CB processes with quadratic competition and conditioning on the non-extinction

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Let $d \geq 0$, $c > 0$, and $\Pi = (\Pi_j, j \in \mathbb{Z}^+)$ such that $0 < \rho := \sum_{i \in \mathbb{Z}^+} \Pi_i < \infty$.

We will consider a continuous time random Markov chain

$$(L_t, \ t \geq 0)$$

that takes values in $\mathbb{Z}^+$, whose infinitesimal generator is given by

$$Q_{i,j} = \begin{cases} i\Pi_{j-i}, & j > i, \& \ i \geq 1; \\ di + c\binom{i}{2}, & i \geq 1, j = i-1; \\ -i\left(d + \rho + \frac{c(i-1)}{2}\right), & j = i. \end{cases}$$

Thus, we start with i-individuals equipped with independent exponential clocks with parameter $\rho + d$, and $\binom{i}{2}$ exponential clocks of parameter c. When either of these clocks ring, if it is of the first type, then the corresponding particle begets a random number of descendants according to a probability proportional to Π, or dies with a probability proportional to d; while if the clock is of the second type then a pair of individuals among the $\binom{i}{2}$ in the population compete and only one of them survives.

- This model was introduced and studied by A. Lambert under the assumption of a finite log moment
$$\sum_{i\geq 2} \log(i)\Pi_i < \infty$$
.
- He proved that if $d = 0$, the process L is positive recurrent.
- While if $d > 0$, then the process hits zero in a finite time a.s. In that case, the process comes down from infinity.
- Under some assumptions, Lambert proved that such processes admit a renormalization of time and space such that the scaling limit is a weak solution to the stochastic differential equation of the type
$$\mathrm{d}Z_t = bZ_t\mathrm{d}t + \sqrt{\gamma Z_t}\mathrm{d}B_t - cZ_t^2\mathrm{d}t;$$
with $b, \gamma \geq 0$, some parameters.

Logistic continuous state branching process

Let $(Z_t, t \geq 0)$ be the size of a population evolving in continuous time and space along the dynamics :

- **branching** : each individual reproduces or dies independently with a same law (classical CB's dynamics).
- **quadratic competition** : pairwise fights at constant rate $c \geq 0$ (quadratic negative drift).

$$\mathrm{d}Z_t = \ll \text{CB dynamics} \gg -\frac{c}{2} Z_t^2 \mathrm{d}t.$$

The competition <u>breaks</u> the branching property.

The process Z has been introduced by Lambert (2005) and is called *logistic* CB process (LCB).

Let Ψ be a branching mechanism :

$$\Psi(x) := \frac{\sigma^2}{2}x^2 - \gamma x + \int_0^\infty \left(e^{-xy} - 1 + xy\mathbb{1}_{\{y\leq 1\}}\right)\pi(\mathrm{d}y) \tag{1}$$

with $\sigma \geq 0, \gamma \in \mathbb{R}$ and π a Lévy measure.

Definition/Theorem

A $LCB(\Psi, c)$ *is solution to the stochastic equation :*

$$\begin{aligned} Z_t = z + \sigma \int_0^t \sqrt{Z_s}\mathrm{d}B_s + \gamma \int_0^t Z_s \mathrm{d}s + \int_0^t \int_0^{Z_{s-}} \int_1^\infty y\mathcal{M}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}y) \\ + \int_0^t \int_0^{Z_{s-}} \int_0^1 y\bar{\mathcal{M}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}y) - \frac{c}{2}\int_0^t Z_s^2 \mathrm{d}s, \end{aligned} \tag{2}$$

with B *a Brownian motion,* $\mathcal{M}$ *an indep. PRM with intensity* $\mathrm{d}s\mathrm{d}u\pi(\mathrm{d}y)$ *and* $\bar{\mathcal{M}}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}y) := \mathcal{M}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}y) - \mathrm{d}s\mathrm{d}u\pi(\mathrm{d}y)$.

Aim : Given a LCB process Z satisfying

$$\mathbb{P}_z(Z_t \underset{t\to\infty}{\longrightarrow} 0) = 1 \qquad \textbf{(almost-sure asymptotic extinction)},$$

we wish to condition the process on the negligible event

$$\mathscr{S} := \{Z_t \underset{t\to\infty}{\longrightarrow} 0\}^c.$$

⚠ $\mathscr{S}$ is of zero probability and there is not a unique way to define such a conditioning !

It depends in general on how the event $\mathscr{S}$ is approached.

(1) When there is *extinction in finite time* a.s., i.e.

$$\{Z_t \underset{t\to\infty}{\longrightarrow} 0\} = \{\zeta_0 < \infty\},$$

with $\zeta_0 := \inf\{t > 0 : Z_t = 0\}$, we could seek a conditioning ≪ along the approximation ≫ :

$$\bigcap_{s>0} \{\zeta_0 > s\}.$$

This is the notion of Q-**process** :

$$\mathbb{Q}_z(\Lambda) = \text{“}\lim_{s\to\infty}\text{”}\, \mathbb{P}_z(\Lambda|\zeta_0 > t + s), \forall \Lambda \in \mathcal{F}_t.$$

(2) We could also try to force the process to go above any levels before being close to 0, i.e. we approach $\mathscr{S}$ by :

$$\bigcap_{b>0} \{\zeta_b^+ < \zeta_0\}, \text{ with } \zeta_b^+ := \inf\{t > 0 : Z_t > b\}.$$

Those methods do not seem to apply to LCBs without strong assumptions on Ψ***. We will approach*** $\mathscr{S}$ ***in a different way.***

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Extinction and total progeny

We call **total progeny**,

$$J := \int_0^\infty Z_t \mathrm{d}t.$$

Proposition (F., Rivero, Winter 24 ($c > 0$), Bingham 75 ($c = 0$))

The following equality holds a.s.

$$\{Z_t \underset{t\to\infty}{\longrightarrow} 0\} = \{J < \infty\}.$$

- $J < \infty$ $\mathbb{P}_z$-*p.s. iff* $\begin{cases} \Psi'(0+) \geq 0 & \text{if } c = 0, \\ \mathbb{H} & \text{if } c > 0. \end{cases}$
- $\mathbb{E}_z(J) < \infty$ *iff* $\begin{cases} \Psi'(0+) > 0 & \text{if } c = 0, \\ \Psi(\infty) = \infty \ \& \ \int^\infty \log y \, \pi(\mathrm{d}y) < \infty & \text{if } c > 0. \end{cases}$

We will approach the survival event $\mathscr{S}$ by forcing the total progeny to be infinite, in the following way

$$\mathscr{S} = \bigcap_{\theta > 0} \{J > \mathbb{e}/\theta\} \tag{3}$$

with $\mathbb{e}$ an exponential r.v. of parameter 1 independent from Z.

Construction of LCB and proof of the proposition

The generator of the LCB Z takes the form :

$$\mathscr{L}f(z) := z\mathrm{L}^{\Psi}f(z) - \frac{c}{2}z^2 f'(z)$$

with L^{Ψ} the generator of a Lévy process Y of Laplace exponent Ψ.

Factorization :

$$\mathscr{L}f(z) = z\left(\mathrm{L}^{\Psi}f(z) - \frac{c}{2}zf'(z)\right) =: z\mathscr{G}f(z)$$

Let $J_s := \int_0^s Z_u \mathrm{d}u$, $J_\infty = J$ and the random clock :

$$C_t := \inf\{s \geq 0 : J_s > t\}$$

Lamperti's Transformation : The time-changed process

$$(R_t := Z_{C_t}, t \leq J_\infty)$$

is a positive Markov process with generator $\mathscr{G}$, it satisfies

$$\mathrm{d}R_t = \mathrm{d}Y_t - \frac{c}{2}R_t\mathrm{d}t, t \leq \sigma_0$$

where $\sigma_0 := \inf\{t > 0 : R_t = 0\}$.

$\rightarrow$ By the *time change*, $Z_t = R_{J_t}, \forall t \geq 0$, $\sigma_0 = J_\infty = J$, and

$$\{J < \infty\} = \{\sigma_0 < \infty\} = \{Z_t \underset{t\to\infty}{\longrightarrow} 0\}.$$

Asymptotic extinction condition :

$$\mathbb{H} : \Psi(\infty) = \infty \text{ and } \mathcal{E} := \int_0^{x_0} \frac{1}{u} e^{\int_u^{x_0} \frac{2\Psi(v)}{cv}\mathrm{d}v}\mathrm{d}u = \infty.$$

Examples satisfying $\mathbb{H}$

- Stable and Neveu mechanisms :

$$\Psi(x) := ax^{\alpha} - \gamma x, \ \forall x \geq 0, \ \text{for } \alpha \in (1,2], \gamma \in \mathbb{R}, a > 0,$$
$$\Psi(x) := x \log x, \ \forall x \geq 0$$

- Let $\alpha \in (0, c/2], a > 0, \beta \in [1,2]$ and Ψ such that

$$\Psi(x) \underset{x \to 0}{\sim} -\alpha / \log(1/x) \text{ and } \Psi(x) \underset{x \to \infty}{\sim} ax^{\beta}.$$

NB : here $\displaystyle\int_0 \frac{|\Psi(x)|}{x} \mathrm{d}x = \infty.$

Fact :

$$\int_0 \frac{\Psi(x)}{x} \mathrm{d}x > -\infty \iff \int^{\infty} \log y \pi(\mathrm{d}y) < \infty \implies \mathcal{E} = \infty.$$

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Without competition

Theorem (Foucart, R., Winter (2024))

Let $c = 0$ *and* Ψ *(sub)-critical :* $\varrho := \Psi'(0+) \geq 0$ $(\Longleftrightarrow J < \infty$ $\mathbb{P}_z$*-a.s.).*

1. *The function* $h(z) = z$ *is* ***excessive*** *and* $\forall z > 0$,

$$\mathbb{P}_z^{\uparrow}(\Lambda, t < \zeta) := \mathbb{E}_z\left(\frac{Z_t}{z}\mathbb{1}_{\Lambda}\right) = \lim_{\theta \to 0} \mathbb{P}_z\left(\Lambda, J_t \leq \mathrm{e}/\theta \,\middle|\, J \geq \mathrm{e}/\theta\right).$$

   - *If* $\varrho = 0$, $\mathbb{E}_z(J) = \infty$, Z *martingale,* $\zeta = \infty$, $\mathbb{P}_z^{\uparrow}$*-a.s.*
   - *If* $\varrho > 0$, $\mathbb{E}_z(J) < \infty$, Z *supermartingale,* $\zeta < \infty$, $\mathbb{P}_z^{\uparrow}$*-a.s.*

2. $(Z, \mathbb{P}_z^{\uparrow})$ *satisfies*

$$Z_t = z + \ll \text{ CB}(\Psi) \text{ dynamics } \gg + \sigma^2 t + \int_0^t \int_0^\infty y\mathcal{I}(\mathrm{d}s, \mathrm{d}y), \; t < \zeta,$$

with $\mathcal{I}$ *a Poisson mes. of intensity* $\mathrm{d}s\, y\pi(\mathrm{d}y)$ *and* $\zeta \stackrel{Law}{=} \mathrm{Exp}(\varrho)$.

- The additional term

$$\left(\sigma^2 t + \int_0^t \int_0^\infty y\mathcal{I}(\mathrm{d}s, \mathrm{d}y), t \geq 0\right)$$

is a subordinator of Laplace exponent Ψ'.

$\to$ *immigration dynamics* independent on the population size,

$\to$ The process $(Z, \mathbb{P}_z^\uparrow)$ is a CBI with mechanisms Ψ and Ψ'.

- If there is *extinction in finite time*, (NASC : $\displaystyle\int^\infty \frac{\mathrm{d}u}{\Psi(u)} < \infty$),

$$(Z, \mathbb{P}_z^\uparrow) \overset{\text{Law}}{=} Q\text{-process } \textbf{killed} \text{ at an indep. time} \sim \mathrm{Exp}(\varrho)$$

- The process $(Z, \mathbb{P}_z^\uparrow)$ starts from $z = 0$ and 0 is interpreted as an *immortal* individual.

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With competition : looking for an excessive function

Theorem (Foucart, R., Winter (2024))

Under the hypothesis $\mathbb{H}$ *and* $c > 0$*, there exists a constant* c_θ *such that*

$$c_\theta \mathbb{P}_z(J \geq \mathbb{e}/\theta),$$

has a limit as $\theta \to 0+$*, for any* $z > 0$*;*

$$\lim_{\theta \to 0} c_\theta \mathbb{P}_z(J \geq \mathbb{e}/\theta) =: h(z) = \int_0^\infty (1 - e^{-xz}) \overbrace{\frac{1}{x} e^{-\int_{x_0}^{x} \frac{2\Psi(u)}{cu} \mathrm{d}u} \mathrm{d}x}^{=:s(\mathrm{d}x)}.$$

Moreover, when

$$\Psi(\infty) = \infty \;\&\; \int^\infty \log y \, \pi(\mathrm{d}y) < \infty,$$

we have $h(z) = \mathbb{E}_z(J) < \infty$, $z > 0$.

Theorem (Foucart, R., Winter (2024))

Assume $\mathbb{H}$ *and* $c > 0$. *Let* $x_0 > 0$ *fixed. Set*

$$\forall z \in [0,\infty),\ \ h(z) := \int_0^\infty (1 - e^{-xz}) \overbrace{\frac{1}{x} e^{-\int_{x_0}^{x} \frac{2\Psi(u)}{cu}\mathrm{d}u}\mathrm{d}x}^{=:s(\mathrm{d}x)}\,.$$

1. *h is of Bernstein form* $\Longrightarrow$ *positive* $\uparrow$, $C^\infty(0,\infty)$,

$$h(0) = 0,\ h(\infty) = \infty,\ h'(0) < \infty \text{ and } \int^\infty h(y)\pi(\mathrm{d}y) < \infty.$$

2. $\forall z \geq 0$,

$$\mathscr{L}h(z) = -\frac{c\ell}{2} z \leq 0$$

with $\ell := \exp\left(\int_0^{x_0} \frac{2\Psi(u)}{cu}\mathrm{d}u\right) \geq 0$, *and*

$$\ell > 0 \text{ iff } \int^\infty \log y\, \pi(\mathrm{d}y) < \infty.$$

Study of the h-transformed process

$\big(h(Z_t), t \geq 0\big)$ is a $\mathbb{P}_z$-supermartingale and we define :

$$\mathbb{1}_{\{t<\zeta\}} \mathrm{d}\mathbb{P}_z^{\uparrow} := \frac{h(Z_t)}{h(z)} \mathrm{d}\mathbb{P}_z, \quad \text{on } \mathcal{F}_t, \ \forall t \geq 0 \text{ and } z > 0,$$

with ζ the lifetime of $(Z, \mathbb{P}_z^{\uparrow})$ and ∞ is the cemetery state.

Theorem (Foucart, R., Winter (2024))

1. $\forall z > 0$, *the law* $\mathbb{P}_z^{\uparrow}$ *satisfies,*

$$\mathbb{P}_z^{\uparrow}(\Lambda, t < \zeta) = \lim_{\theta \to 0} \mathbb{P}_z\left(\Lambda, J_t \leq \mathbb{e}/\theta \,\middle|\, J \geq \mathbb{e}/\theta\right), \forall \Lambda \in \mathcal{F}_t, \forall t \geq 0$$

2. $(Z, \mathbb{P}_z^{\uparrow})$ *is a* $(0, \infty]$*-valued Feller process,* $\zeta < \infty$ $\mathbb{P}_z^{\uparrow}$*-a.s., and*
3. $$\mathbb{P}_z^{\uparrow}\Big(\inf_{0 \leq s < \zeta} Z_s \leq a\Big) = \frac{h(a)}{h(z)}, \ \forall z > a \geq 0.$$

In particular, $\inf_{0 \leq t < \zeta} Z_t > 0$, $\mathbb{P}_z^{\uparrow}$*-a.s. for all* $z > 0$.

For all $z > 0$ and $y > 0$, let

$$b(z) := z\frac{h'(z)}{h(z)},\ q(z,y) := \frac{z}{h(z)}\big(h(z+y) - h(z)\big) \text{ and } k(z) := \frac{c\ell}{2}\frac{z}{h(z)}.$$

Theorem (Foucart, R., Winter (2024))

$(Z, \mathbb{P}_z^{\uparrow})$ *has same law as the weak solution of the stochastic equation below, killed at time* $\zeta := \inf\{t > 0 : \int_0^t k(Z_s)\mathrm{d}s \geq \mathbb{e}\}$.

$$Z_t = z + \ll \ LCB(\Psi, c) \text{ dynamics } \gg$$
$$+ \sigma^2 \int_0^t b(Z_s)\mathrm{d}s + \int_0^t \int_0^{q(Z_{s-},y)} \int_0^{\infty} y\mathcal{N}(\mathrm{d}s, \mathrm{d}u, \mathrm{d}y),\ t < \zeta,$$

where $\mathbb{e}$ *is a standard exponential r.v. ,* $\mathcal{N}$ *a Poisson measure of intensity* $\mathrm{d}s\,\mathrm{d}u\,\pi(\mathrm{d}y)$, *everything is mutually indep.*

$\rightarrow$ size-dependent immigration, see also Z. Li's work (2019).

Starting from zero : immortal individual

Theorem (Foucart, R., Winter (2024))

- $$\mathbb{P}_z^{\uparrow} \underset{z \to 0+}{\Longrightarrow} \mathbb{P}_0^{\uparrow}, \ \textit{in Skorokhod's sense,}$$

  with $\mathbb{P}_0^{\uparrow}$ *s.t.* $\mathbb{P}_0^{\uparrow}\big(Z_0 = 0, \exists t > 0 : \forall s \geq t, Z_s > 0\big) = 1.$
- $(Z, \mathbb{P}_0^{\uparrow})$ *is weak solution to the SDE with* $z = 0$*, where* b, q, k *are defined at* 0 *by :*

  $$b(0) := 1, \quad \forall y > 0, \ q(0, y) := \frac{h(y)}{h'(0)}, \ \textit{and } k(0) := \frac{c\ell}{2} \frac{1}{h'(0)}.$$

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Study of $(Z, \mathbb{P}_z)$ and $(Z, \mathbb{P}_z^{\uparrow})$

We use **two duality relationships**.

$$(Z, \mathscr{L}) \overset{\text{Laplace dual}}{\longleftrightarrow} (U, \mathscr{A}) \overset{\text{Siegmund dual}}{\longleftrightarrow} (V, \mathscr{G})$$

$$\mathbb{E}_z(e^{-xZ_t}) = \mathbb{E}_x(e^{-zU_t}) \text{ and } \mathbb{P}_x(U_t > y) = \mathbb{P}_y(x > V_t)$$

with U et V diffusions, weak solutions to

$$\mathrm{d}U_t = \sqrt{cU_t}\mathrm{d}B_t - \Psi(U_t)\mathrm{d}t, \ U_0 = x$$
$$\mathrm{d}V_t = \sqrt{cV_t}\mathrm{d}B_t + \big(c/2 + \Psi(V_t)\big)\mathrm{d}t, \ V_0 = y.$$

We call V the **bidual** process of Z.

$\mathbb{H}$ "=" NASC (Feller's tests) for $V_t \underset{t\to\infty}{\longrightarrow} \infty$ a.s.

By combining the dualities, we get

$$\mathbb{E}_z(e^{-xZ_t}) = \int_0^\infty z e^{-zy} \mathbb{P}_y(V_t > x) \mathrm{d}y.$$

Recall the excessive function h and $s(\mathrm{d}x) = \frac{1}{x} e^{-\int_x^{x_0} \frac{2\Psi(u)}{cu} \mathrm{d}u}$. The *scale function* of V vanishing at ∞ is $S(y) := \int_y^\infty s(\mathrm{d}x)$ and

$$h(z) = \int_0^\infty z e^{-zy} S(y) \mathrm{d}y \text{ and } \mathbb{E}_z\big(h(Z_t)\big) = \int_0^\infty y e^{-zy} \mathbb{E}_y\big(S(V_t)\big) \mathrm{d}y.$$

Lemma (Foucart, R., Winter (2024))

$\big(h(Z_t), t \geq 0\big)$ *under* $\mathbb{P}_z$,

- *is a strict supermartingale (i.e. this is not a local martingale) when*
$$\int^\infty \log y \, \pi(\mathrm{d}y) < \infty \; (\Longleftrightarrow \ell > 0 \Longleftrightarrow \mathbb{E}_z(J) < \infty),$$
- *is a strict local martingale (i.e. this is not a martingale) when*
$$\int^\infty \log y \, \pi(\mathrm{d}y) = \infty \; (\Longleftrightarrow \ell = 0 \Longleftrightarrow \mathbb{E}_z(J) = \infty).$$

About h-transforms and locally harmonic functions

If T is an $(\mathcal{F}_t)_{t\geq 0}$-stopping time then for all $A \in \mathcal{F}_T$ et $z \in (0,\infty)$,

$$\mathbb{P}_z^{\uparrow}(A, T < \zeta) = \frac{1}{h(z)}\mathbb{E}_z\left(h(Z_T)\mathbb{1}_A\right). \tag{4}$$

Three different situations :

1. If $(h(Z_t), t \geq 0)$ is a $\mathbb{P}_z$-martingale, then $(Z, \mathbb{P}_z^{\uparrow})$ has an infinite lifetime : $\zeta = \infty$, $\mathbb{P}_z^{\uparrow}$-a.s.
2. If $(h(Z_t), t \geq 0)$ is a $\mathbb{P}_z$- *strict* supermartingale (i.e. this is not a local martingale), $(Z, \mathbb{P}_z^{\uparrow})$ has a finite lifetime and it is killed with positive probability. One has $\mathbb{P}_z^{\uparrow}(Z_{\zeta-} < \infty) > 0$.
3. If $(h(Z_t), t \geq 0)$ is a $\mathbb{P}_z$- *strict* local martingale (i.e. this is not a true martingale), $(Z, \mathbb{P}_z^{\uparrow})$ has a finite lifetime but is not killed. It explodes : $Z_{\zeta-} = \infty$, $\mathbb{P}_z^{\uparrow}$-a.s..

Thank you for you attention !