Branching processes in nearly degenerate varying environment

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$$X_0 = 1, \quad X_n = \sum_{j=1}^{X_{n-1}} \xi_{n,j}, \quad n \in \mathbb{N},$$

where $\{\xi_{n,j}\}_{n,j\in\mathbb{N}}$ are nonnegative independent random variables such that for each n $\{\xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed; let ξ_n denote a generic copy.

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- Church, Fearn (1970s)
- Kersting (2020)
- Bhattacharya and Perlman, Cardona-Tobón and Palau, ...

Nearly degenerate branching processes

Let $f_n(s) = \mathbf{E}(s^{\xi_n})$ be the generating function (g.f.) of the offspring distribution in the nth generation and let $\overline{f}_n := f_n'(1) = \mathbf{E}(\xi_n)$.

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The process $(X_n)_{n\in\mathbb{N}}$ is *nearly degenerate* if

(C1) $\overline{f}_n < 1$, $\lim_{n\to\infty} \overline{f}_n = 1$, $\sum_{n=1}^{\infty}(1-\overline{f}_n) = \infty$,
(or more generally $\lim_{n\to\infty} \overline{f}_n = 1$, $\sum_{n=1}^{\infty}(1-\overline{f}_n)_+ = \infty$,
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(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0,\infty)$,
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hence $(X_n)_{n\in\mathbb{N}}$ dies out a.s.

Yaglom-type limit theorems

Theorem (Kevei & K [1, Theorem 1])

Let $\bar{f}_n = f_n'(1)$, $\bar{f}_{0,n} = \prod_{i=1}^{n} \bar{f}_i = \mathbf{E}(X_n)$ *and suppose that*

(C1) $\bar{f}_n < 1$, $\lim_{n\to\infty} \bar{f}_n = 1$, $\sum_{n=1}^{\infty}(1-\bar{f}_n) = \infty$,

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0,\infty)$,

(C3) *if* $\nu > 0$, *then* $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$.

Then

$$\mathcal{L}(X_n | X_n > 0) \xrightarrow{\mathcal{D}} \mathrm{Geom}\left(\frac{2}{2+\nu}\right), \quad \text{as } n \to \infty.$$

Example (Linear fractional offspring distribution)

Let $\mathbf{P}(\xi = k) = a(1-p)^{k-1}p, \quad \mathbf{P}(\xi = 0) = 1 - a$ be the probability distribution of a linear fractional variable.

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$$\frac{1}{1-f(s)} = \frac{1}{\bar{f}(1-s)} + \frac{1}{2}\frac{f''(1)}{\bar{f}^2},$$

and with $f_{j+1} \circ f_{j+2} \circ \cdots \circ f_n(s) =: f_{j,n}(s)$

$$\frac{1}{1-f_{0,n}(s)} = \frac{1}{1-f_1(f_{1,n}(s))} = \frac{1}{\bar{f}_1(1-f_{1,n}(s))} + \frac{1}{2}\frac{f_1''(1)}{\bar{f}_1^2}$$

$$\frac{1}{1-f_{0,n}(s)} = \frac{1}{\bar{f}_{0,n}(1-s)} + \frac{1}{2}\sum_{k=1}^{n}\frac{f_k''(1)}{\bar{f}_k\bar{f}_{0,k}}$$

$$\frac{\bar{f}_{0,n}}{1-f_{0,n}(s)} = \frac{1}{1-s} + \frac{1}{2}\sum_{k=1}^{n}\bar{f}_{k,n}\frac{f_k''(1)}{\bar{f}_k} \to \frac{1}{1-s} + \frac{\nu}{2}$$

Proof of the general case - shape function

For a g.f. f, with mean $\bar{f}$ and $f''(1) < \infty$, define the *shape function* as

$$\varphi(s) = \frac{1}{1 - f(s)} - \frac{1}{\bar{f}(1 - s)},\ 0 \leq s < 1, \quad \varphi(1) = \frac{f''(1)}{2\,(\bar{f})^2}.$$

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Let φ_j be the shape function of f_j. By definition of $f_{j,n}$, iteration gives

$$\frac{1}{1 - f_{j,n}(s)} = \frac{1}{\bar{f}_{j,n}(1-s)} + \varphi_{j,n}(s), \quad \varphi_{j,n}(s) := \sum_{k=j+1}^{n} \frac{\varphi_k(f_{k,n}(s))}{\bar{f}_{j,k-1}}.$$

Functional limit theorems for BPVEs

Theorem (Kevei & K [2, Theorem 1])

For a BPVE $(X_n)_{n\in\mathbb{N}}$ *satisfying conditions*

(C1') $\overline{f}_n = 1 - \frac{1}{n}$, $n \geq 2$,

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\overline{f}_n} = \nu \in [0, \infty)$,

(C3) *if* $\nu > 0$, *then* $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\overline{f}_n} = 0$,

for every $0 < \varepsilon < 1$,

$$\mathcal{L}((X_{\lfloor nt \rfloor})_{t\geq\varepsilon} | X_n > 0) \xrightarrow{\mathcal{D}} \mathcal{L}((Z(\log t))_{t\geq\varepsilon} | Z(0) > 0),$$

where $(Z(t))_{t\geq\log\varepsilon}$ *is a simple birth and death process with initial distribution* $Z(\log\varepsilon) \sim \mathrm{Geom}(\frac{2}{2+\nu})$, *birth rate* $\lambda = \frac{\nu}{2}$ *and death rate* $\mu = 1 + \frac{\nu}{2}$.

The distribution Geom$(\cdot)$ is the extremal quasi-stationary distribution of the birth-and-death process Z: $\mathbf{E}_\varepsilon[s^{Z(0)} | Z(0) > 0] = \mathbf{E}_\varepsilon(s^{Z(\log\varepsilon)})$.

For $a \in [0,1]$, $\nu \geq 0$, introduce the notation $h_a(s) = 1 - a\left(\frac{1}{1-s} + \frac{\nu}{2}(1-a)\right)^{-1}$. Put $p = \frac{2}{2+\nu}$, $q = 1-p$ and define the generating function

$$g_t(s) = \frac{ps}{1-sq} \frac{t^{-1}(1-h_t(0))}{1-h_t(0)qs} = \sum_{x=1}^{\infty} pq^{x-1} t^{-1}(1-h_t(0)^x)s^x.$$

Note that $t^{-1}(1-h_t(0)) = \frac{p}{1-qt} \to p$, as $t \downarrow 0$.

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Let $U(t) = Z(\log t)$.

Lemma

For $0 < u < t \leq 1$, $x_0 \in \mathbb{N}$ the g.f. of the conditional transition probabilities is

$$\mathbf{E}\left[s^{U(t)} | U(u) = x_0, U(1) > 0\right] = \frac{(h_{u/t}(s))^{x_0} - (h_{u/t}(h_t(0)s))^{x_0}}{1-(h_u(0))^{x_0}} =: k_{u,t;x_0}(s).$$

The family of laws with g.f.'s $(g_t)_{t\in(0,1]}$ is an entrance law for the transition g.f.'s $k_{u,t;x}$, that is, for $\varepsilon \in (0,1]$ and for any $t \in [\varepsilon, 1]$,

$$\mathbf{E}_{\varepsilon}\left[s^{U(t)} | U(1) > 0\right] = g_t(s).$$

Corollary

There exists a càdlàg Markov process $(\widetilde{U}(t))_{t\in(0,1]}$ *such that its transition probabilities are given by the generating functions*

$$k_{u,t;x}(s) = \mathbf{E}[s^{\widetilde{U}(t)}|\widetilde{U}(u) = x],$$

and $\mathbf{E}(s^{\widetilde{U}(t)}) = g_t(s)$ *for each* $t \in (0,1]$.

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Corollary

For a nearly degenerate BPVE $(X_n)_{n\in\mathbb{N}}$,

$$\mathcal{L}((X_{\lfloor \frac{n}{t} \rfloor})_{t\geq 1} | X_n > 0) \xrightarrow{\mathcal{D}} \mathcal{L}((Z(-\log t))_{t\geq 1} | Z(0) > 0), \quad \text{as } n \to \infty.$$

Branching processes in varying environment with immigration

A *branching process in varying environment with immigration* (BPVEI) $(Y_n)_{n\in\mathbb{N}_0}$ is defined as follows. Let

$$Y_0 = 0, \quad \text{and } Y_n = \sum_{j=1}^{Y_{n-1}} \xi_{n,j} + \varepsilon_n, \quad n \in \mathbb{N},$$

where $\{\xi_n, \xi_{n,j}, \varepsilon_n\}_{n,j\in\mathbb{N}}$ are non-negative, independent random variables such that $\{\xi_n, \xi_{n,j}\}_{j\in\mathbb{N}}$ are identically distributed.

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- Györfi, Ispány, Pap, Varga (2007) - INAR
- Kevei (2011)

Nearly degenerate BPVEIs

For $n, k \in \mathbb{N}$ we define

$$m_{n,k} = \mathbf{E}[\varepsilon_n(\varepsilon_n - 1)\cdots(\varepsilon_n - k + 1)].$$

Recall that

(C1') $\bar{f}_n = 1 - \frac{1}{n}$, $n \geq 2$,

(C2) $\lim_{n\to\infty} \frac{f_n''(1)}{1-\bar{f}_n} = \nu \in [0, \infty)$,

(C3) if $\nu > 0$, then $\lim_{n\to\infty} \frac{f_n'''(1)}{1-\bar{f}_n} = 0$,

and introduce the conditions

(C4) $\lim_{n\to\infty} \frac{m_{n,k}}{k!(1-\bar{f}_n)} = \lambda_k, \quad k = 1, 2, \ldots, K$ and $\lambda_K = 0$, or

(C4') $\lim_{n\to\infty} \frac{m_{n,k}}{k!(1-\bar{f}_n)} = \lambda_k, \quad k = 1, 2, \ldots$ and $\limsup_{n\to\infty} \lambda_n^{1/n} \leq 1$.

Theorem (Kevei & K [1, Theorem 2 & 3] [2, Theorem 5])

Suppose that the BPVEI $(Y_n)_{n\in\mathbb{N}}$ *satisfies conditions (C1'), (C2), (C3) and (C4) or (C4'). Then, for any* $0 < \varepsilon \le 1$,

$$\mathcal{L}((Y_{\lfloor nt \rfloor})_{t\ge\varepsilon}) \xrightarrow{\mathcal{D}} \mathcal{L}((W(\log t))_{t\ge\varepsilon}), \quad \text{as } n\to\infty,$$

where $(W(u))_{u\ge\log\varepsilon}$ *is a continuous time branching process with immigration with initial distribution having g.f.*

$$\log g_Y(s) = \begin{cases} -\sum_{k=1}^{\kappa} \frac{2^k\lambda_k}{\nu^k}\left(\log\left(1+\frac{\nu}{2}(1-s)\right)+\sum_{i=1}^{k-1}\frac{\nu^i}{i2^i}(s-1)^i\right), & \nu>0,\\ \sum_{k=1}^{\kappa}\frac{\lambda_k}{k}(s-1)^k, & \nu=0,\end{cases}$$

with offspring and immigration rates $\alpha = 1+\nu$, $\beta = \sum_{k=1}^{\kappa}(-1)^{k+1}\lambda_k$, *and*

$$f(s) = (1+\nu)^{-1}\left(1+\frac{\nu}{2}+\frac{\nu}{2}s^2\right), \quad h(s) = 1+\beta^{-1}\sum_{k=1}^{\kappa}\lambda_k(s-1)^k,$$

g.f.'s, where $\kappa = K-1$ *(C4) or* $\kappa=\infty$ *(C4').*

The limiting process

Let $(W(t))_{t\geq -w}$ denote the resulting continuous time branching process with immigration on the time interval $t \in [-w, \infty)$, $w \geq 0$, and introduce the notation $G(s,t) = \mathbf{E}(s^{W(t+u)} | W(u) = 0)$, $t \geq 0$, $s \in [0,1]$. Then $G(s,t)$ satisfies the Kolmogorov forward equation

$$\frac{\partial}{\partial t} G(s,t) = a(s) \frac{\partial}{\partial s} G(s,t) + b(s) G(s,t)$$

with boundary condition $G(s,0) = 1$, where

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with boundary condition $G(s,0)=1$, where

$$a(s)=\alpha\left(f(s)-s\right),\quad b(s)=\beta\left(h(s)-1\right).$$

Corollary

Under the assumptions of the previous theorem,

$$\mathcal{L}((Y_{\lfloor \frac{n}{t}\rfloor})_{t\geq 1})\xrightarrow{\mathcal{D}}\mathcal{L}((W(-\log t))_{t\geq 1}),\quad \textit{as } n\to\infty.$$

References

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P. Kevei and K. Kubatovics.
Functional limit theorem for branching processes in nearly degenerate varying environment.
arXiv:2412.03325, 2024.

Thank you for your attention!