Axiomatic characterisation of generalized ψ-estimators

Mátyás Barczy

HUN-REN–SZTE Analysis and Applications Research Group
University of Szeged, Hungary

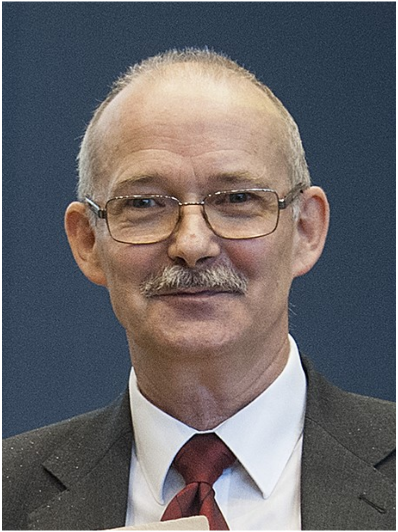
19th International Summer Conference on Probability and Statistics
Sofia, Bulgaria

24–27 July, 2025

Joint work with Zsolt Páles (University of Debrecen, Hungary)

Supported by the project TKP2021-NVA-09.

Coauthor



Zsolt Páles
Budapest, 2016

Outline of my talk

- M-estimators, ψ-estimators (also called Z-estimators).
- We introduce *generalized* ψ-estimators.
- Questions that we investigate:
  1. existence and uniqueness,
  2. axiomatic characterisation.
- Main tool in the proof of characterisation theorem:
  a separation theorem for Abelian subsemigroups.

This topic has a close connection to the *theory of means in analysis.*

M-estimators, ψ-estimators (Z-estimators)

The M-estimators were introduced by Huber (1963, 1967).

The letter M refers to ”maximum likelihood-type”.

Let

- $(X, \mathcal{X})$ be a measurable space (sample space),
- Θ be a Borel subset of $\mathbb{R}$ (parameter set),
- $\varrho : X \times \Theta \to \mathbb{R}$ be a function, measurable in its first variable, i.e., for each $t \in \Theta$, the function $X \ni x \mapsto \varrho(x, t)$ is measurable,
- $(\xi_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables with values in X such that the law of ξ_1 depends on an unknown parameter $\vartheta \in \Theta$.

M-estimators, ψ-estimators (Z-estimators)

The M-estimators were introduced by Huber (1963, 1967).

The letter M refers to "maximum likelihood-type".

Let

- $(X, \mathcal{X})$ be a measurable space (sample space),
- Θ be a Borel subset of $\mathbb{R}$ (parameter set),
- $\varrho : X \times \Theta \to \mathbb{R}$ be a function, measurable in its first variable, i.e., for each $t \in \Theta$, the function $X \ni x \mapsto \varrho(x, t)$ is measurable,
- $(\xi_n)_{n \in \mathbb{N}}$ be a sequence of i.i.d. random variables with values in X such that the law of ξ_1 depends on an unknown parameter $\vartheta \in \Theta$.

For each $n \in \mathbb{N}$, Huber introduced an estimator of ϑ based on the observations $\xi_1, \ldots, \xi_n$ as a solution $\widehat{\vartheta}_n := \widehat{\vartheta}_n(\xi_1, \ldots, \xi_n)$ of the minimization problem:

$$\inf_{t \in \Theta} \sum_{i=1}^{n} \varrho(\xi_i, t),$$

provided that such a solution exists.

M-estimators, ψ-estimators (Z-estimators)

One calls $\widehat{\vartheta}_n$ an M-estimator of the unknown parameter $\vartheta \in \Theta$ based on the i.i.d. observations $\xi_1, \ldots, \xi_n$.

Under suitable regularity assumptions, this minimization problem can be solved by setting the derivative of the objective function (w.r.t the unknown parameter) equal to 0:

$$\sum_{i=1}^{n} \partial_2 \varrho(\xi_i, t) = 0, \qquad t \in \Theta.$$

In the statistical literature, $\partial_2 \varrho$ is often denoted by ψ, and hence, in this case, an M-estimator is often called a ψ-estimator.

While other authors call it a Z-estimator (the letter Z refers to ”zero”).

For a detailed exposition of M-estimators and ψ-estimators, see, e.g., Kosorok (2008) or van der Vaart (1998).

Generalized ψ-estimators

Let

- X be a nonempty set,
- Θ be a nonempty open interval of $\mathbb{R}$,
- $\Psi(X, \Theta)$ be the set

$$\Big\{\psi : X \times \Theta \to \mathbb{R} : \text{ for each } x \in X, \text{ there exist } t_+, t_- \in \Theta \\ \text{ such that } t_+ < t_- \text{ and } \psi(x, t_+) > 0 > \psi(x, t_-)\Big\}.$$

That is, $\psi \in \Psi(X, \Theta)$ if for each $x \in X$, the function $\Theta \ni t \mapsto \psi(x, t)$ changes sign (from positive to negative) on the interval Θ at least once.

Generalized ψ-estimators

Let

- X be a nonempty set,
- Θ be a nonempty open interval of $\mathbb{R}$,
- $\Psi(X, \Theta)$ be the set

$$\Big\{\psi : X \times \Theta \to \mathbb{R} : \text{ for each } x \in X, \text{ there exist } t_+, t_- \in \Theta \text{ such that } t_+ < t_- \text{ and } \psi(x, t_+) > 0 > \psi(x, t_-)\Big\}.$$

That is, $\psi \in \Psi(X, \Theta)$ if for each $x \in X$, the function $\Theta \ni t \mapsto \psi(x, t)$ changes sign (from positive to negative) on the interval Θ at least once.

Given a function $\psi \in \Psi(X, \Theta)$, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$, let us consider the equation:

$$\psi_{\boldsymbol{x}}(t) := \sum_{i=1}^{n} \psi(x_i, t) = 0, \qquad t \in \Theta.$$

Generalized ψ-estimators

Task: find necessary as well as sufficient conditions for the unique solvability of this equation.

In a broader context, find necessary as well as sufficient conditions for the existence of a point of sign change for the function $\psi_{\boldsymbol{x}}$.

Point of sign change

Given a nonempty open interval Θ of $\mathbb{R}$ and a function $f : \Theta \to \mathbb{R}$, we say that $\vartheta \in \Theta$ is a point of sign change (of decreasing type) for f if

$$f(t) > 0 \quad \text{for} \quad t < \vartheta, \qquad \text{and} \qquad f(t) < 0 \quad \text{for} \quad t > \vartheta.$$

Generalized ψ-estimators

Task: find necessary as well as sufficient conditions for the unique solvability of this equation.

In a broader context, find necessary as well as sufficient conditions for the existence of a point of sign change for the function $\psi_{\boldsymbol{x}}$.

Point of sign change

Given a nonempty open interval Θ of $\mathbb{R}$ and a function $f : \Theta \to \mathbb{R}$, we say that $\vartheta \in \Theta$ is a point of sign change (of decreasing type) for f if

$$f(t) > 0 \quad \text{for} \quad t < \vartheta, \qquad \text{and} \qquad f(t) < 0 \quad \text{for} \quad t > \vartheta.$$

Remark.

(i) There can exist at most one point of sign change for f.

(ii) If f is continuous and $\vartheta \in \Theta$ is a point of sign change for f, then ϑ is the unique zero of f.

Generalized ψ-estimators

Generalized ψ-estimator

We say that a function $\psi \in \Psi(X, \Theta)$ *possesses the property* $[T_n]$ *(briefly,* ψ *is a* T_n*-function) for some* $n \in \mathbb{N}$ if there exists a mapping $\vartheta_{n,\psi} : X^n \to \Theta$ such that, for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and $t \in \Theta$,

$$\psi_{\boldsymbol{x}}(t) = \sum_{i=1}^{n} \psi(x_i, t) \begin{cases} > 0 & \text{if } t < \vartheta_{n,\psi}(\boldsymbol{x}), \\ < 0 & \text{if } t > \vartheta_{n,\psi}(\boldsymbol{x}). \end{cases}$$

Generalized ψ-estimators

Generalized ψ-estimator

We say that a function $\psi \in \Psi(X, \Theta)$ *possesses the property* $[T_n]$ *(briefly,* ψ *is a* T_n*-function) for some* $n \in \mathbb{N}$ if there exists a mapping $\vartheta_{n,\psi} : X^n \to \Theta$ such that, for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and $t \in \Theta$,

$$\psi_{\boldsymbol{x}}(t) = \sum_{i=1}^{n} \psi(x_i, t) \begin{cases} > 0 & \text{if } t < \vartheta_{n,\psi}(\boldsymbol{x}), \\ < 0 & \text{if } t > \vartheta_{n,\psi}(\boldsymbol{x}). \end{cases}$$

In other words, for each $\boldsymbol{x} \in X^n$, the value $\vartheta_{n,\psi}(\boldsymbol{x})$ is a point of sign change for the function $\psi_{\boldsymbol{x}}$.

We call $\vartheta_{n,\psi}(\boldsymbol{x})$ as a generalized ψ-estimator for some unknown parameter in Θ based on the realization $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$.

In many cases, instead of $\vartheta_{n,\psi}$, we simply write ϑ_n.

Generalized ψ-estimators

Generalized ψ-estimator

We say that a function $\psi \in \Psi(X, \Theta)$ *possesses the property* $[T_n]$ *(briefly,* ψ *is a* T_n*-function) for some* $n \in \mathbb{N}$ if there exists a mapping $\vartheta_{n,\psi} : X^n \to \Theta$ such that, for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$ and $t \in \Theta$,

$$\psi_{\boldsymbol{x}}(t) = \sum_{i=1}^{n} \psi(x_i, t) \begin{cases} > 0 & \text{if } t < \vartheta_{n,\psi}(\boldsymbol{x}), \\ < 0 & \text{if } t > \vartheta_{n,\psi}(\boldsymbol{x}). \end{cases}$$

In other words, for each $\boldsymbol{x} \in X^n$, the value $\vartheta_{n,\psi}(\boldsymbol{x})$ is a point of sign change for the function $\psi_{\boldsymbol{x}}$.

We call $\vartheta_{n,\psi}(\boldsymbol{x})$ as a generalized ψ-estimator for some unknown parameter in Θ based on the realization $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$.

In many cases, instead of $\vartheta_{n,\psi}$, we simply write ϑ_n.

Property $[Z_n]$ for some $n \in \mathbb{N}$

Property $[T_n]$ and $\sum_{i=1}^{n} \psi(x_i, \vartheta_n(\boldsymbol{x})) = 0$ for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in X^n$.

Existence and uniqueness

Necessary as well as sufficient conditions for $[T_n]$, $n \in \mathbb{N}$

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $\psi \in \Psi(X, \Theta)$ be a T_1-function.

(i) *Necessity*: If ψ is a T_n-function for infinitely many $n \in \mathbb{N}$, then, for each $x, y \in X$ with $\vartheta_1(x) < \vartheta_1(y)$, the auxiliary function

$$(\vartheta_1(x), \vartheta_1(y)) \ni t \mapsto -\frac{\psi(x,t)}{\psi(y,t)} \qquad (*)$$

is increasing.

Existence and uniqueness

Necessary as well as sufficient conditions for $[T_n]$, $n \in \mathbb{N}$

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $\psi \in \Psi(X, \Theta)$ be a T_1-function.

(i) *Necessity*: If ψ is a T_n-function for infinitely many $n \in \mathbb{N}$, then, for each $x, y \in X$ with $\vartheta_1(x) < \vartheta_1(y)$, the auxiliary function

$$(\vartheta_1(x), \vartheta_1(y)) \ni t \mapsto -\frac{\psi(x,t)}{\psi(y,t)} \qquad (*)$$

is increasing.

(ii) *Sufficiency*: If

- ψ is a Z_1-function (i.e., $\psi(x, \vartheta_1(x)) = 0$, $x \in X$),
- for each $x, y \in X$ with $\vartheta_1(x) < \vartheta_1(y)$, the auxiliary function $(*)$ is *strictly* increasing,

then ψ is a T_n-function for each $n \in \mathbb{N}$.

Existence and uniqueness

Necessary as well as sufficient conditions for $[T_n]$, $n \in \mathbb{N}$

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $\psi \in \Psi(X, \Theta)$ be a T_1-function.

(i) *Necessity*: If ψ is a T_n-function for infinitely many $n \in \mathbb{N}$, then, for each $x, y \in X$ with $\vartheta_1(x) < \vartheta_1(y)$, the auxiliary function

$$(\vartheta_1(x), \vartheta_1(y)) \ni t \mapsto -\frac{\psi(x,t)}{\psi(y,t)} \qquad (*)$$

is increasing.

(ii) *Sufficiency*: If

- ψ is a Z_1-function (i.e., $\psi(x, \vartheta_1(x)) = 0$, $x \in X$),
- for each $x, y \in X$ with $\vartheta_1(x) < \vartheta_1(y)$, the auxiliary function $(*)$ is *strictly* increasing,

then ψ is a T_n-function for each $n \in \mathbb{N}$.

Remark. If $\psi \in \Psi(X, \Theta)$ is *continuous in its second variable* as well, then part (ii) of the previous result provides a sufficient condition for the existence and uniqueness of a *usual* ψ-estimator.

Existence and uniqueness

Another sufficient condition for $[T_n]$, $n \in \mathbb{N}$

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $\psi \in \Psi(X, \Theta)$ be a T_1-function.

If for all $x \in X$, the function $\Theta \ni t \mapsto \psi(x, t)$ is *strictly* decreasing, then

the function ψ has the property $[T_n]$ for each $n \in \mathbb{N}$.

Below, we provide some examples.

Examples

- empirical α-quantile, where $\alpha \in (0,1)$:

$$\psi(x,t) := \begin{cases} \alpha & \text{if } t < x, \\ 0 & \text{if } t = x, \\ \alpha - 1 & \text{if } t > x. \end{cases}$$

  Special case: empirical median with $\alpha = 1/2$.

- empirical α-expectile, where $\alpha \in (0,1)$:

$$\psi(x,t) := \begin{cases} \alpha(x-t) & \text{if } t < x, \\ 0 & \text{if } t = x, \\ (1-\alpha)(x-t) & \text{if } t > x. \end{cases}$$

  It is a smoothed version of empirical α-quantile.

Examples

- empirical α-quantile, where $\alpha \in (0, 1)$:

$$\psi(x, t) := \begin{cases} \alpha & \text{if } t < x, \\ 0 & \text{if } t = x, \\ \alpha - 1 & \text{if } t > x. \end{cases}$$

  Special case: empirical median with $\alpha = 1/2$.

- empirical α-expectile, where $\alpha \in (0, 1)$:

$$\psi(x, t) := \begin{cases} \alpha(x - t) & \text{if } t < x, \\ 0 & \text{if } t = x, \\ (1 - \alpha)(x - t) & \text{if } t > x. \end{cases}$$

  It is a smoothed version of empirical α-quantile.

- Mathieu-type ψ-estimator: $\psi(x, t) := \operatorname{sign}(x - t) f(|x - t|)$, $x, t \in \mathbb{R}$, where $f : [0, \infty) \to [0, \infty)$. In particular, Catoni-type ψ-estimator with $f(z) := \ln\left(1 + z + \frac{z^2}{2}\right)$, $z \geqslant 0$.

Examples

- empirical α-quantile, where $\alpha \in (0,1)$:

$$\psi(x,t) := \begin{cases} \alpha & \text{if } t < x, \\ 0 & \text{if } t = x, \\ \alpha - 1 & \text{if } t > x. \end{cases}$$

  Special case: empirical median with $\alpha = 1/2$.

- empirical α-expectile, where $\alpha \in (0,1)$:

$$\psi(x,t) := \begin{cases} \alpha(x-t) & \text{if } t < x, \\ 0 & \text{if } t = x, \\ (1-\alpha)(x-t) & \text{if } t > x. \end{cases}$$

  It is a smoothed version of empirical α-quantile.

- Mathieu-type ψ-estimator: $\psi(x,t) := \operatorname{sign}(x-t) f(|x-t|)$, $x, t \in \mathbb{R}$, where $f : [0,\infty) \to [0,\infty)$. In particular, Catoni-type ψ-estimator with $f(z) := \ln\left(1 + z + \frac{z^2}{2}\right)$, $z \geqslant 0$.

- MLE for absolutely continuous distributions: $\psi(x,t) := \partial_2(\ln(f(x,t))) = \frac{\partial_2 f(x,t)}{f(x,t)}$, $(x,t) \in \mathcal{X}_f \times \Theta$, where, for each $t \in \Theta$, the function $\mathbb{R} \ni x \mapsto f(x,t)$ is a density function.

Motivation for searching for axiomatic characterization

We present a question that prompted us to search for an axiomatic characterization of generalized ψ-estimators.

Question. Let $X := (0, \infty)$, $\Theta := (0, \infty)$, and define the estimator $\kappa : \bigcup_{n=1}^{\infty} (0, \infty)^n \to (0, \infty)$ by

$$\kappa(x_1, \ldots, x_n) := \frac{1}{2}\Big(\frac{x_1 + \cdots + x_n}{n} + \sqrt[n]{x_1 \cdots x_n}\Big), \quad n \in \mathbb{N},\ x_1, \ldots, x_n > 0.$$

Does there exist a function $\psi \in \Psi((0, \infty), (0, \infty))$ such that

$$\kappa(x_1, \ldots, x_n) = \vartheta_{n,\psi}(x_1, \ldots, x_n), \qquad n \in \mathbb{N},\ x_1, \ldots, x_n > 0\ ?$$

Motivation for searching for axiomatic characterization

We present a question that prompted us to search for an axiomatic characterization of generalized ψ-estimators.

Question. Let $X := (0,\infty)$, $\Theta := (0,\infty)$, and define the estimator $\kappa : \bigcup_{n=1}^{\infty}(0,\infty)^n \to (0,\infty)$ by

$$\kappa(x_1,\ldots,x_n) := \frac{1}{2}\Big(\frac{x_1+\cdots+x_n}{n} + \sqrt[n]{x_1\cdots x_n}\Big), \quad n\in\mathbb{N},\ x_1,\ldots,x_n>0.$$

Does there exist a function $\psi \in \Psi((0,\infty),(0,\infty))$ such that

$$\kappa(x_1,\ldots,x_n) = \vartheta_{n,\psi}(x_1,\ldots,x_n), \qquad n\in\mathbb{N},\ x_1,\ldots,x_n>0\ ?$$

In other words,

is κ a generalized ψ-estimator with some function ψ?

We will answer this question later on.

Motivation for searching for axiomatic characterization

More generally, one can formulate the following problem:

given an *arbitrary* estimator κ for the unknown parameter $\vartheta \in \Theta$, can one find a function $\psi : X \times \Theta \to \mathbb{R}$ such that

$$\kappa \text{ is a generalized } \psi\text{-estimator?}$$

A similar question can be formulated for (usual) ψ-estimators as well.

Motivation for searching for axiomatic characterization

More generally, one can formulate the following problem:

given an *arbitrary* estimator κ for the unknown parameter $\vartheta \in \Theta$, can one find a function $\psi : X \times \Theta \to \mathbb{R}$ such that

$$\kappa \text{ is a generalized } \psi\text{-estimator?}$$

A similar question can be formulated for (usual) ψ-estimators as well.

For the solutions, the characterisation theorem of quasi-arithmetic means due to Kolmogorov (1930), Nagumo (1930) and de Finetti (1931) served as motivation for us.

Axiomatic characterisation

Notations.

- Property $[T]$: property $[T_n]$ holds for each $n \in \mathbb{N}$,
- Property $[Z]$: property $[Z_n]$ holds for each $n \in \mathbb{N}$,
- Property $[C]$: ψ is continuous in its second variable,
- Given a function $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ and $m \in \mathbb{N}$, we will denote by M_m the restriction of M onto X^m, i.e.,

$$M\Big|_{X^m} =: M_m.$$

Axiomatic characterisation

Characterisation of generalized ψ-estimators

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

(i) There exists a function $\psi \in \Psi[T](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.

(ii) The function M possesses the following properties:

Axiomatic characterisation

Characterisation of generalized ψ-estimators

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

- (i) There exists a function $\psi \in \Psi[T](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.
- (ii) The function M possesses the following properties:
  - (a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$, that is, $M_n(x_1, \ldots, x_n) = M_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $x_1, \ldots, x_n \in X$ and each permutation π.

Axiomatic characterisation

Characterisation of generalized ψ-estimators

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

(i) There exists a function $\psi \in \Psi[T](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.

(ii) The function M possesses the following properties:

(a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$, that is, $M_n(x_1, \ldots, x_n) = M_n(x_{\pi(1)}, \ldots, x_{\pi(n)})$ for all $x_1, \ldots, x_n \in X$ and each permutation π.

(b) Mean-type property (internality): for each $n, k \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$, $(y_1, \ldots, y_k) \in X^k$, we have

$$\begin{aligned}\min(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)) &\leqslant M_{n+k}(x_1, \ldots, x_n, y_1, \ldots, y_k)\\ &\leqslant \max(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)),\end{aligned}$$

Axiomatic characterisation

Characterisation of generalized ψ-estimators

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

(i) There exists a function $\psi \in \Psi[T](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \dots, x_n) = M_n(x_1, \dots, x_n)$.

(ii) The function M possesses the following properties:

(a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$, that is, $M_n(x_1, \dots, x_n) = M_n(x_{\pi(1)}, \dots, x_{\pi(n)})$ for all $x_1, \dots, x_n \in X$ and each permutation π.

(b) Mean-type property (internality): for each $n, k \in \mathbb{N}$ and $(x_1, \dots, x_n) \in X^n$, $(y_1, \dots, y_k) \in X^k$, we have

$$\begin{aligned}\min(M_n(x_1, \dots, x_n), M_k(y_1, \dots, y_k)) &\leqslant M_{n+k}(x_1, \dots, x_n, y_1, \dots, y_k) \\ &\leqslant \max(M_n(x_1, \dots, x_n), M_k(y_1, \dots, y_k)),\end{aligned}$$

(c) Asymptotic idempotency: for each $k \in \mathbb{N}$ and $x_1, \dots, x_k, y \in X$,

$$\lim_{n\to\infty} M_{1+kn}(y, \underbrace{x_1, \dots, x_1}_{n}, \dots, \underbrace{x_k, \dots, x_k}_{n}) = M_k(x_1, \dots, x_k).$$

Answer to the motivating question

Motivating question: Is

$$\kappa(x_1,\dots,x_n) := \frac{1}{2}\left(\frac{x_1+\cdots+x_n}{n} + \sqrt[n]{x_1\cdots x_n}\right), \quad n\in\mathbb{N},\ x_1,\dots,x_n>0,$$

a generalized ψ-estimator with some function ψ?

Answer to the motivating question

Motivating question: Is

$$\kappa(x_1,\ldots,x_n) := \frac{1}{2}\left(\frac{x_1+\cdots+x_n}{n} + \sqrt[n]{x_1\cdots x_n}\right), \quad n\in\mathbb{N},\ x_1,\ldots,x_n>0,$$

a generalized ψ-estimator with some function ψ?

Answer: No. On the contrary, suppose that κ is a generalized ψ-estimator. Then, by the mean-type property with

$$(x_1,x_2) := (1,81) \qquad \text{and} \qquad (y_1,y_2) = (25,25),$$

we should obtain that

$$\min(\kappa(1,81),\kappa(25,25)) \leqslant \kappa(1,81,25,25) \leqslant \max(\kappa(1,81),\kappa(25,25)).$$

Answer to the motivating question

Motivating question: Is

$$\kappa(x_1,\ldots,x_n) := \frac{1}{2}\left(\frac{x_1+\cdots+x_n}{n} + \sqrt[n]{x_1\cdots x_n}\right), \quad n\in\mathbb{N},\ x_1,\ldots,x_n>0,$$

a generalized ψ-estimator with some function ψ?

Answer: No. On the contrary, suppose that κ is a generalized ψ-estimator. Then, by the mean-type property with

$$(x_1,x_2) := (1,81) \qquad \text{and} \qquad (y_1,y_2) = (25,25),$$

we should obtain that

$$\min(\kappa(1,81),\kappa(25,25)) \leqslant \kappa(1,81,25,25) \leqslant \max(\kappa(1,81),\kappa(25,25)).$$

However, the left hand side inequality above is violated, since

$$\kappa(1,81) = \kappa(25,25) = 25 \qquad \text{and} \qquad \kappa(1,81,25,25) = 24.$$

It leads us to a contradiction.

Axiomatic characterisation

Characterisation of (usual) ψ-estimators with ψ continuous in its 2nd variable

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

- (i) There exists a function $\psi \in \Psi[Z, C](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.
- (ii) The function M possesses the following properties:

Axiomatic characterisation

Characterisation of (usual) ψ-estimators with ψ continuous in its 2nd variable

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

(i) There exists a function $\psi \in \Psi[Z, C](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.

(ii) The function M possesses the following properties:

(a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$.

Axiomatic characterisation

Characterisation of (usual) ψ-estimators with ψ continuous in its 2nd variable

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

- (i) There exists a function $\psi \in \Psi[Z, C](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.
- (ii) The function M possesses the following properties:
  - (a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$.
  - (b) Strict mean-type property (strict internality): for each $n, k \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$, $(y_1, \ldots, y_k) \in X^k$, we have

$$\begin{aligned}\min(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)) &\leqslant M_{n+k}(x_1, \ldots, x_n, y_1, \ldots, y_k)\\ &\leqslant \max(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)),\end{aligned}$$

    and if $M_n(x_1, \ldots, x_n) \neq M_k(y_1, \ldots, y_k)$, then both inequalities are strict.

Axiomatic characterisation

Characterisation of (usual) ψ-estimators with ψ continuous in its 2nd variable

Let X be a nonempty set, Θ be a nonempty open interval of $\mathbb{R}$, and $M : \bigcup_{n=1}^{\infty} X^n \to \Theta$ be a function such that $\inf M_1(X) = \inf \Theta$ and $\sup M_1(X) = \sup \Theta$. Then the following two statements are equivalent:

- (i) There exists a function $\psi \in \Psi[Z, C](X, \Theta)$ such that, for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have $\vartheta_{n,\psi}(x_1, \ldots, x_n) = M_n(x_1, \ldots, x_n)$.
- (ii) The function M possesses the following properties:
  - (a) Symmetry: M_n is symmetric for each $n \in \mathbb{N}$.
  - (b) Strict mean-type property (strict internality): for each $n, k \in \mathbb{N}$ and $(x_1, \ldots, x_n) \in X^n$, $(y_1, \ldots, y_k) \in X^k$, we have

$$\min(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)) \leqslant M_{n+k}(x_1, \ldots, x_n, y_1, \ldots, y_k) \leqslant \max(M_n(x_1, \ldots, x_n), M_k(y_1, \ldots, y_k)),$$

    and if $M_n(x_1, \ldots, x_n) \neq M_k(y_1, \ldots, y_k)$, then both inequalities are strict.
  - (c) Asymptotic idempotency: for each $k \in \mathbb{N}$ and $x_1, \ldots, x_k, y \in X$,

$$\lim_{n\to\infty} M_{1+kn}(y, \underbrace{x_1, \ldots, x_1}_{n}, \ldots, \underbrace{x_k, \ldots, x_k}_{n}) = M_k(x_1, \ldots, x_k).$$

Main tool in the proof of characterisation theorem

Proof of part $(ii) \Rightarrow (i)$ is based on a separation theorem for Abelian subsemigroups.

Core of an Abelian subsemigroup

Let $(S, \oplus)$ be an Abelian semigroup, and A be a subsemigroup of S. The (algebraic) core of A is defined as the subsemigroup

$$\mathrm{cor}(A) := \left\{ a \in A : \forall\, s \in S \;\; \exists\, n \in \mathbb{N} \text{ such that } \underbrace{a \oplus \ldots \oplus a}_{n} \oplus s \in A \right\}.$$

Main tool in the proof of characterisation theorem

Separation theorem for Abelian subsemigroups (Páles (1989))

Let $(S, \oplus)$ be an Abelian semigroup,
let A and B are disjoint subsemigroups of S such that
$\mathrm{cor}(A) \neq \emptyset$ and $\mathrm{cor}(B) \neq \emptyset$.

Then there exists a function $F : S \to \mathbb{R}$ such that

(i) it is a homomorphism, i.e., $F(s_1 \oplus s_2) = F(s_1) + F(s_2)$, $s_1, s_2 \in S$,

(ii)
$$F(a) \geqslant 0 \geqslant F(b), \qquad a \in A, \ \ b \in B,$$

(iii)
$$F(a) > 0 > F(b), \qquad a \in \mathrm{cor}(A), \ \ b \in \mathrm{cor}(B).$$

This result already played an important role in the characterisation of (strongly) internal means due to Páles (1989).

Proof of part $(ii) \Rightarrow (i)$:

Let $(S(X), \oplus)$ be the free Abelian semigroup generated by the elements of X, that is,

- $S(X)$ consists of all the finite (unordered) sequences (or unordered strings) of X having positive lengths.
- furnish $S(X)$ with the string concatenation $\oplus$ as an operation.

Proof of part $(ii) \Rightarrow (i)$:

Let $(S(X), \oplus)$ be the free Abelian semigroup generated by the elements of X, that is,

- $S(X)$ consists of all the finite (unordered) sequences (or unordered strings) of X having positive lengths.
- furnish $S(X)$ with the string concatenation $\oplus$ as an operation.

Note that, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in X$ and each permutation $(\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$, we have

$$x_1 \oplus \cdots \oplus x_n = x_{\pi(1)} \oplus \cdots \oplus x_{\pi(n)}.$$

Proofs, characterisation of generalized ψ-estimators

Introduce the mapping $\mu : S(X) \to \Theta$ by

$$\mu(x_1 \oplus \cdots \oplus x_n) := M_n(x_1, \ldots, x_n), \qquad n \in \mathbb{N}, \ \ x_1, \ldots, x_n \in X.$$

Proofs, characterisation of generalized ψ-estimators

Introduce the mapping $\mu : S(X) \to \Theta$ by

$$\mu(x_1 \oplus \cdots \oplus x_n) := M_n(x_1, \ldots, x_n), \qquad n \in \mathbb{N}, \ \ x_1, \ldots, x_n \in X.$$

Due to the symmetry of M, the mapping μ is well-defined.

Proofs, characterisation of generalized ψ-estimators

Introduce the mapping $\mu : S(X) \to \Theta$ by

$$\mu(x_1 \oplus \cdots \oplus x_n) := M_n(x_1, \ldots, x_n), \qquad n \in \mathbb{N}, \ \ x_1, \ldots, x_n \in X.$$

Due to the symmetry of M, the mapping μ is well-defined.

Let $t \in \Theta$ be fixed and define

$$A_t := \{s \in S(X) : \mu(s) < t\}, \qquad B_t := \{s \in S(X) : \mu(s) > t\}.$$

Proofs, characterisation of generalized ψ-estimators

Introduce the mapping $\mu : S(X) \to \Theta$ by

$$\mu(x_1 \oplus \cdots \oplus x_n) := M_n(x_1, \ldots, x_n), \qquad n \in \mathbb{N}, \;\; x_1, \ldots, x_n \in X.$$

Due to the symmetry of M, the mapping μ is well-defined.

Let $t \in \Theta$ be fixed and define

$$A_t := \{s \in S(X) : \mu(s) < t\}, \qquad B_t := \{s \in S(X) : \mu(s) > t\}.$$

One can verify the conditions of the separation theorem with the choices

$$S := S(X), \qquad A := A_t, \qquad B := B_t.$$

It turns out that

$$\mathrm{cor}(A_t) = A_t \qquad \text{and} \qquad \mathrm{cor}(B_t) = B_t.$$

Possible future work

Some questions that may be studied:

- given $\psi \in \Psi(X, \Theta)$ and a sequence of i.i.d. random variables ξ_n, $n \in \mathbb{N}$, investigate the asymptotic properties of $\vartheta_n(\xi_1, \ldots, \xi_n)$ as $n \to \infty$ (e.g., consistency and central limit theorem),

Possible future work

Some questions that may be studied:

- given $\psi \in \Psi(X, \Theta)$ and a sequence of i.i.d. random variables ξ_n, $n \in \mathbb{N}$, investigate the asymptotic properties of $\vartheta_n(\xi_1, \ldots, \xi_n)$ as $n \to \infty$ (e.g., consistency and central limit theorem),
- investigation of a multidimensional parameter set instead of an open interval Θ of $\mathbb{R}$,

Some questions that may be studied:

- given $\psi \in \Psi(X, \Theta)$ and a sequence of i.i.d. random variables ξ_n, $n \in \mathbb{N}$, investigate the asymptotic properties of $\vartheta_n(\xi_1, \ldots, \xi_n)$ as $n \to \infty$ (e.g., consistency and central limit theorem),
- investigation of a multidimensional parameter set instead of an open interval Θ of $\mathbb{R}$,
- given an estimator κ for the unknown parameter $\vartheta \in \Theta$, is it a MLE for some absolutely continuous distribution?

  More precisely, is there a family of density functions $\mathbb{R} \ni x \mapsto f(x, t)$, $t \in \Theta$ such that the MLE for θ coincides with κ?

Possible future work

Some questions that may be studied:

- given $\psi \in \Psi(X, \Theta)$ and a sequence of i.i.d. random variables ξ_n, $n \in \mathbb{N}$, investigate the asymptotic properties of $\vartheta_n(\xi_1, \ldots, \xi_n)$ as $n \to \infty$ (e.g., consistency and central limit theorem),
- investigation of a multidimensional parameter set instead of an open interval Θ of $\mathbb{R}$,
- given an estimator κ for the unknown parameter $\vartheta \in \Theta$, is it a MLE for some absolutely continuous distribution?

  More precisely, is there a family of density functions $\mathbb{R} \ni x \mapsto f(x, t)$, $t \in \Theta$ such that the MLE for θ coincides with κ?

  Using the characterisation theorem, we can check that whether κ is a generalized ψ-estimator with some function ψ. However, since in the proof of the characterisation theorem, our construction for ψ is not explicit, we can not see whether the functional equation

$$\psi(x, t) = \frac{\partial_2 f(x, t)}{f(x, t)}, \qquad (x, t) \in \mathcal{X}_f \times \Theta,$$

  could be solved for f or not.

References

The talk is based on the following three papers:

MÁTYÁS BARCZY, ZSOLT PÁLES:
Existence and uniqueness of weighted generalized ψ-estimators.
Lithuanian Mathematical Journal 65(1), 14-49 (2025).

MÁTYÁS BARCZY, ZSOLT PÁLES:
Basic properties of generalized ψ-estimators.
Publicationes Mathematicae Debrecen 106/3-4, 499-524 (2025).

MÁTYÁS BARCZY, ZSOLT PÁLES:
Axiomatic characterisation of generalized ψ-estimators.
arXiv 2409.16240 (2024)

References

The talk is based on the following three papers:

- MÁTYÁS BARCZY, ZSOLT PÁLES:
  Existence and uniqueness of weighted generalized ψ-estimators.
  Lithuanian Mathematical Journal 65(1), 14-49 (2025).

- MÁTYÁS BARCZY, ZSOLT PÁLES:
  Basic properties of generalized ψ-estimators.
  Publicationes Mathematicae Debrecen 106/3-4, 499-524 (2025).

- MÁTYÁS BARCZY, ZSOLT PÁLES:
  Axiomatic characterisation of generalized ψ-estimators.
  arXiv 2409.16240 (2024)

Thank you for your attention!