Branching process models for integer-valued periodic and vector time series

Márton Ispány

Faculty of Informatics, University of Debrecen
Hungary

joint work with
Pascal Bondon, Valdério A. Reisen

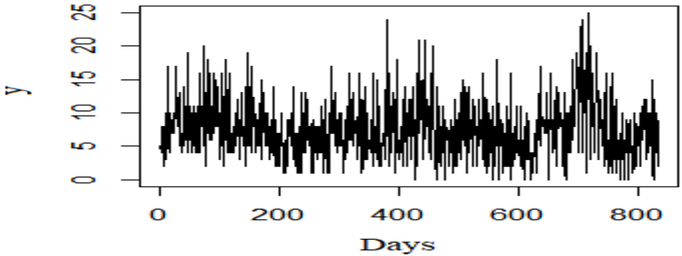
19th International Summer Conference on Probability and Statistics - ISCPS2025
Sofia, Bulgaria
July 25, 2025

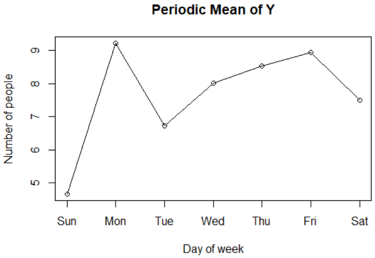
Outline

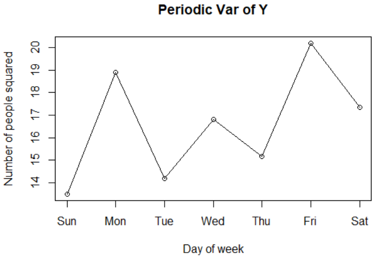
1. Data1: daily number of people who got antibiotics for the treatment of respiratory problems
2. Data2: daily number of parcels picked up from one pickup point (PUP)
3. Branching process with immigration (BPI) and integer-valued autoregression (INAR)
4. PINARMA model: a periodic integer-valued ARMA model
5. VINARMA model: a stationary vector integer-valued ARMA model (multitype BPI, MBPI)
6. Equivalence, existence and properties
7. Simulation study
8. Real data applications
9. Conclusions

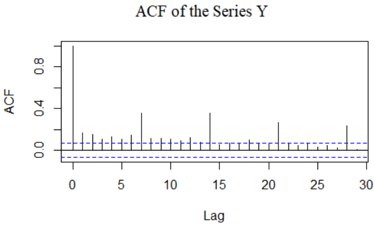
Data1

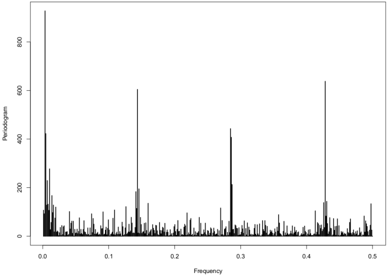
- Time series: The daily number of people who got antibiotics for treating respiratory problems from the public health care system in the emergency service.
- Duration: May 26, 2013, to September 05, 2015, resulting in $T = 833$ daily ($n = 119$ weeks) observations.
- Source:This real data set was obtained from the network records system welfare of the municipality Vitória-ES, Brazil. (Filho et al., 2021)




Periodic Mean of Y
Number of people
5
6
7
8
9
Sun
Mon
Tue
Wed
Thu
Fri
Sat
Day of week


Periodic Var of Y
Number of people squared
14
15
16
17
18
19
20
Sun
Mon
Tue
Wed
Thu
Fri
Sat
Day of week


ACF of the Series Y
ACF
0.0
0.4
0.8
0
5
10
15
20
25
30
Lag


Periodogram
0
200
400
600
800
0.0
0.1
0.2
0.3
0.4
0.5
Frequency

Periodic ACF and PACF of Data1

Autocovariance function of periodic time series with period $S \in \mathbb{N}$: $\gamma_s(h) = \mathrm{Cov}(Y_t, Y_{t-h})$, where $s = 1, \dots, S$, $h \in \mathbb{N}_0$, such that $t \equiv s \mod S$.

Periodic ACF

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
Sunday	0.01	**0.26**	**0.18**	**0.24**	**0.28**	0.11	0.15	0.02	0.07	**0.29**
Monday	**0.38**	-0.12	0.14	**0.23**	**0.18**	**0.19**	**0.29**	0.13	-0.11	0.04
Tuesday	**0.33**	**0.34**	-0.02	0.10	**0.23**	**0.39**	**0.42**	**0.18**	**0.37**	0.03
Wednesday	**0.27**	0.05	**0.17**	0.10	0.16	**0.33**	**0.29**	**0.23**	0.14	0.14
Thursday	**0.18**	**0.36**	**0.23**	**0.31**	0.01	**0.18**	**0.29**	**0.22**	**0.25**	0.11
Friday	**0.25**	0.16	**0.20**	0.14	0.16	**0.17**	**0.18**	**0.30**	**0.23**	0.13
Saturday	**0.20**	0.10	-0.03	-0.05	**−0.18**	0.03	**0.30**	0.10	-0.07	0.16

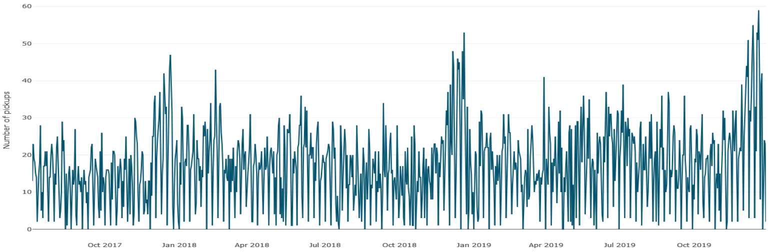
Periodic PACF

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
Sunday	0.01	**0.26**	0.12	**0.20**	**0.18**	0.00	0.03	-0.02	-0.01	0.16
Monday	**0.38**	-0.14	0.08	**0.18**	0.07	0.01	**0.22**	-0.06	-0.08	-0.02
Tuesday	**0.33**	**0.24**	0.00	-0.00	0.15	**0.32**	**0.29**	0.01	**0.26**	0.04
Wednesday	**0.27**	-0.04	0.10	0.10	0.11	**0.27**	**0.18**	0.03	0.09	-0.07
Thursday	**0.18**	**0.33**	0.13	**0.18**	0.01	0.10	**0.17**	0.04	0.02	-0.02
Friday	**0.25**	0.12	0.10	0.06	0.03	**0.18**	0.08	**0.18**	0.13	-0.05
Saturday	**0.20**	0.05	-0.07	-0.11	**−0.21**	0.08	**0.26**	0.03	-0.13	**0.21**

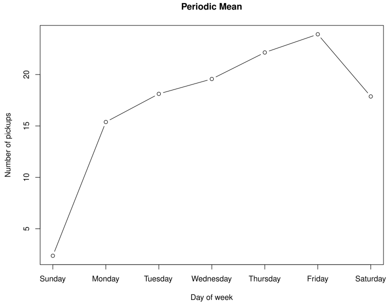
bold: significant correlation

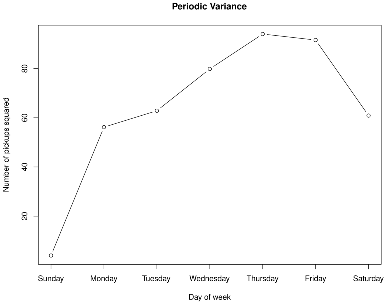
Data2

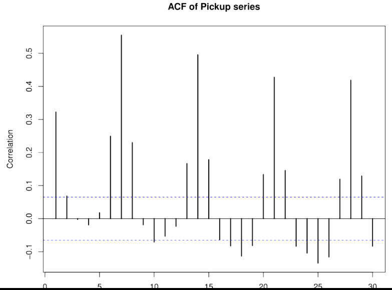
- Time series: The daily number of parcels picked up from one pickup point (PUP) at a PUP management company.
- Duration: July 3, 2017 to December 29, 2019, resulting in $T = 910$ daily ($n = 130$ weeks) observations.
- Source: https://github.com/cabani/ForecastingParcels (Nguyen, et al., 2023)

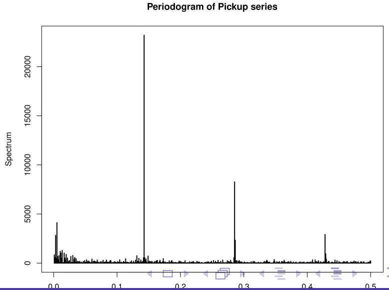


Basic plots of Data2









Periodic ACF and PACF of Data2

Periodic ACF

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
Sunday	0.072	0.008	0.118	-0.021	0.036	0.120	-0.042	0.000	-0.058	-0.012
Monday	**0.261**	**0.215**	**0.370**	**0.287**	**0.321**	0.075	0.169	0.084	0.184	0.186
Tuesday	**0.328**	**0.438**	**0.241**	**0.208**	0.081	**0.281**	0.060	0.168	**0.205**	0.115
Wednesday	**0.548**	**0.479**	**0.373**	**0.215**	**0.342**	0.171	**0.222**	**0.238**	**0.238**	**0.232**
Thursday	**0.486**	**0.450**	**0.196**	**0.278**	**0.196**	**0.222**	**0.308**	**0.406**	**0.245**	0.096
Friday	**0.521**	0.149	**0.381**	**0.351**	**0.314**	**0.398**	**0.368**	**0.363**	0.097	**0.337**
Saturday	**0.244**	**0.332**	**0.238**	**0.341**	**0.312**	**0.443**	**0.406**	**0.260**	**0.234**	0.135

Periodic PACF

	$h=1$	$h=2$	$h=3$	$h=4$	$h=5$	$h=6$	$h=7$	$h=8$	$h=9$	$h=10$
Sunday	0.072	-0.011	**0.174**	**0.237**	0.009	-0.037	-0.068	-0.045	-0.088	-0.021
Monday	**0.261**	0.142	0.114	**0.240**	0.091	0.015	0.033	0.081	0.067	-0.022
Tuesday	**0.328**	**0.326**	**0.260**	-0.116	0.085	0.142	0.118	-0.045	0.027	0.005
Wednesday	**0.548**	**0.290**	0.002	0.134	-0.004	**0.297**	0.065	**0.204**	-0.086	-0.156
Thursday	**0.486**	**0.264**	0.064	-0.008	0.152	0.117	0.152	0.063	-0.104	0.017
Friday	**0.521**	0.027	0.096	0.109	-0.021	-0.025	-0.081	**0.202**	-0.080	0.010
Saturday	**0.244**	**0.327**	**0.258**	0.074	0.164	0.157	0.003	-0.052	0.020	0.158

bold: significant correlation

Observations:

- periodicity (varying environment)
- strong seasonality

Overviews on count time series

- Davis RA, Fokianos K, Holan SH, Joe H, Livsey J, Lund R, Pipiras V, Ravishanker N. (2021) Count time series: a methodological review. Journal of the American Statistical Association 116:1533–1547.
- Davis RA, Holan SH, Lund R, Ravishanker N. (2016) Handbook of Discrete-Valued Time Series CRC Press, New York, NY.
- Fokianos, K. (2012) Count time series models. Handbook of Statistics., C. R. Rao, C. Rao, and V. Govindaraju (eds); 30, 315–347, Elsevier, Amsterdam.
- Weiss CH. (2018) An Introduction to Discrete-Valued Time Series. John Wiley & Sons, Boca Raton, FL.

Branching process with immigration (BPI)

$$X_k = \sum_{j=1}^{X_{k-1}} \xi_j^k + \varepsilon_k, \quad k \in \mathbb{N}, \quad X_0 = 0$$

$\xi^k := \{\xi_j^k \mid j \in \mathbb{N}\}$ and $\{\varepsilon_k \mid k \in \mathbb{N}\}$ i.i.d. sequences of r.v.'s
To avoid degeneracy: $\mathsf{P}(\varepsilon = 0) < 1$
Reformulation by (generalized) thinning operator:

$$X_k = \xi^k \circ X_{k-1} + \varepsilon_k, \quad k \in \mathbb{N}, \quad X_0 = 0$$

Parameters: $m := \mathsf{E}[\xi]$, $\sigma^2 := \mathrm{Var}[\xi]$, $\lambda := \mathsf{E}[\varepsilon]$, $b^2 := \mathrm{Var}[\varepsilon]$
Existence: always since $k \in \mathbb{N}_0$

Classification:	$m < 1$	$m = 1$	$m > 1$
	subcritical	**critical**	**supercritical**

Integer valued autoregression (INAR)

INAR(1) model, Al-Osh and Alzaid (1987), McKenzie (1985)

$$Y_k = \alpha_k \circ Y_{k-1} + \varepsilon_k, \quad k \in \mathbb{Z}$$

$\{\alpha_k \circ \mid k \in \mathbb{Z}\}$, are i.i.d. binomial thinning operators with parameter $\alpha \in [0, 1]$, $\{\varepsilon_k \mid k \in \mathbb{Z}\}$ are i.i.d. $\mathbb{N}_0$-valued r.v.'s
To avoid degeneracy: $\mathrm{P}(\varepsilon = 0) < 1$
Reformulation in BPI form:

$$Y_k = \sum_{j=1}^{Y_{k-1}} \xi_j^k + \varepsilon_k, \quad k \in \mathbb{Z}$$

$\{\xi_j^k \mid k \in \mathbb{Z}, j \in \mathbb{N}\}$ i.i.d. Bernoulli r.v.'s with mean α

Parameters: $\alpha := \mathrm{E}[\xi]$, $\lambda := \mathrm{E}[\varepsilon]$, $b^2 := \mathrm{Var}[\varepsilon]$

Existence: $\alpha < 1$ (subcritical)
In the critical case $\alpha = 1$, there is no solution!

Conditional structure

Filtration: $\mathcal{F}_k := \sigma(X_0, X_1, \dots, X_k), \quad k \in \mathbb{Z}_+$

Conditioning: $\operatorname{E}(X_k \mid \mathcal{F}_{k-1}, \varepsilon_k) = mX_{k-1} + \varepsilon_k$

$M_k := X_k - \operatorname{E}(X_k \mid \mathcal{F}_{k-1}, \varepsilon_k) = X_k - mX_{k-1} - \varepsilon_k =$
$\xi^k \circ X_{k-1} - mX_{k-1}$ martingale differences

We have an AR(1) model:

$$X_k = mX_{k-1} + \varepsilon_k + M_k$$

$m \geq 0$ autoregressive coefficient and $\{\varepsilon_k + M_k\}$ innovation

Conditional variance: $\operatorname{E}(M_k^2 \mid \mathcal{F}_{k-1}) = \sigma^2 X_{k-1}$

Remark

- Each INAR(1) process is a BPI with Bernoulli offsprings.
- Each BPI can be interpreted as an AR(1) process with non-negative autoregressive coefficient and conditionally heteroscedastic and dependent innovation.

A periodic integer-valued ARMA (PINARMA) model

$\{Y_t\}$ is called a PINARMA process with period $S \in \mathbb{N}$ and autoregressive and moving average orders $\boldsymbol{p} = (p_s) \in \mathbb{N}_0^S$ and $\boldsymbol{q} = (q_s) \in \mathbb{N}_0^S$ if it satisfies the stochastic recursion

$$Y_{kS+s} = \sum_{i=1}^{p_s} \xi_{s,i}^k \circ Y_{kS+s-i} + \sum_{j=0}^{q_s} \eta_{s,j}^k \circ \varepsilon_{kS+s-j},$$

$k \in \mathbb{Z}$, where $\{\xi_{s,i}^k \circ\}$ and $\{\eta_{s,j}^k \circ\}$ are i.i.d. thinning operators with mean $\alpha_{s,i}, \beta_{s,j} \geq 0$ for all $i = 1, \ldots, p_s$, $j = 1, \ldots, q_s$, $s = 1, \ldots, S$.

Autoregressive coefficients: $\{\alpha_{s,i} \mid i = 1, \ldots, p_s, s = 1, \ldots, S\}$
Moving average coefficients: $\{\beta_{s,j} \mid j = 1, \ldots, q_s, s = 1, \ldots, S\}$
Input process $\{\varepsilon_t\}$: periodic sequence of $\mathbb{N}_0$-valued r.v.'s, i.e., for each s, $\{\varepsilon_{kS+s} \mid k \in \mathbb{Z}\}$ are i.i.d.r.v.'s
Input parameters: $\lambda_s = \mathrm{E}(\varepsilon_{kS+s}) \geq 0$, $\sigma_s^2 = \mathrm{Var}(\varepsilon_{kS+s}) > 0$.

Periodic time series models

Periodic and seasonal AR, PAR(1,S), model:

$$X_{kS+s} = \alpha_s X_{kS+s-1} + \beta_s X_{kS+s-S} + \varepsilon_{kS+s}$$

Examples for PINARMA models (with Bernoulli offspring):

- PINAR(1) model, Monteiro et al. (2010)
- INAR(1)$_S$ model, Bourguignon, Vasconcellos, Reisen, I (2014), Buteikis and Leipus (2020)
- PINAR(2) model with periodic immigration, Morina et al. (2011)
- PINAR(1, S) model, Filho et al. and I (2021)
- PINARMA(p,q) model Bentarzi and Aries (2020)

Periodic models (PARMA): Gladyshev (1961), Gardner et al. (2006), and Hurd and Miamee (2007). Inference: Lund and Basawa (2000) and Basawa and Lund (2001).

Seasonal models (SARMA): Chatfield and Prothero (1973)

SPARIMA models: Basawa, Lund, and Shao (2004), Koopman et al. (2006), Hindrayanto et al. (2010)

BP(I) in varying environment

General inhomogeneous branching processes:

- Domain of peer-to-peer file sharing networks, Adar and Huberman (2000), Zhao et al. (2005)
- Modeling biodiversity or macroevolution, Aldous and Popovic (2005), Haccou and Iwasa (1996)
- Epidemic–type Aftershock Sequence (ETAS) in seismology, Farrington et al. (2003)

Heterogeneous INAR models (Bernoulli offspring):

- Understanding and predicting consumers' buying behavior, Böckenholt (1999)
- Modeling the premium in the bonus-malus scheme of car insurance, Gourieroux and Jasiak (2004)

The supercritical case was studied by Goettge (1976), Cohn and Hering (1983), Jagers and Nerman (1985), D'Souza and Biggins (1992, 1993), D'Souza (1994).

Vector representation by multitype BPI (MBPI)

Define the matricial thinning operator $\Xi\circ = (\xi_{i,j}\circ)$, Latour (1997), as

$$(\Xi \circ \boldsymbol{Y})_i = \sum_{j=1}^{S} \xi_{i,j} \circ Y_j, \quad i = 1, \dots, S$$

We only assume that Y_j and $\xi_{i,j}\circ$ are independent for all i, j.
Example: $\mathrm{PINAR}_2(2,3)$ model

$$\begin{aligned} Y_{kS+2} &= \xi^k_{2,1} \circ Y_{kS+1} + \xi^k_{2,2} \circ Y_{kS} + \xi^k_{2,3} \circ Y_{kS-1} + \varepsilon_{kS+2} \\ Y_{kS+1} &= \qquad\qquad\qquad\qquad \xi^k_{1,1} \circ Y_{kS} + \xi^k_{1,2} \circ Y_{kS-1} + \varepsilon_{kS+1} \end{aligned}$$

Let $\boldsymbol{Y}_k = (Y_{kS+2}, Y_{kS+1})^\top$ and $\boldsymbol{\varepsilon}_k = (\varepsilon_{kS+2}, \varepsilon_{kS+1})^\top$ and let

$$\boldsymbol{A}^k_0\circ := \begin{bmatrix} 0\circ & \xi^k_{2,1}\circ \\ 0\circ & 0\circ \end{bmatrix}, \quad \boldsymbol{A}^k_1\circ := \begin{bmatrix} \xi^k_{2,2}\circ & \xi^k_{2,3}\circ \\ \xi^k_{1,1}\circ & \xi^k_{1,2}\circ \end{bmatrix}$$

$\boldsymbol{A}^k_0\circ$ is a strictly upper triangular matricial thinning operator.
Then

$$\boldsymbol{Y}_k = \boldsymbol{A}^k_0 \circ \boldsymbol{Y}_k + \boldsymbol{A}^k_1 \circ \boldsymbol{Y}_{k-1} + \boldsymbol{\varepsilon}_k$$

A vector integer-valued ARMA (VINARMA) model

$\{\boldsymbol{Y}_k\}$ is called a VINARMA process of dimension $S \in \mathbb{N}$ and autoregressive and moving average orders $p, q \in \mathbb{N}_0$ if it satisfies the stochastic recursion

$$\boldsymbol{Y}_k = \sum_{i=0}^{p} \boldsymbol{A}_i^k \circ \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} \boldsymbol{B}_j^k \circ \boldsymbol{\varepsilon}_{k-j} \qquad k \in \mathbb{Z}$$

where $\{\boldsymbol{A}_i^k \circ\}$ and $\{\boldsymbol{B}_j^k \circ\}$ are i.i.d. matricial thinning operators with finite mean matrices A_i and B_j, $i = 0, \ldots, p$, $j = 0, \ldots, q$.
The input process $\{\boldsymbol{\varepsilon}_k\}$ is a sequence of i.i.d. $\mathbb{N}_0^S$-valued r.v.'s.
Suppose that $\boldsymbol{A}_0^k \circ$'s are strictly upper triangular and $\boldsymbol{B}_0^k \circ$'s are upper triangular.

Implicit model: $\boldsymbol{A}_0^k \circ \neq O \circ$
$\{\boldsymbol{Y}_k\}$ is a generalized MBPI with ordered types $\{1, \ldots, S\}$ where an individual of type j may have offspring of type i at the 'same' time if $i > j$.

Equivalence of PINARMA and VINARMA models

Let $[\boldsymbol{p}] := \max_{1\le s\le S}[(p_s + S - s)/S]$, $[\cdot]$ is the integer part

Theorem

Let the $\mathbb{N}_0$-valued stochastic process $\{Y_t\}$ be a non-anticipative solution to a PINARMA$_S(\boldsymbol{p}, \boldsymbol{q})$ model. Then, there exists a VINARMA$_S(p, q)$ model, where $p = [\boldsymbol{p}]$ and $q = [\boldsymbol{q}]$, of which the $\mathbb{N}_0^S$-valued stochastic process $\{\boldsymbol{Y}_k\}$, where $\boldsymbol{Y}_k := (Y_{kS+S}, Y_{kS+S-1}, \ldots, Y_{kS+1})^\top$, $k \in \mathbb{Z}$, is a non-anticipative solution.
Conversely, let the $\mathbb{N}_0^S$-valued stochastic process $\{\boldsymbol{Y}_k\}$ be a solution to the VINARMA$_S(p, q)$ model, then there exists a PINARMA$_S(\boldsymbol{p}, \boldsymbol{q})$ model of which the $\mathbb{N}_0$-valued stochastic process $\{Y_t\}$, where $Y_t := (\boldsymbol{Y}_k)_{S+1-s}$ provided that $t = kS + s$, $k \in \mathbb{Z}$ and $s \in \{1, \ldots, S\}$, for all $t \in \mathbb{Z}$, is a solution.

The vector model is implicit if there exist $s \in \{2, \ldots, S\}$ and $i \in \{1, \ldots, s-1\}$ that $\xi_{s,i}\circ \neq 0\circ$.

Notations and assumptions

Autoregressive and moving average polynomials:

$$P(z) := I - \sum_{i=0}^{p} z^i A_i, \quad Q(z) := \sum_{j=0}^{q} z^j B_j$$

Define the matrices $A := I - P(1) = \sum_{i=0}^{p} A_i$ and $B := Q(1) = \sum_{j=0}^{q} B_j$.

Assumption 1. In VINARMA$_S(p, q)$ model, all matricial thinning operators and the input process have a finite first moment.

Let $\rho(M)$ denote the spectral radius of a matrix M.
M is called stable if $\rho(M) < 1$.

Assumption 2. The sum $A = A_0 + A_1 + \ldots + A_p$ of the autoregressive coefficient matrices of VINARMA$_S(p, q)$ model is stable, i.e., $\rho(A) < 1$.

Existence of VINARMA process (MBPI view)

Theorem

Under the Assumptions, the VINARMA$_S(p,q)$ model has a unique non-anticipative solution $\{\boldsymbol{Y}_k\}$ that possesses a stationary mean vector. This process can be expressed as the almost sure and mean convergent infinite series

$$\boldsymbol{Y}_k = \sum_{j=1}^{\infty} \boldsymbol{Z}_k^{(j)} \qquad k \in \mathbb{Z}$$

where $\boldsymbol{Z}_k^{(n)}$ denotes the number of nth generation offspring of the input at time k. (generation representation)

$\{\boldsymbol{Y}_k\}$ is a strictly stationary and ergodic, pth order homogeneous Markov chain with mean vector $\boldsymbol{\mu}$, which satisfies the implicit non-negative linear vector equation

$$\boldsymbol{\mu} = A\boldsymbol{\mu} + B\boldsymbol{\lambda} \quad thus \quad \boldsymbol{\mu} = (I - A)^{-1} B\boldsymbol{\lambda}$$

Examples

Consider the $\text{PINAR}_S(\mathbf{1}_S)$ model with autoregressive coefficients $\alpha_s \geq 0$, $s = 1, \ldots, S$.

The characteristic polynomial of this model simplifies to $P(z) = z^S - \prod_{j=1}^S \alpha_j$ and the condition $\rho(A) < 1$ is equivalent to $\prod_{j=1}^S \alpha_j < 1$. The process can be supercritical in a season!

Consider the $\text{PINAR}_2((2,2))$ model with autoregressive coefficients $\alpha_{i,j}$, $i, j = 1, 2$. Then

$$A_0 = \begin{bmatrix} 0 & \alpha_{2,1} \\ 0 & 0 \end{bmatrix}, \; A_1 = \begin{bmatrix} \alpha_{2,2} & 0 \\ \alpha_{1,1} & \alpha_{1,2} \end{bmatrix}, \; A = A_0 + A_1 = \begin{bmatrix} \alpha_{2,2} & \alpha_{2,1} \\ \alpha_{1,1} & \alpha_{1,2} \end{bmatrix}$$

The characteristic polynomial is given by

$$P(z) = (z - \alpha_{2,2})(z - \alpha_{1,2}) - \alpha_{1,1}\alpha_{2,1}$$

A necessary and sufficient stationarity condition:

$$\alpha_{1,2}, \alpha_{2,2} < 1 \quad \alpha_{1,2} + \alpha_{2,2} - \alpha_{1,2}\alpha_{2,2} + \alpha_{1,1}\alpha_{2,1} < 1$$

The latter condition can be rewritten as $\alpha_{1,1}\alpha_{2,1} < (1 - \alpha_{1,2})(1 - \alpha_{2,2})$. See Darolles et al. (2019).

Representation of VINARMA processes

Introduce the vector martingale difference:

$$\boldsymbol{M}_k := (I - A_0)(\boldsymbol{Y}_k - \mathrm{E}(\boldsymbol{Y}_k \mid \mathcal{F}_{k-1}, \varepsilon_k))$$

Implicit VARMA$_S(p,q)$ representation with martingale difference perturbation:

$$\boldsymbol{Y}_k = \sum_{i=0}^{p} A_i \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} B_j \varepsilon_{k-j} + \boldsymbol{M}_k$$

Explicit form of VINARMA$_S(p,q)$ model:

$$\boldsymbol{Y}_k = (I - \boldsymbol{A}_0^k)^{\circ -1} \left(\sum_{i=1}^{p} \boldsymbol{A}_i^k \circ \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} \boldsymbol{B}_j^k \circ \varepsilon_{k-j} \right)$$

Explicit VARMA$_S(p,q)$ representation:

$$\boldsymbol{Y}_k = (I - A_0)^{-1} \left(\sum_{i=1}^{p} A_i \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} B_j \varepsilon_{k-j} + \boldsymbol{M}_k \right)$$

Moving average representation (TS view)

Consider the implicit non-negative linear matrix recursions:

$$C_n := \sum_{i=0}^{p} A_i C_{n-i} + B_n, \quad n \in \mathbb{N}_0,$$

and

$$D_n := \sum_{i=0}^{p} A_i D_{n-i} + \delta_{n,0} I, \quad n \in \mathbb{N}_0,$$

where $B_n := O$ if $n > q$ and $C_n = D_n = O$ if $n = -1, \ldots, -p$.

Under Assumption 2, the generating functions of $\{C_n\}$ and $\{D_n\}$ exist and they are equal to the matricial functions $P^{-1}(z)Q(z)$ and $P^{-1}(z)$ on the closed complex unit disc.

Theorem

Under the Assumptions, $\{\boldsymbol{Y}_k\}$ can be expressed as the almost surely and mean square convergent infinite series (two-part infinite moving average representation)

$$\boldsymbol{Y}_k = \sum_{i=0}^{\infty} C_i \boldsymbol{\varepsilon}_{k-i} + \sum_{j=0}^{\infty} D_j \boldsymbol{M}_{k-j} \qquad k \in \mathbb{Z}$$

where the coefficient $\{C_i\}$ and $\{D_j\}$ are non-negative matrices. If the matricial thinning operators have finite second moment, then $\{\boldsymbol{M}_k\}$ is a vector MDS with diagonal conditional heteroscedastic variance matrix

$$\mathrm{Var}(\boldsymbol{M}_k \mid \mathcal{F}_{k-1}, \boldsymbol{\varepsilon}_k) = \mathrm{diag}(V_0 \boldsymbol{X}_k + \sum_{i=1}^{p} V_i \boldsymbol{Y}_{k-i} + \sum_{j=0}^{q} W_j \boldsymbol{\varepsilon}_{k-j}),$$

where V_i, $i = 0, 1, \ldots, p$, and W_j, $j = 0, 1, \ldots, q$, are the variance matrices of the matricial thinning operators.

Moving average representation (TS view)

Theorem (cont.)

If $\{\varepsilon_k\}$ has a finite second moment, then $\{\boldsymbol{Y}_k\}$ is a weakly stationary process and the infinite series converges in the mean square. The two stochastic processes $\{\varepsilon_k\}$ and $\{\boldsymbol{M}_k\}$ are uncorrelated, and $\mathrm{Cov}(\boldsymbol{M}_k, \boldsymbol{Y}_j) = \mathrm{E}(\boldsymbol{M}_k \boldsymbol{Y}_j^\top) = O$ for all $j < k$, $j, k \in \mathbb{Z}$.

The processes $\{\varepsilon_k\}$ and $\{\boldsymbol{M}_k\}$ are uncorrelated but not independent!

The unconditional variance matrix $\Sigma_{\boldsymbol{M}} := \mathrm{Var}(\boldsymbol{M}_0)$ of the process $\{\boldsymbol{M}_k\}$ can be expressed as

$$\Sigma_{\boldsymbol{M}} = \mathrm{diag}\left(V\boldsymbol{\mu} + W\boldsymbol{\lambda}\right)$$

where $V := \sum_{i=0}^{p} V_i$ and $W := \sum_{j=0}^{q} W_j$.

Estimation in a sparse PINAR model

Consider the PINAR(1,S)= $\text{PINAR}_S((1, 0, \ldots, 0, 1)_S)$ model: $\alpha_{s,1} = \alpha_s$ and $\alpha_{s,S} = \beta_s$, $s = 1, \ldots, S$, with Bernoulli offspring
Parameters: $\boldsymbol{\vartheta} = (\vartheta_1^\top, \ldots, \vartheta_S^\top)^\top \in ((0,1)^2 \times (0,\infty)^2)^S$ where $\vartheta_s = (\alpha_s, \beta_s, \lambda_s, \sigma_s^2)^\top$, $s = 1, \ldots, S$, $4S$ parameters!

CQML method: Gaussian type penalty $L(\boldsymbol{\vartheta}) = \sum_{s=1}^S l_s(\vartheta_s)$ where

$$l_s(\vartheta_s) = \sum_{k=1}^{n-1} \log\{\sigma_{k,s}^2(\vartheta_s)\} + (Y_{kS+s} - m_{k,s}(\vartheta_s))^2 / \sigma_{k,s}^2(\vartheta_s)$$

where

$$m_{k,s}(\vartheta_s) = \alpha_s Y_{kS+s-1} + \beta_s Y_{kS+s-S} + \lambda_s$$

is the conditional mean and

$$\sigma_{k,s}^2(\vartheta_s) = \alpha_s(1-\alpha_s) Y_{kS+s-1} + \beta_s(1-\beta_s) Y_{kS+s-S} + \sigma_s^2$$

is the conditional variance, respectively.

Simulation study

The performance of the CQML method is investigated for small sample sizes with $S = 4, 7$ and $T = nS$ observations, $n = 50, 200, 500$.
The immigrations are independent and follow Poisson distribution with periodic mean λ_s, $s = 1, \ldots, S$.
The empirical bias (Bias) and mean square error (MSE) correspond to the mean of 1000 replications.
All simulations were carried out using the R software.

Scenarios:

- S1: $S = 4$, ϑ_0 = (0.10, 0.47, 4.00; 0.42, 0.25, 3.00; 0.23, 0.36, 2.00; 0.39, 0.30, 1.00)
- S2: $S = 7$, ϑ_0 = (0.31, 0.27,4.00; 0.35, 0.25, 3.30; 0.29, 0.26, 2.1; 0.29, 0.39, 2.50; 0.37, 0.27, 3.10; 0.29, 0.22, 2.60; 0.28, 0.33, 3.50)

Simulation results for S1

	$n = 50, T = 200$		$n = 200, T = 800$		$n = 500, T = 2000$	
	Bias	MSE	Bias	MSE	Bias	MSE
α_1	0.025	(0.018)	-0.002	(0.005)	-0.004	(0.003)
α_2	0.021	(0.014)	0.007	(0.004)	-0.004	(0.002)
α_3	0.009	(0.013)	0.002	(0.003)	0.001	(0.001)
α_4	0.004	(0.010)	0.006	(0.002)	0.000	(0.001)
β_1	-0.028	(0.015)	-0.008	(0.003)	0.002	(0.001)
β_2	-0.024	(0.017)	-0.007	(0.004)	-0.005	(0.002)
β_3	-0.035	(0.017)	-0.006	(0.004)	-0.003	(0.002)
β_4	-0.011	(0.015)	-0.005	(0.004)	-0.003	(0.002)
λ_1	0.081	(1.324)	0.085	(0.278)	-0.008	(0.151)
λ_2	0.003	(1.427)	0.017	(0.342)	0.068	(0.157)
λ_3	0.11	(1.16)	0.005	(0.208)	0.015	(0.091)
λ_4	0.058	(0.455)	-0.02	(0.096)	0.01	(0.042)

Simulation results for S2

	$n = 50, T = 350$		$n = 100, T = 700$		$n = 200, T = 1400$	
	Bias	MSE	Bias	MSE	Bias	MSE
α_1	0.017	(0.021)	0.005	(0.009)	0.003	(0.002)
α_2	0.013	(0.017)	0.011	(0.007)	0.004	(0.001)
α_3	0.003	(0.011)	0.006	(0.006)	0.000	(0.001)
α_4	0.005	(0.019)	0.007	(0.009)	0.003	(0.001)
α_5	0.012	(0.016)	-0.001	(0.007)	0.002	(0.002)
α_6	0.004	(0.013)	0.002	(0.006)	0.001	(0.001)
α_7	0.012	(0.019)	0.010	(0.010)	0.004	(0.002)
β_1	-0.032	(0.019)	-0.010	(0.008)	-0.003	(0.001)
β_2	-0.017	(0.016)	-0.014	(0.009)	-0.006	(0.002)
β_3	-0.038	(0.018)	-0.009	(0.008)	0.001	(0.002)
β_4	-0.028	(0.018)	-0.012	(0.007)	-0.006	(0.001)
β_5	-0.034	(0.018)	-0.008	(0.008)	-0.001	(0.002)
β_6	-0.016	(0.016)	-0.010	(0.009)	0.000	(0.002)
β_7	-0.019	(0.017)	-0.011	(0.008)	0.003	(0.001)
λ_1	0.139	(2.096)	0.052	(0.920)	0.002	(0.166)
λ_2	0.036	(1.997)	0.014	(0.849)	0.016	(0.177)
λ_3	0.188	(1.194)	0.012	(0.581)	0.013	(0.096)
λ_4	0.170	(1.110)	0.046	(0.522)	0.024	(0.090)
λ_5	0.177	(1.269)	0.083	(0.662)	-0.024	(0.120)
λ_6	0.069	(1.048)	0.044	(0.594)	-0.001	(0.108)
λ_7	0.047	(1.431)	0.045	(0.639)	-0.046	(0.121)

Application of PINAR(1, 7) model to Data1.
The parameters were estimated by the CQML method.

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
α	0.095 (0.039)	0.012 (0.074)	0.209 (0.045)	0.211 (0.061)	0.133 (0.060)	0.083 (0.056)	0.126 (0.045)
β	0.192 (0.047)	0.108 (0.054)	0.217 (0.055)	0.280 (0.056)	0.150 (0.061)	0.169 (0.053)	0.097 (0.051)
λ	3.031 (0.360)	8.209 (0.654)	3.364 (0.551)	4.361 (0.562)	6.182 (0.616)	6.739 (0.640)	5.649 (0.562)

standard errors are inside the parenthesis

Goodness-of-fit

Model	AIC	BIC
PINAR(1, 7)	7812.329	2363.470
PINAR(1)$_7$-Poisson	7906.284	2378.189
PINARMA$_7$(7, 0)-Poisson	7889.946	2552.595
PINGARCH$_7$(1, 1)	8385.838	2936.979

Fitting the model to Data2

Application of PINAR$(1, 7)$ model to Data2.
The parameters were estimated by the YW method.

	Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
α	0.065	0.224	0.280	0.337	0.547	0.398	0.346
β	-0.072	0.165	-0.014	0.171	0.196	0.207	0.218
λ	1.393	12.321	14.072	10.122	7.092	10.137	5.698

Conclusions

- Real count time series are presented which possess periodicity and seasonality.
- A general periodic non-negative integer-valued ARMA (PINARMA) model is proposed.
- A vector representation (VINARMA model) is introduced.
- Necessary and sufficient condition is given for the existence of VINARMA process.
- Two infinite series representations are derived.
- The proposed model is successfully fitted to real data.

Thank you for your attention!

References:

[1] Filho, Reisen, Bondon, Ispány, Melo and Serpa (2021) A periodic and seasonal statistical model for non-negative integer-valued time series with an application to dispensed medications in respiratory diseases. In: Appl Math Model, 545-558.

[2] Ispány, Bondon, Reisen and Filho (2023) Existence of a periodic and seasonal non-negative integer-valued autoregressive process. J Time Ser Anal, 45(6), 980-1005.

[3] Ispány, Bondon, Reisen: On the existence of thinning-based non-negative vector ARMA processes with application to periodic count time series, manuscript