2,1}\int_0^t \mathcal{X}_{s,1}\,\mathrm{d}s,$$
$$\mathcal{X}_{t,3} = a_{3,1}\int_0^t \mathcal{X}_{s,1}\,\mathrm{d}s.$$

- In case (4), for all $t \in \mathbb{R}_+$, we have

$$\mathcal{X}_{t,2} = a_{2,1}\int_0^t \mathcal{X}_{s,1}\,\mathrm{d}s,$$
$$\mathcal{X}_{t,3} = a_{3,2}\int_0^t \mathcal{X}_{s,2}\,\mathrm{d}s = a_{3,2}a_{2,1}\int_0^t \left(\int_0^s \mathcal{X}_{r,1}\,\mathrm{d}r\right)\mathrm{d}s.$$

Ingredients for the proofs

Let $(\mathcal{F}_k)_{k\in\mathbb{Z}_+}$ be the natural filtration of $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$. Defining

$$\boldsymbol{M}_k := \boldsymbol{X}_k - \mathbb{E}(\boldsymbol{X}_k \,|\, \mathcal{F}_{k-1}) = \boldsymbol{X}_k - \boldsymbol{A}\boldsymbol{X}_{k-1} - \boldsymbol{b}, \qquad k \in \mathbb{N},$$

we arrive at

$$\boldsymbol{X}_k = \sum_{j=1}^{k} \boldsymbol{A}^{k-j} \left(\boldsymbol{M}_j + \boldsymbol{b}\right), \qquad k \in \mathbb{Z}_+.$$

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Let $(\mathcal{F}_k)_{k\in\mathbb{Z}_+}$ be the natural filtration of $(\boldsymbol{X}_k)_{k\in\mathbb{Z}_+}$. Defining

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we arrive at

$$\boldsymbol{X}_k = \sum_{j=1}^{k} \boldsymbol{A}^{k-j}\left(\boldsymbol{M}_j + \boldsymbol{b}\right), \qquad k \in \mathbb{Z}_+.$$

In particular, one can show that

$$\boldsymbol{X}_k = \begin{bmatrix} X_{k,1}^{(0)} \\ a_{2,1}X_{k,1}^{(1)} + X_{k,2}^{(0)} \\ a_{3,2}a_{2,1}X_{k,1}^{(2)} + a_{3,1}X_{k,1}^{(1)} + a_{3,2}X_{k,2}^{(1)} + X_{k,3}^{(0)} \end{bmatrix}, \qquad k \in \mathbb{Z}_+,$$

where for $i \in \{1,2,3\}$ and $m \in \{0,1,2\}$,

$$X_{k,i}^{(m)} := \sum_{j=1}^{k} \binom{k-j}{m}\left(M_{j,i} + b_i\right), \qquad k \in \mathbb{Z}_+.$$

Sketch of the proof for case (2):

1. Using a result of Ispány and Pap, we prove

$$\left(n^{-1} \begin{bmatrix} X_{\lfloor nt \rfloor,1} \\ X_{\lfloor nt \rfloor,2} \end{bmatrix} \right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \left(\begin{bmatrix} \mathcal{X}_{t,1} \\ \mathcal{X}_{t,2} \end{bmatrix} \right)_{t\in\mathbb{R}_+} \qquad \text{as } n \to \infty.$$

Ispány and Pap (2010)

Let $\beta : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$ and $\gamma : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^{d \times r}$ be continuous functions. Assume that uniqueness in the sense of probability law holds for the SDE

$$\mathrm{d}\mathcal{U}_t = \beta(t, \mathcal{U}_t)\,\mathrm{d}t + \gamma(t, \mathcal{U}_t)\,\mathrm{d}\mathcal{W}_t, \qquad t \in \mathbb{R}_+,$$

with initial value $\mathcal{U}_0 = \boldsymbol{u}_0$ for all $\boldsymbol{u}_0 \in \mathbb{R}^d$, where $(\mathcal{W}_t)_{t\in\mathbb{R}_+}$ is an r-dimensional standard Wiener process. Let $(\mathcal{U}_t)_{t\in\mathbb{R}_+}$ be a solution of this SDE with initial value $\mathcal{U}_0 = \mathbf{0} \in \mathbb{R}^d$. For each $n \in \mathbb{N}$, let $(\boldsymbol{U}_k^{(n)})_{k\in\mathbb{Z}_+}$ be a sequence of square-integrable d-dimensional random vectors adapted to a filtration $(\mathcal{F}_k^{(n)})_{k\in\mathbb{Z}_+}$ (that is, $\boldsymbol{U}_k^{(n)}$ is $\mathcal{F}_k^{(n)}$-measurable and $\mathbb{E}(\|\boldsymbol{U}_k^{(n)}\|^2) < \infty$). Let

$$\mathcal{U}_t^{(n)} := \sum_{k=0}^{\lfloor nt \rfloor} \boldsymbol{U}_k^{(n)}, \qquad t \in \mathbb{R}_+, \quad n \in \mathbb{N}.$$

Suppose that $\mathcal{U}_0^{(n)} = \boldsymbol{U}_0^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{0}$ as $n \to \infty$ and that for each $T > 0$,

(i) $\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathbb{E}\big(\boldsymbol{U}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}\big) - \int_0^t \boldsymbol{\beta}(s, \mathcal{U}_s^{(n)}) \mathrm{d}s \right\| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$,

(ii) $\sup_{t\in[0,T]} \left\| \sum_{k=1}^{\lfloor nt \rfloor} \mathrm{Var}\big(\boldsymbol{U}_k^{(n)} \,|\, \mathcal{F}_{k-1}^{(n)}\big) - \int_0^t \boldsymbol{\gamma}(s, \mathcal{U}_s^{(n)}) \boldsymbol{\gamma}(s, \mathcal{U}_s^{(n)})^\top \mathrm{d}s \right\| \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$,

(iii) $\sum_{k=1}^{\lfloor nT \rfloor} \mathbb{E}\left(\|\boldsymbol{U}_k^{(n)}\|^2 \mathbb{1}_{\{\|\boldsymbol{U}_k^{(n)}\| > \theta\}} \,\Big|\, \mathcal{F}_{k-1}^{(n)} \right) \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ for all $\theta > 0$.

Then $\mathcal{U}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{U}$ as $n \to \infty$.

2. Using a version of the continuous mapping theorem, we prove

$$\left(\begin{bmatrix} n^{-1}X_{\lfloor nt\rfloor,1} \\ n^{-1}X_{\lfloor nt\rfloor,2} \\ n^{-2}\left(a_{3,1}X^{(1)}_{\lfloor nt\rfloor,1} + a_{3,2}X^{(1)}_{\lfloor nt\rfloor,2}\right)\end{bmatrix}\right)_{t\in\mathbb{R}_+} \xrightarrow{\mathcal{D}} \left(\begin{bmatrix} \mathcal{X}_{t,1} \\ \mathcal{X}_{t,2} \\ \mathcal{X}_{t,3}\end{bmatrix}\right)_{t\in\mathbb{R}_+}$$

as $n \to \infty$.

For $d \in \mathbb{N}$, $\mathrm{D}(\mathbb{R}_+, \mathbb{R}^d)$ denotes the set of càdlàg functions from $\mathbb{R}_+$ to $\mathbb{R}^d$. For functions $f \in \mathrm{D}(\mathbb{R}_+, \mathbb{R}^d)$ and $f_n \in \mathrm{D}(\mathbb{R}_+, \mathbb{R}^d)$, $n \in \mathbb{N}$, we write $f_n \xrightarrow{\mathrm{lu}} f$ as $n \to \infty$ if

$$\sup_{t \in [0,T]} \|f_n(t) - f(t)\| \to 0 \qquad \text{as } n \to \infty \text{ for all } T > 0.$$

A version of the continuous mapping theorem

Let d, $q \in \mathbb{N}$. Let $(\mathcal{U}_t)_{t \in \mathbb{R}_+}$ and $(\mathcal{U}_t^{(n)})_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, be $\mathbb{R}^d$-valued stochastic processes with càdlàg paths such that $\mathcal{U}^{(n)} \xrightarrow{\mathcal{D}} \mathcal{U}$ as $n \to \infty$. Let $\Phi : \mathrm{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathrm{D}(\mathbb{R}_+, \mathbb{R}^q)$ and $\Phi_n : \mathrm{D}(\mathbb{R}_+, \mathbb{R}^d) \to \mathrm{D}(\mathbb{R}_+, \mathbb{R}^q)$, $n \in \mathbb{N}$, be measurable mappings such that there exists $C \in \mathcal{B}(\mathrm{D}(\mathbb{R}_+, \mathbb{R}^d))$ so that $\mathbb{P}(\mathcal{U} \in C) = 1$ and for all $f \in C$, $f_n \in \mathrm{D}(\mathbb{R}_+, \mathbb{R}^d)$, $n \in \mathbb{N}$, if $f_n \xrightarrow{\mathrm{lu}} f$ as $n \to \infty$, then $\Phi_n(f_n) \xrightarrow{\mathrm{lu}} \Phi(f)$ as $n \to \infty$. Then $\Phi_n(\mathcal{U}^{(n)}) \xrightarrow{\mathcal{D}} \Phi(\mathcal{U})$ as $n \to \infty$.

3. We show that the remaining terms disappear in the limit using a kind of Slutsky's lemma for stochastic processes with trajectories in $D(\mathbb{R}_+, \mathbb{R}^d)$.

Jacod and Shiryaev (2003)

Let $d \in \mathbb{N}$. Let $(\boldsymbol{\mathcal{Y}}_t^{(n)})_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, $(\boldsymbol{\mathcal{Y}}_t)_{t \in \mathbb{R}_+}$, and $(\boldsymbol{\mathcal{Z}}_t^{(n)})_{t \in \mathbb{R}_+}$, $n \in \mathbb{N}$, be $\mathbb{R}^d$-valued stochastic processes with càdlàg paths on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. Suppose that $\boldsymbol{\mathcal{Y}}^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{Y}}$ as $n \to \infty$ and

$$\lim_{n \to \infty} \mathbb{P}\left(\sup_{t \in [0,T]} \|\boldsymbol{\mathcal{Z}}_t^{(n)}\| > \varepsilon \right) = 0 \qquad \text{for all } T > 0 \text{ and } \varepsilon > 0.$$

Then $\boldsymbol{\mathcal{Y}}^{(n)} + \boldsymbol{\mathcal{Z}}^{(n)} \xrightarrow{\mathcal{D}} \boldsymbol{\mathcal{Y}}$ as $n \to \infty$.

This talk is based on our paper available on arXiv:

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Asymptotic behavior of some strongly critical decomposable 3-type Galton-Watson processes with immigration.
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Thank you for your attention!