Multitype continuous-time branching processes with immigration generated by point processes

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Contents

Introduction

- In this study we aim to generalize the immigration pattern occurring into the Multi-type Markov branching processes (MMBP).
- Suppose that, in general, a given **point process** (PP) is driving the immigration flux
- We aim to introduce the general framework of Laplace functionals of random PP within the context of branching processes. We believe that this approach will be valuable for researchers looking to extend their work beyond the Poisson process.
- For example, our method simplifies the derivation of the generating function for a branching process with immigration, which often serves as the foundation for many studies. Traditionally, this derivation was based on the exact distribution of points over a given interval, which is tractable in the case of a Poisson process, as demonstrated in Mitov,Yanev, Hyrien (2018), Rabehasaina and Woo (2021) and Butkovsky (2012).
- Our framework provides a more unified and often simpler approach to addressing problems involving immigration

Introduction

- Multitype Markov branching processes were introduced by Kolmogorov and Dmitriev (1947) where the terminology *branching process* officially appeared for the first time.
- *Kolmogorov, A., and N. Dmitriev. 1947. Branching stochastic processes. Proceedings of AS-SSSR 56 (1):7–10.*
- The first branching process with immigration was proposed by Sevastyanov (1957). He studied a single-type Markov processes in which immigration occurs according to a **time-homogeneous Poisson process**, and proved limiting distributions in the sub-, super-, and critical cases.
- *Sevastyanov, B. A. 1957. Limit theorems for branching random processes of special type. Theory of Probability and Its Applications 2:339–87 (in Russian).*

Introduction

- More recent results for multiple variants of branching processes with immigration were obtained by
- *Barczy, M., Nedényi F. K., Pap G. (2020). On aggregation of subcritical Galton-Watson branching processes with regularly varying immigration. Lith. Math. J. 60 (4):425–51.*
- *Barczy, M., Palau S., Pap G. (2021). Asymptotic behavior of projections of supercritical multi-type continuous-state and continuous-time branching processes with immigration. Adv. in Appl. Probab. 53 (4):1023–60.*
- *González, M., del Puerto I.,Yanev G. . (2018). Controlled branching processes. London: ISTE and John Wiley & Sons.*
- *Li, D., Vatutin V. , Zhang M. (2021). Subcritical branching processes in random environment with immigration stopped at zero. J. Theoret. Probab. 34 (2):874–96.*

Point processes on the real line

- Let $(\Omega, \mathcal{F}, P)$ be a probability space,
- $\mathcal{B}(\mathbb{R})$ be the usual Borel σ-algebra on $\mathbb{R}$, and $\mathbf{N}_{<\infty}$ be the set of measures $\widetilde{\Phi}$ on $\mathbb{R}_+$ such that for each $B \in \mathcal{B}(\mathbb{R}_+), \widetilde{\Phi}(B) \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- $\mathbf{N}$ be the set of measures which can be represented as a countable sum of elements of $\mathbf{N}_{<\infty}$,
- $\mathcal{N}$ be the σ-algebra generated by the sets

$$\{\Phi \in \mathbf{N} : \Phi(B) = k \text{ for some } B \in \mathcal{B}(\mathbb{R}_+) \text{ and } k \in \mathbb{N}_0\}.$$

Point processes on the real line

- We call Φ a random measure or a PP if it is a random element of $(\mathbf{N}, \mathcal{N})$, that is, a measurable mapping $\Phi : \Omega \to \mathbf{N}$.
- In this work, we will consider PPs which are *proper*, i.e., such that there exist random variables $\kappa, X_1, X_2, \ldots$ such that
- $$\Phi = \sum_{i \leq \kappa} \delta_{X_i},$$

  where δ_x is the Dirac mass at x, so Φ places unit mass at the random locations $X_1, X_2, \ldots, X_\kappa$.

Point processes on the real line

Some of the characteristics which help describing a PP include:

- its *intensity measure* Λ, defined by $\Lambda(B) = \mathbb{E}[\Phi(B)]$, where $B \in \mathcal{B}(\mathbb{R}_+)$;
- its *Laplace functional*, which characterizes the process completely, for all test functions f,

$$\mathcal{L}_\Phi(f) := \mathbb{E}\left[e^{-\int f d\Phi}\right] = \mathbb{E}[e^{-\sum_{i\leq\kappa} f(X_i)}].$$

  The set of test functions can be all positive measurable ones, like in the case of a Poisson process, or these of compact support, like in the case of determinantal point processes;
- its *joint intensities* ρ_k, if they exist, defined as $\rho_k : {\mathbb{R}_+}^k \to \mathbb{R}_+$ are such that for each disjoint $B_1, \ldots, B_k \in \mathcal{B}(\mathbb{R}_+)$,

$$\mathbb{E}\left[\prod_{i=1}^{k} \Phi(B_i)\right] = \int_{B_1\times\cdots\times B_k} \rho_k(x_1, \ldots, x_k)\Lambda(dx_1)\ldots\Lambda(dx_k).$$

Poisson processes

Poisson process is characterized by the property that its *intensity measure* Λ is such that:

1. for every $B \in \mathcal{B}(\mathbb{R}_+)$, $\Phi(B) \sim Pois(\Lambda(B))$ and
2. for every disjoint $B_1, \ldots, B_m \in \mathcal{B}(\mathbb{R}_+)$, $\Phi(B_1), \ldots, \Phi(B_m)$ are independent.

It is a consequence that the Laplace functional of this PP is then

$$\mathcal{L}_\Phi(f) = \exp\left(- \int_{\mathbb{R}_+} \left(1 - e^{-f(x)}\right) \Lambda(dx) \right).$$

If $\Lambda(dx) = \lambda dx$, then we say that the Poisson process is *homogeneous* of rate λ.

Determinantal point processes (DPPs)

Determinantal point processes (DPPs) were introduced by Macchi (1975) under the name *fermion process*, due to their repulsive behavior. Since then they have arisen in various contexts including random matrix theory, zeros of random analytic functions, statistical mechanics, and even machine learning.

1. We call a point process Φ on $\mathbb{R}_+$ (Λ, K)-*determinantal* if Λ is a locally finite measure on $\mathbb{R}_+$, and its joint intensities satisfy, for $\boldsymbol{x} = (x_1, \dots, x_n)$,

$$\rho_n(x_1, \dots, x_n) = \det K(x_i, x_j)_{1 \le i,j \le n} =: D(\boldsymbol{x}),$$

   where $K\colon {\mathbb{R}_+}^2 \to \mathbb{R}_+$ is symmetric; for Λ^n-almost all $\boldsymbol{x}$, $K(x_i, x_j)_{1 \le i,j \le n}$ is non-negative definite, and for each bounded $D \in \mathcal{B}(\mathbb{R}_+)$, $\int_D K(x,x)\Lambda(dx)$ is finite.
2. Further, the Laplace functional of Φ is given, for any nonnegative f of compact support, and

$$\varphi_n(\boldsymbol{x}) := \prod_{i=1}^{n} (1 - e^{-f(x_i)}), \qquad \Lambda(d\boldsymbol{x}) := \Lambda(dx_1) \dots \Lambda(dx_n)$$

Determinantal point processes (DPPs)

- $$\begin{aligned}
&\mathcal{L}_\Phi(f) \\
&= 1 + \sum_{n\geq 1} \frac{(-1)^n}{n!} \int_{\mathbb{R}_+{}^n} \rho_n(x_1,\ldots,x_n)\varphi_n(x_1,\ldots,x_n)\Lambda(dx_1)\ldots\Lambda(dx_n) \\
&= 1 + \sum_{n\geq 1} \frac{(-1)^n}{n!} \int_{\mathbb{R}_+{}^n} D(\boldsymbol{x})\varphi_n(\boldsymbol{x})\Lambda(d\boldsymbol{x}),
\end{aligned}$$

Cox Processes

Cox processes, also known as doubly stochastic Poisson processes, extend the ordinary Poisson processes by introducing randomness into its intensity measure. They are commonly used to model phenomena where the occurrence of events is influenced by underlying stochastic factors, as demonstrated in Moller (2003), Moller et all. (1998) and Yannaros (1988), particularly in spatial and spatio-temporal data analysis.

Formally, let η be a random σ-finite measure on $\mathbb{R}_+$. A point process Φ is called a Cox process with *directing measure* η if, conditional on η,

$$\Phi \mid \eta \sim \text{Poisson process with intensity measure } \eta.$$

Cox Process

- The Laplace functional is obtained by averaging over η

$$\mathcal{L}_\Phi(f) = \mathbb{E}[\exp(-\int_{\mathbb{R}_+} (1 - e^{-f(x)})\eta(dx))],$$

  for every nonnegative measurable f.

- The first two moment measures follow by conditioning on η and using the Poisson moment formula, as in Last and Penrose (2018)

$$\mathbb{E}[\Phi(B)] = \mathbb{E}[\eta(B)], \qquad Var[(\Phi(B)] = Var[\eta(B)] + \mathbb{E}[\eta(B)],$$

  for all $B \in \mathcal{B}(\mathbb{R}_+)$.

Fractional Poisson Processes

- The fractional Poisson process (FPP), introduced by Mainardi et al. (2004), is a non-Markovian generalisation of the standard homogeneous Poisson process. It features heavy-tailed interarrival times; see also the monograph by Meerschaert and Sikorskii (2019).
- The FPP depends on parameters $\beta \in (0, 1]$ and $\lambda > 0$, and is of renewal type. The atoms of the associated random measure $\Phi_{\beta,\lambda}$ are located at $X_n = T_1 + \cdots + T_n$, where the T_i are i.i.d. random variables with

$$\mathbb{P}(T_i > t) = E_\beta(-\lambda t^\beta),$$

where

$$E_\beta(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)}$$

is the Mittag-Leffler function.

Fractional Poisson Processes

- Define the counting process by $N_{\beta,\lambda}(t) := \Phi_{\beta,\lambda}((0,t])$. For $\beta = 1$, we get $T_i \sim \mathrm{Exp}(\lambda)$, so the FPP reduces to a homogeneous Poisson process with rate λ. However, for $\beta \in (0,1)$, $\mathbb{P}(T_i > t) \sim \frac{1}{t^\beta}$ as $t \to \infty$, and therefore $\mathbb{E}[T_i]$ is infinite.
- The work of Meerschaert et al. (2011) extends the connection to Poisson processes for $\beta < 1$ by showing that

$$N_{\beta,\lambda}(t) = \mathcal{N}_\lambda(Y_\beta(t)),$$

where $\mathcal{N}_\lambda$ is a homogeneous Poisson process of rate $\lambda > 0$, and Y_β is the inverse of an independent β-stable subordinator.
- The Laplace functional of $\Phi_{\beta,\lambda}$ is given by

$$\mathcal{L}_{\Phi_{\beta,\lambda}}(f) = \mathbb{E}\left[\exp\left(-\lambda \int_{\mathbb{R}_+} (1 - e^{-f(t)})\, \mathrm{d}Y_\beta(t)\right)\right].$$

Branching Processes with Immigration (Single-Type)

The basic BP without immigration, denoted $Z_\times$, is a continuous-time Markov branching process starting from $Z_\times(0) = I$, where I is a rv representing the size of a single immigrant group.

- Immigration occurs at times $T_1 < T_2 < \dots$, with incoming groups $I_1, I_2, \dots$ such that $I_k \sim I$ are i.i.d. We assume that $\mathbb{E}[I] < \infty$.
- Each particle evolves independently. After an exponentially distributed lifetime with mean μ (i.e., rate $1/\mu$), it dies and produces ν offspring.
- generating function $G_{Z_\times}(t,s) := \mathbb{E}[s^{Z_\times(t)}]$ Athreya and Ney (1972) satisfies

$$\frac{\partial}{\partial t} G_{Z_\times}(t,s) = f_\nu(G_{Z_\times}(t,s)),$$

where $f_\nu(s) = \frac{G_\nu(s)-s}{\mu}$. Also, the mean function satisfies

$$M_\times(t) := \mathbb{E}[Z_\times(t)] = e^{\rho t}, \quad \text{with } \rho := f'_\nu(1) = \frac{\mathbb{E}[\nu]-1}{\mu}.$$

Branching Processes with Immigration (Multi-Type Case)

- Consider now a generalisation of the previous process that involves d types of particles.
- The generating function of the offspring distribution $\boldsymbol{\nu}$ is given by

$$G_{\boldsymbol{\nu}}(\boldsymbol{s}) := \sum_{\boldsymbol{n}\in\mathbb{N}_0^d} \mathbb{P}(\boldsymbol{\nu} = \boldsymbol{n}) \prod_{i=1}^{d} s_i^{n_i}, \quad \text{for} \quad \boldsymbol{s} = (s_1, \ldots, s_d).$$

- The multi-type branching process $\boldsymbol{Z}$ starts from $\boldsymbol{Z}(0) = \boldsymbol{I}^{(0)} \sim \boldsymbol{I}$, where $\boldsymbol{I}$ is a random vector representing the initial immigrant group.
- Immigration occurs at times $0 = T_0 < T_1 < T_2 < \ldots$, where the incoming groups $\boldsymbol{I}^{(0)}, \boldsymbol{I}^{(1)}, \boldsymbol{I}^{(2)}, \ldots$ are i.i.d. and distributed as $\boldsymbol{I}$.
- Each particle of type i lives an exponentially distributed amount of time with mean μ_i (i.e., lifetime $\sim \mathrm{Exp}(1/\mu_i)$), after which it dies and produces offspring with type counts distributed as $\boldsymbol{\nu}_i$. That is, $(\nu_i)_j$ denotes the number of type-j particles produced by a type-i particle.

Branching Processes with Immigration (Multi-Type Case)

Assumptions (H): In this work we assume that

1. the immigrants and progeny, I_i and $(\nu_i)_j$, have finite expectation. For some of the results we impose the following stronger restriction:

$$I_i \text{ and } (\nu_i)_j \text{ have finite second moments.}$$

2. the mean offspring matrix

$$M := (\mathbb{E}(\nu_i)_j)_{1 \leq i,j \leq d}$$

is assumed *primitive*, i.e. there exists an integer $p \geq 1$ such that M^p has strictly positive entries.

Theorem for PGF of BPI (Single-Type)

Theorem 1.

1. In the case of immigration governed by a (Λ, K)-DPP process,

$$\mathbb{E}[s^{Z(t)}] = 1+\sum_{n\geq 1} \frac{(-1)^n}{n!} \int_{(0,t]^n} D(\boldsymbol{x}) \prod_{i=1}^{n} \left(1 - G_{Z_\times}(t-x_i, s)\right) \; \Lambda(d\boldsymbol{x}).$$

2. In the case of immigration governed by a Cox process with directing measure η,

$$\mathbb{E}[s^{Z(t)}] = \mathbb{E}\left[\exp\left(-\int_{(0,t]} \left(1 - G_{Z_\times}(t-x, s)\right) \eta(dx)\right)\right].$$

3. In the case of immigration governed by a (β, λ)-fractional Poisson process (FPP),

$$\mathbb{E}[s^{Z(t)}] = \mathbb{E}\left[\exp\left(-\lambda\int_{(0,t]} \left(1 - G_{Z_\times}(t-x, s)\right) dY_\beta(x)\right)\right],$$

where Y_β is the inverse of an independent β-stable subordinator.

To obtain the respective Laplace transforms, replace $G_{Z_\times}$ with $\mathcal{L}_{Z_\times}$.

Theorem for PGF of BPI (Multi-Type)

Theorem 2.
The statements of Theorem 1 extend to the vector case by replacing $s \to \boldsymbol{s}$, $Z \to \boldsymbol{Z}$, in all regimes (subcritical, critical, supercritical). The same holds if one replaces generating functions with Laplace transforms: $G_{\mathbf{Z}} \to \mathcal{L}_{\mathbf{Z}}$ and $G_{\mathbf{Z}_\times} \to \mathcal{L}_{\mathbf{Z}_\times}$.

Moments of BP with DPP Immigration

Theorem 3. In the case of immigration governed by a (Λ, K)-determinantal point process (DPP):

1. The first moment satisfies

$$\mathbb{E}[Z_i(t)] = \int_{(0,t]} K(x,x)\, \mathbb{E}[Z_{\times,i}(t-x)]\, \Lambda(dx).$$

2. The covariance is given by

$$\begin{aligned}
&\mathrm{Cov}(Z_i(t), Z_j(t)) \\
&= \int_{(0,t]} K(x,x)\, \mathbb{E}[Z_{\times,i}(t-x)\, Z_{\times,j}(t-x)]\, \Lambda(dx) - \\
&\quad \int_{(0,t]^2} K^2(x,y)\, \mathbb{E}[Z_{\times,i}(t-x)]\, \mathbb{E}[Z_{\times,j}(t-y)]\, \Lambda(dx)\, \Lambda(dy).
\end{aligned}$$

Asymptotics of the MMBP with DPPI

Theorem 4.

- Let ρ be the Perron–Frobenius root of the matrix A. In the case of immigration governed by a (Λ, K)-DPP Φ with kernel K such that

$$\int_{(0,\infty)} e^{-\rho x} K(x,x)\,\Lambda(dx) < \infty,$$

there exists an $\mathbb{R}^d$-valued random variable $\boldsymbol{W}$ such that

- $$\frac{\boldsymbol{Z}(t)}{e^{\rho t}} \xrightarrow[t\to\infty]{d} \boldsymbol{W}.$$

Moreover,

$$\boldsymbol{W} \overset{d}{=} \sum_i \boldsymbol{v} \cdot W_\times^{(i)} e^{-\rho T_i},$$

where $W_\times^{(i)}$ are i.i.d. copies of $W_\times$, and T_i are the atoms of the DPP Φ.

- The Laplace transform of $\boldsymbol{W}$ is given by:

$$\mathbb{E}\left[\exp\left(-\langle \boldsymbol{W}, \boldsymbol{s}\rangle\right)\right] = \mathcal{L}_\Phi\left(-\ln\left(\mathcal{L}_{\boldsymbol{v}W_\times}\left(\boldsymbol{s}e^{-\rho x}\right)\right)\right).$$

Asymptotics of the MMBP with DPPI

Theorem 5.

- Assume that immigration is driven by a stationary (Λ, K)-DPP with $\Lambda(dx) = \lambda(x)\,dx$, and that $\lambda(x) \sim \lambda_\infty e^{\delta x}$ for some $\lambda_\infty > 0$ and $\delta \in \mathbb{R}$.
  - Under **(H)**, if $\delta > \max\{\rho, 0\}$, then
  $$\frac{Z_i(t)}{e^{\delta t}} \xrightarrow[t\to\infty]{L^2} A_i := K_* \lambda_\infty \int_0^\infty e^{-\delta x} \mathbb{E}[Z_{\times,i}(x)]\,dx,$$
  where $K_* := K(0,0)$.
  - Under **(H)**, if the process is **supercritical**, that is, $\rho > 0$ and $\delta = \rho$, then
  $$\frac{Z_i(t)}{te^{\delta t}} \xrightarrow[t\to\infty]{L^2} A_i' := K_* \lambda_\infty \langle \boldsymbol{u}, \mathbb{E}[\boldsymbol{I}] \rangle v_i.$$

Asymptotics of the MMBP with DPPI

- If the process is **critical**, that is, $\rho = 0$, $\mathbb{E}[\|\boldsymbol{\nu}\|^2]$ is finite, λ is bounded, $\delta = \rho$, and the covariance function $K(x, 0)$ of the DPP tends to 0 as $x \to \infty$, then

$$\frac{\boldsymbol{Z}(t)}{t} \xrightarrow[t\to\infty]{d} Y\boldsymbol{\nu},$$

where $Y \sim \Gamma(K(0,0)\lambda_\infty \beta, 1/Q)$ with

$$Q := \frac{1}{2}\sum_{i,j,k=1}^{d} \frac{\partial^2 G_{\nu_i}}{\partial x_j \partial x_k}\bigg|_{\boldsymbol{x}=\boldsymbol{1}} \mu_i^{-1} v_i u_j u_k, \qquad \beta := \frac{\langle \boldsymbol{u}, \mathbb{E}[\boldsymbol{I}]\rangle}{Q},$$

and $\Gamma(\alpha, \beta)$ denotes the Gamma distribution with shape parameter α and rate parameter β, i.e., its density for $x > 0$ is

$$f_{\alpha,\beta}(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

Asymptotics of the MMBP with DPPI

- If the process is **subcritical**, that is, $\rho < 0$, and $\delta = 0$, then

$$\boldsymbol{Z}(t) \xrightarrow[t\to\infty]{d} \boldsymbol{X},$$

where $\boldsymbol{X}$ is a random variable with Laplace transform satisfying

$$\mathcal{L}_{\boldsymbol{X}}(t, \boldsymbol{s}) \xrightarrow[t\to\infty]{} 1+\sum_{n\geq 1} \frac{(-1)^n \lambda_\infty^n}{n!} \int_{(0,\infty)^n} D(\boldsymbol{x}) \prod_{i=1}^{n} \left[1 - \mathcal{L}_{\boldsymbol{Z}_\times}(x_i, \boldsymbol{s})\right] d\boldsymbol{x}.$$

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